Technische Universität München
Zentrum Mathematik

Winter Semester 2023
Sheet 5

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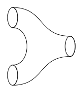
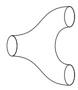
Bordisms and Topological Field Theories [MA5133]

Exercise 1. ***TFTs valued in*** $(\boldsymbol{Vect}_{\Bbbk}, \otimes)$

Let $\mathcal{Z} : \mathrm{Bord}_{n,n-1} \to \mathrm{Vect}_{\Bbbk}$ be an n-dimensional topological field theory. Prove that for every closed $(n-1)$-manifold Y, the vector space $F(Y)$ is finite dimensional.

Exercise 2. ***Algebra and coalgebra structure from a 2-dimensional TFT***

Let $\mathcal{Z} : \mathrm{Bord}_{2,1} \to \mathrm{Vect}_{\Bbbk}$ denote a 2-dimensional topological field theory. Set $A = \mathcal{Z}(S^1)$,

$m := \mathcal{Z}\Big($  $\Big)$ and $\Delta := \mathcal{Z}\Big($  $\Big)$.

(a) Show that m is a unital, associative and commutative product on A.

(b) Show that Δ is a coassociative and cocommutative coproduct

$$\Delta\colon A \longrightarrow A \otimes A.$$

That is, show that Δ satisfies

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta, \qquad (\textit{coassociativity}),$$
$$\beta \circ \Delta = \Delta, \qquad (\textit{cocommutativity}),$$

where β denotes the braiding in $\mathrm{Vect}_{\Bbbk}$.

(c) Find a counit ε for A, i.e. a map $\varepsilon : A \to \Bbbk$ such that

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \Delta.$$

Exercise 3. ***Embedding diffeomorphisms in the bordism category.***

Definition 1. Let Y be a closed $(n-1)$-manifold. An *isotopy* is a smooth map $F\colon [0,1] \times Y \to Y$ such that $F(t,-)\colon Y \to Y$ is a diffeomorphism for all $t \in [0,1]$. A *pseudoisotopy* is a diffeomorphism $\widetilde{F} : [0,1] \times Y \to [0,1] \times Y$ which preserves the submanifolds $\{0\} \times Y$ and $\{1\} \times Y$. For a manifold Y, the group of diffeomorphisms modulo isotopies is called the *mapping class group* $MCG(Y)$ of Y.

(a) The goal of the first part of the exercises is to construct a homomorphism $MCG(Y) \to \mathrm{Bord}_{n,n-1}(Y, Y)$.

  (i) Let $f : Y \to Y$ be a diffeomorphism of Y. Construct a cobordism X_f from Y to Y associated to f.

  (ii) Show that (pseudo-)isotopic diffeomorphisms of Y produce equal associated cobordisms in $\mathrm{Hom}_{\mathrm{Bord}_{n,n-1}}(Y, Y)$.

  (iii) Is homomorphism from above injective for $n = 1$ and $Y = * \amalg *$?

(b) Given a category $\mathcal{C}$, one can form the quotient $\pi_0\mathcal{C} = \mathrm{Ob}\,\mathcal{C}/\sim$, where two objects are equivalent if there is a morphism between them (in either direction). What is $\pi_0\mathrm{Bord}_{n,n-1}$?