

# Knots

**Time and place:** March 26 & 27

**Website:** <http://scheimbauer.at/teaching/knots25/knots.html>

## 1. GENERAL

Every talk should be **75 minutes** long. Using the outline, first find references that you plan to use. You should always look at several references to see different viewpoints on your topic. Please send us an outline of your talk including the references you used 2-3 weeks before the talk. We will have a Q & A session on March 3 and encourage you to use that time to ask us questions about your talk. Every talk should contain (roughly):

- 3 examples and a counterexample
- a proof of a mathematical statement
- at least one drawing/illustration/demonstration – with a topic like this, several are usually possible and encouraged! Be creative!

There are different ways to give a talk. Be creative! For instance, you can bring materials (ropes, string, etc.); for a lot of concepts, there are nice animations which you can find online (e.g. [Numberphile](#)). We will have blackboards and a beamer. You could:

- give a board talk.
- write on a tablet if you have one.
- prepare a presentation e.g. with LaTeX. We do not recommend this method, because it makes it very difficult to react to questions, and almost always, students are too fast. I'd rather have you spend time on learning mathematics and thinking about how to present rather than spend time on making slides pretty, which can turn into a rabbit hole very fast.
- In all options, you can and should show short videos, photos, or animations. [Numberphile](#) is a wonderful resource.
- Please prepare a handout for everyone. If you send it to us by (at the latest) March 25, we will print it for everyone.

As a participant, we encourage you to participate actively by asking questions. This helps the speaker, in particular in our small group, and creates a friendly atmosphere.

After every talk, we will give **constructive feedback**, starting with something positive. Everyone is required to give feedback twice and ask a question twice throughout the seminar. That requires some courage in the beginning, but you will see that once you start, it gets easier!

## 2. TOPICS

Overview: The references we added are the first sources of information. Find your own as well!

- (1) Definition of knots and links, knot diagram/projection, Reidemeister moves, first invariants: alternating knots, unknotting number, tricoloring, optional: towards Alexander polynomial [?], [?], [?]
- (2) Jones polynomial via bracket polynomial [?]
- (3) Conway's tangles and continued fractions [?]
- (4) Braids, the braid group, braid knots (every link is closure of a braid), and torus knots [?, Chapter 5]

- (5) Borromean rings and Brunnian links: “they do not exist in  $\mathbb{R}^3$ ”, they are linked [?]
- (6) “Applications” to biology, DNA, ... [?] [?, Chapter 13], fun database: <https://knotprot.cent.uw.edu.pl/>
- (7) Seifert surface of a knot, knot genus, Seifert matrix, Alexander polynomial and relation to tricoloring (needs classification of surfaces) [?], [?, Chapters 5 and 6]
- (8) Wirtinger presentation of a knot and computability (needs fundamental group of the knot complement) [?, Chapter 8]
- (9) Witten’s brilliant realization: From Chern-Simons 3-dimensional topological field theory to the Jones polynomial [?] *alternative/additional*: Jones polynomial from 6-vertex model [?, Chapter 12] (both need some basics QFT ideas)
- (10) Khovanov homology: categorifying the Jones polynomial. [?], [?]

**2.1. Knots, Reidemeister moves, tricoloring.** Explain the definition of knots and links – for this, you will have to take an intuitive approach, while being honest about the precise definition (as some students, including yourself, may not know the mathematically precise definition of embedding and isotopy). Discuss the equivalence of knots via ambient isotopy. Explain knot diagrams/projections and Reidemeister moves. If you like, you may sketch why the Reidemeister moves are enough. Introduce some first invariants and types of knots: alternating knots, unknotting number, tricoloring. generalize to  $p$ -colorings and take first steps towards the Alexander polynomial.

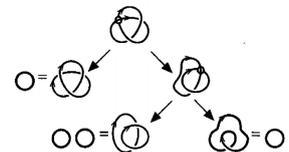


If time permits,

*References:* [?], [?], [?]

*Useful prerequisites:* embedding, smooth map, isotopy, fundamental group

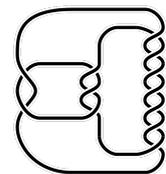
**2.2. Jones polynomial and the Kauffman bracket.** Explain the bracket polynomial (Kauffman bracket). Define the Jones polynomial and show that it is a knot invariant. Compute some examples and show that two knots which are not obviously different are indeed different.



*References:* [?], every standard text on knots!

*Useful prerequisites:* ring, module

**2.3. Conway’s tangles and continued fractions.** Starting with a finite sequence of integers, construct a tangle. Such a tangle is called a *rational tangle*. Explain what a continued fraction is and how to relate rational tangles and continued fractions. You may want to use ropes and helpers to demonstrate the “square dance”.



*References:* [?], [?]

*Useful prerequisites:* continued fraction

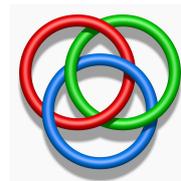
**2.4. Braids and torus knots.**



Explain what a braid is, define the braid group, and explain the connection. Show that every knot can be obtained as the closure of some braid [?]. Introduce torus knots. Explain why  $K(p, q) \cong K(q, p)$  [?]. If you have extra time, find some fun facts about braids and/or torus knots.

*Useful prerequisites:* surface (torus), fundamental group, embedding

**2.5. Borromean rings and Brunnian links.** Define Brunnian links. Special cases are the Borromean rings and the Tait link. Explain the following non-topological result: Borromean rings do not exist! In this case, we want each ring to lie in a plane. Topologically (if we are allowed to bend the rings), they do exist. Show that they are indeed linked.



*References:* [?]

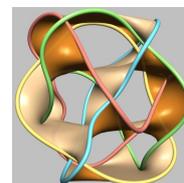
*Useful prerequisites:* surface, embedding

**2.6. Applications of knots, e.g. biology.** Explain some “applications” to biology, DNA, . . . You may have to introduce more concepts from biology rather than show mathematical proofs. You may find the talk about tangles to be helpful. Explore this topic! There are also plenty of fun videos of knots appearing in nature, but keep in mind that it should still be a mathematical talk.



*References:* [?] [?, Chapter 13], [?], [?] fun database is [here](#), picture by Javier Arsuaga was taken from [this](#) blog post.

**2.7. Seifert surfaces.** Introduce the Seifert surface of a knot, the knot genus, and the Seifert matrix. (Re)define the Alexander polynomial and explain the relation to tricoloring. Do some computations! (You may need to recollect/explain the classification of surfaces.)



*References:* [?], [?, Chapters 5 and 6]

*Useful prerequisites:* surface, genus, homology

**2.8. Knot groups.**

$$\pi_1(\mathbb{R}^3 \setminus 3_1) = \langle x, y \mid xyx = yxy \rangle$$

Introduce the knot group and explain the Wirtinger presentation of a knot. Touch upon computability (You will need to at least heuristically introduce the fundamental group.)

*References:* [?, Chapter 8]

*Useful prerequisites:* fundamental group, embedding

**2.9. Witten’s paper: Chern-Simons theory and the Jones polynomial.**

$$S(A) = \frac{k}{4\pi} \int_M \text{Tr} A \wedge dA + \frac{2}{3} A \wedge A \wedge A$$

Explain Witten’s brilliant realization that we can recover the Jones polynomial from Chern-Simons 3-dimensional topological field theory [?]

*alternative/additional:* Jones polynomial from 6-vertex model [?, Chapter 12]

(Both subjects need some basic QFT ideas – be gentle on the audience.)

*Useful prerequisites:* manifold, smooth map, cobordism

**2.10. Khovanov homology: categorifying the Jones polynomial.** Introduce Khovanov homology. Explain the idea of categorification: instead of assigning a polynomial to a knot, one assigns a graded vector space. Explain the relation between Khovanov homology and the Jones polynomial (Euler characteristic, give examples). Describe the construction at a diagrammatic level. Do some computations! Give ideas of the invariance of Khovanov homology under Reidemeister moves. If time permits, discuss functoriality.

*References:* [?], [?]

*Useful prerequisites:* manifold, smooth map, chain complex, graded vector space, homology, cobordism