Technische Universität München
Zentrum Mathematik

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Sheet 4

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Bordisms and Topological Field Theories [MA5133]

Exercise 1. ***Elementary cobordisms***
Definition 1. An *elementary cobordism* is a manifold with boundary posessing an admissible Morse function f with exactly one critical point p.

(a) Find all the elementary orientable bordisms for $n = 1, 2$.

(b) Find three non-examples of elementary cobordisms.

(c) Decompose Σ_2, i.e. the genus 2 surface, into elementary cobordisms.

(d) Which elementary bordisms you found in part (a) can be framed?

Exercise 2. ***Euler characteristic (mod 2) as a bordism invariant***

(a) Let M, N be two n-dimensional compact closed manifolds. Prove (or look up) the following formula for the Euler characteristic:

(i) $\chi(M \times N) = \chi(M)\chi(N)$,

Let M be an n-dimensional compact manifold with boundary. Prove (or look up) the following formula for the Euler characteristic:

(ii) $\chi(\partial M) = (1 - (-1)^{\dim M})\chi(M)$.

(b) Prove that the Euler characteristic modulo 2 is an unoriented cobordism invariant.

Remark: If you are unfamiliar with the Euler characteristic, you can use the following formula for the above exercise for $n = 2$. Let M be a connected compact surface of genus g with b boundary components. Then

$$\chi(M) = 2 - 2g - b.$$

Exercise 3. ***Monoidal categories***
Show that the following categories are monoidal. How would you define the "symmetric monoidal" category (in analogy to commutative monoids and discussion from the lectures)? Check if the examples below are symmetric monoidal for your definition. Can you find any extra structure in the braid group example?

(a) $(\mathrm{Vect}_{\mathbb{k}}, \oplus)$

(b) $(\mathrm{Vect}_{\Bbbk}, \otimes)$

Remark. Given a family of groups $\{G_n\}_{n\in\mathbb{N}}$ such that $G_0 = \{1\}$ we can build a category G whose objects are the non-negative integers and whose morphisms are given by

$$\mathrm{Hom}_G(n, m) = \begin{cases} \varnothing & \text{if } m \neq n \\ G_n & \text{if } m = n. \end{cases}$$

Definition 2. The *braid group* B_n is the group with generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ subject to the relations $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$ for all $1 \leqslant i \leqslant n-2$, and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i - j \geqslant 2$.

(c) Let $B = \big(\{B_n\}_{n\in\mathbb{N}}, \otimes\big)$ be a category constructed as discussed in the remark above using the braid groups. Monoidal structure $\otimes : B \times B \to B$ is defined by addition on objects and group homomorphisms $\rho_{m,n} : B_n \times B_m \to B_{n+m}$ given by $\rho_{m,n}(\sigma_i, \sigma_j) = \sigma_i\sigma_{m+j}$ for all $1 \leqslant i \leqslant m-1$, $1 \leqslant j \leqslant n-1$.

Definition 3. The *symmetric group* S_n is the group with generators $s_1, s_2, ..., s_{n-1}$ subject to the relations $s_{i+1}s_is_{i+1} = s_is_{i+1}s_i$ for all $1 \leqslant i \leqslant n-2$, $s_is_j = s_js_i$ for $i - j \geqslant 2$ and $s_i^2 = 1$ for all $i \in \{1, ..., n-1\}$.

(d) Let $S = \big(\{S_n\}_{n\in\mathbb{N}}, \otimes\big)$ be a category constructed as discussed in the remark above using the symmetric groups, with the same monoidal structure as in (c).