Technische Universität München
Zentrum Mathematik

Winter Semester 2023
Sheet 3

Prof. Dr. Claudia Scheimbauer
Anja Švraka

Bordisms and Topological Field Theories [MA5133]

Exercise 1. ***Morse functions***

(a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x^3 - 3xy^2$. Find all the critical points and check if f is a Morse function. If it does not meet the criteria, perturb it in such a way that it becomes a Morse function.

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto x^2y^2$. Find all the critical points and check if f is a Morse function. If it does not meet the criteria, perturb it in such a way that it becomes a Morse function.

(c) Show that if $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ are Morse functions, then $f + g : M \times N \to \mathbb{R}$ is also a Morse function, and the critical points are pairs of critical points of f and g. Visualize this for $M = N = S^1$ and $f : S^1 \subset \mathbb{R}^2 \to \mathbb{R}$ the projection onto the first coordinate.

Exercise 2. ***Handle decomposition***

Definition 1. Let M be a compact 2-manifold. A **handle decomposition** of M is a finite sequence of manifolds

$$\varnothing = W_{-1} \subseteq W_0 \subseteq W_1 \subseteq W_2 = M$$

such that each W_i is obtained from W_{i-1} by attaching i-handles.

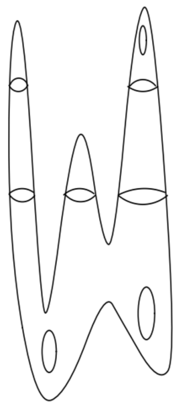
(a) Find two different handle decompositions of S^2.

(b) Find a handle decomposition of $\mathbb{RP}^2$.

(c) Find a handle decomposition of the Klein bottle.

(d) Explain why for any non-empty closed connected surface we can start a handle decomposition with a single 0-handle. *Hint*: The key argument was mentioned in lectures as "*handle cancellation*".

(e) Using the idea of handle cancellation, bring the surface below into *normal form*, i.e. such that read from bottom to top the index of the critical points are non-decreasing.



Exercise 3. ***Reading exercise - Classification of closed 1-manifolds***
Prove the following theorem using Morse theory and/or read through the proof in `https://www.math.csi.cuny.edu/~abhijit/papers/classification.pdf` [Theorem 15].

Theorem 1. Any closed 1-manifold is homeomorphic to S^1.