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Sheet 6

Bordisms and Topological Field Theories [MA5133]

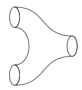
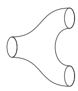
Exercise 1. ***Euler TFTs***
Fix a non-zero number $\lambda \in \mathbb{C}$. Let F be a map

$$F : \mathrm{Bord}_{n,n-1} \to \mathrm{Vect}_{\mathbb{C}}$$

such that for any closed $(n-1)$-manifold X we assign $F(X) = \mathbb{C}$. Let Σ be an n-manifold where $\Sigma : X \to Y$ in $\mathrm{Bord}_{n,n-1}$, then $F(\Sigma) : \mathbb{C} \to \mathbb{C}$ is given by multiplication with $\lambda^{\chi(\Sigma)-\chi(X)}$. Check that this is a well-defined invertible TFT.

Exercise 2. ***TFTs as categorified bordism invariants***
Let TFT_n be a category of n-dimensional topological field theories with natural transformations, namely $\mathrm{TFT}_n = \mathrm{Fun}^{\otimes}(\mathrm{Bord}_{n,n-1}, \mathcal{C})$. Show that an endomorphism of the trivial TFT $\mathbb{1}$ in TFT_n yields a bordism invariant. Which categories $\mathcal{C}$ should we use?

Exercise 3. ***2-dimensional TFTs and Frobenius algebras***
Let $\mathcal{Z} : \mathrm{Bord}_{2,1} \to \mathcal{C}$ denote a 2-dimensional topological field theory with an arbitrary target category. Set

$m := \mathcal{Z}\Big($  $\Big)$ and $\Delta := \mathcal{Z}\Big($  $\Big)$.

Show that A is a commutative Frobenius algebra, i.e. an algebra and coalgebra such that

$$(m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = \Delta \circ m = (\mathrm{id} \otimes m) \circ (\Delta \otimes \mathrm{id}).$$

Exercise 4. ***Guided proof of classification theorem***
Definition 1. Let x, y be two dualizable objects in a symmetric monoidal category $\mathcal{C}$, and $f : x \to y$ be a morphism. Then the dual morphism $f^\vee$ is given by

$$f^\vee : y^\vee \xrightarrow{\mathrm{id}_{y^\vee} \otimes \mathrm{coev}_x} y^\vee \otimes x \otimes x^\vee \xrightarrow{\mathrm{id}_{y^\vee} \otimes f \otimes \mathrm{id}_{x^\vee}} y^\vee \otimes y \otimes x^\vee \xrightarrow{\mathrm{ev}_y \otimes \mathrm{id}_{x^\vee}} x^\vee.$$

The goal of the exercise is to prove the following theorem.

Theorem 1. Let $\mathcal{C}$ be a symmetric monoidal category. Then the map

$$\begin{aligned}\Phi : \mathrm{TFT}^{or}_{1,0}(\mathcal{C}) &\to \left(\mathcal{C}^{\text{dualizable}}\right)^{\sim} \\ \mathcal{Z} &\mapsto \mathcal{Z}(\bullet +)\end{aligned}$$

is an equivalence of groupoids.

(a) Let $\eta : F \Rightarrow G$ be a symmetric monoidal natural transformation between two symmetric monoidal functors $F, G : \mathcal{C} \to \mathcal{D}$. Show that for any dualizable object $x \in \mathcal{C}$, we have that $\eta(x)$ is invertible and

$$\eta\left(x^{\vee}\right) = \left(\eta(x)^{-1}\right)^{\vee} = \left(\eta(x)^{\vee}\right)^{-1}.$$

(b) Conclude that $\mathrm{TFT}^{\mathrm{or}}_{1,0}$ is a groupoid.

(c) Prove that Φ is faithful.

(d) Prove that Φ is full.

(e) Prove that Φ is essentially surjective, i.e. every object in the target category $\left(\mathcal{C}^{\text{dual.}}\right)^{\sim}$ is isomorphic to an object in the image of Φ.