Technische Universität München
Zentrum Mathematik

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Sheet 2

Prof. Dr. Claudia Scheimbauer
Anja Švraka

Bordisms and Topological Field Theories [MA5133]

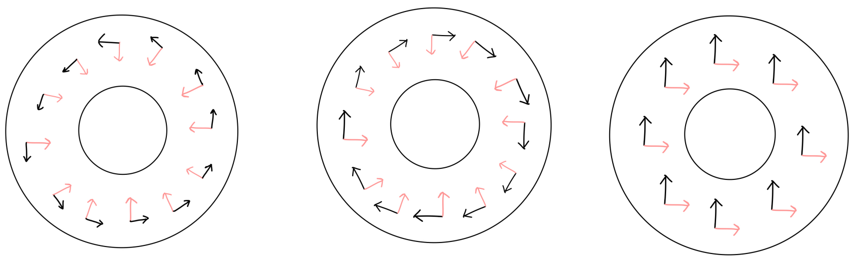
Exercise 1. ***Framings***
Justify your answers to the following questions:

(a) Can a Klein bottle be framed?

(b) Can S^2 be framed?

(c) An **isotopy between the framings** is a deformation given by a family of framings parameterized by the interval. Are any of the framed cylinders below isotopic?

(d) Which of the framings induce the same orientations?



Exercise 2. ***Attaching handles***
Definition. Let B^n denote the n-dimensional ball as a manifold with boundary and S^n the n-dimensional sphere.

Given a 2-dimensional manifold M, we *attach a j-handle* $H^j := B^j \times B^{2-j}$, for $j \in \{0, 1, 2\}$ via and a smooth embedding $f : S^{j-1} \times D^{2-j} \hookrightarrow \partial M$ as follows:

$$M \cup_f H^j := \big(M \sqcup (B^j \times B^{2-j})\big)/\sim$$

where for $(p, x) \in S^{j-1} \times B^{2-j} \subset B^j \times B^{2-j}$, we set $f(p, x) \sim (p, x)$.

(a) Convince yourself that there is a smooth structure on $M \cup_f H^j$.

(b) Which surface is obtained from attaching a 1-handle to a disk?

(c) Which surface is obtained from attaching two 1-handles to a disk, i.e. from attaching an additional 1-handle to the surface obtained in part (a)?

(d) Build the torus by successively attaching handles to a disk.

Exercise 3. ***Properties of the connected sum of manifolds***

(a) Given n-manifolds M, M', and M'', show that the connected sum satisfies the following properties.

   (i) $M\#S^n \cong M$, *(neutral element)*

   (ii) $M\#M' \cong M'\#M$, and *(commutativity)*

   (iii) $(M\#M')\#M'' \cong M\#(M'\#M'')$. *(associativity)*

(b) If M and M' are smooth n-manifolds, construct a smooth structure on the connected sum $M\#M'$. Note that this is not unique, but defines a well-defined diffeomorphism class. You may like to read more details using isotopies in Chapter 8, Section 2 in Hirsch, Differential Topology[1].

Exercise 4. ***Reading exercise***

Below is a list of several proofs of the classification theorem of 1-dimensional manifolds, using different tools. Read through one (or several) of them or find your own.

(i) `https://pnp.mathematik.uni-stuttgart.de/igt/eiserm/lehre/2014/Topologie/Gale%20-%201-manifolds.pdf`

(ii) Appendix of `https://www.maths.ed.ac.uk/~v1ranick/papers/milnortop.pdf`, starting at p.55.

[1]Can e.g. be accessed at `https://www.researchgate.net/publication/268035774_Differential_Topology`.