Technische Universität München
Zentrum Mathematik

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Sheet 8

Prof. Dr. Claudia Scheimbauer
Anja Švraka

Bordisms and Topological Field Theories [MA5133]

Solutions for the following exercises can be found on the links `https://upennig.weebly.com/uploads/7/4/0/3/74037187/2d-tqft.pdf` and `https://math.berkeley.edu/~qchu/TQFT.pdf`.

Exercise 1. ***Class functions as Frobenius algebra***
Let G be a finite group of order n. A *class function* on G is a function $G \to \Bbbk$ which is constant on each conjugacy class. The class functions on G form a ring $\mathrm{Map}(G, \Bbbk)^G$ under the convolution product, i.e.

$$\phi * \psi(x) := \sum_{x_1 x_2 = x} \phi(x_1)\psi(x_2).$$

(a) Show that the bilinear pairing

$$\kappa(\phi, \psi) := \frac{1}{n} \sum_{t \in G} \phi(t)\psi(t^{-1})$$

gives $\mathrm{Map}(G, \Bbbk)^G$ the structure of a κ-Frobenius algebra.

(b) Show that under the identification

$$\begin{aligned} \mathrm{Map}(G, \Bbbk)^G &\to \Bbbk[G] \\ f &\mapsto \sum_{g \in G} f(g)g, \end{aligned}$$

$\mathrm{Map}(G, \Bbbk)^G$ bijectively corresponds to the center $Z(\Bbbk[G])$ of the group algebra.

Exercise 2. ***Principal G-bundles***
Let G denote a topological group, and let $\mathrm{Prin}_G(X)$ denote the isomorphism classes of principal G-bundles over the space X.

(a) Show that

$$\mathrm{Prin}_G(S^1) \cong \mathrm{Hom}(\mathbb{Z}, G)/_\sim \; = G/_\sim \,,$$

where the equivalence relation $\sim$ is generated by conjugation in G.

(b) Show that for any surface X there is a bijection

$$\mathrm{Hom}(\pi_1(X), G)/G \xrightarrow{\sim} \mathrm{Prin}_G(X).$$

Exercise 3. ***Dijkgraaf-Witten theory***

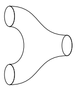
Recollection. For every M a bordism in $\mathrm{Bord}_2^{\mathrm{or}}$ with boundary $\partial M = \Sigma_1 \sqcup \Sigma_2$, restricting a principal G-bundle on M to either of the boundary components Σ_i, $i \in \{1,2\}$, yields a principal G-bundle $P\big|_{\Sigma_i} \to \Sigma_i$. If $P \to M$ is a principal G-bundle, then we denote by $\mathrm{Aut}(P)$ the G-equivariant homeomorphisms $P \to P$ that cover the identity of M.

Dijkgraaf-Witten theory is an oriented 2-TFT $\mathcal{Z}$ constructed by sending objects M to $\mathcal{Z}(M) := \mathrm{Map}(\mathrm{Prin}_G(M), \Bbbk)$. For Σ an oriented 2-dimensional bordism from M_1 to M_2 the assignment is

$$\mathcal{Z}(\Sigma) : \quad \mathcal{Z}(M_1) \longrightarrow \mathcal{Z}(M_2)$$

$$\mathcal{Z}(\Sigma)(f)(p_2) = \sum_{p_1 \in \mathrm{Prin}_G(M_1)} \sum_{p \in C_\Sigma(p_1,p_2)} f(p_1) \frac{\#\mathrm{Aut}(p_2)}{\#\mathrm{Aut}(p)},$$

where $C_\Sigma(p_1,p_2) = \{p \in \mathrm{Prin}_G(\Sigma) \mid p_{|M_i} = p_i, i \in \{1,2\}\}$.

(a) Compute $\mathcal{Z}\Big($  $\Big)$.

(b) Compute the value of this $\mathcal{Z}$ on the genus g surface with 2 disks removed, i.e. compute $\mathcal{Z}(\Sigma_g \backslash D \sqcup D)$.