Technische Universität München
Zentrum Mathematik

Winter Semester 2023
<u>Sheet 12</u>

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Bordisms and Topological Field Theories [MA5133]

Exercise 1. ***Finite dimensional modules of a Hopf algebra***
Let H be a Hopf algebra with invertible antipode S. Show that any finite-dimensional H-module has a left and right dual.

Solutions can be found on the following link `https://www.math.uni-hamburg.de/home/schweigert/skripten/hskript.pdf` as part of the Proposition 2.5.16.

Exercise 2. ***Sweedler's Hopf algebra***
Consider the $\mathbb{C}$-algebra H generated by two elements C and X subject to the relations

$$C^2 = 1, \quad X^2 = 0 \quad \text{and} \quad CX + XC = 0.$$

The comultiplication, count, and antipode are defined by

$$\Delta(C) = C \otimes C, \quad \Delta(X) = 1 \otimes X + X \otimes C,$$
$$\varepsilon(C) = 1, \quad \varepsilon(X) = 0, \quad S(C) = C \quad \text{and} \quad S(X) = CX.$$

(a) Prove that this indeed defines a Hopf algebra.

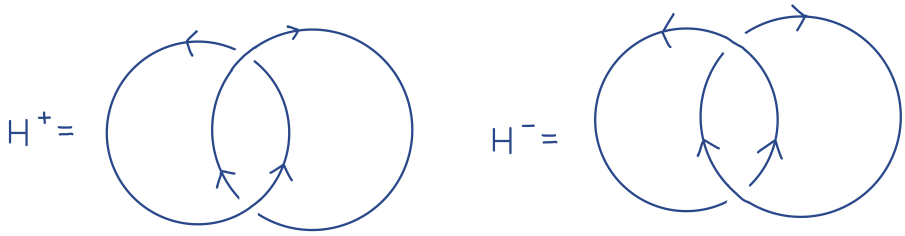
(b) Is this Hopf algebra (co)commutative?

Solutions can be found on the following link `https://www.math.uni-hamburg.de/home/robert/Hopf/ws14hopf9sol.pdf` as Problem 3.

Exercise 3. ***Jones polynomial of Hopf links***

(a) Compute the Jones polynomial of the positive Hopf link H^+ in two different ways: first by the Kaufmann bracket and then using the skein relations.

(b) Compute the Jones polynomial of the negative Hopf link H^-. Do you see any relation between this result and the result of (b)?



Reminder

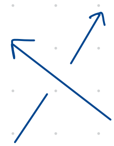
Kaufmann bracket

- $\langle O \rangle = 1$
- $\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \cdot \langle$  $\rangle$
- $\langle L \cup O \rangle = (-A^2 - A^{-2}) \langle L \rangle$

$$w(D) = \sum_{\substack{c \text{ crossing} \\ \text{in } D}} \operatorname{sign}(c)$$

$$X(L) = (-A^3)^{-w(p(L))} \langle p(L) \rangle$$

$$V(L) = X(L)\Big|_{A = t^{-1/4}}$$ ~ Jones polynomial

$\operatorname{sign}\Big($  $\Big) = +1$, $\quad \operatorname{sign}\Big($  $\Big) = -1$

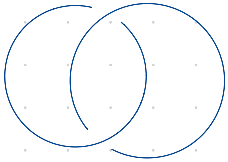
- skein relation

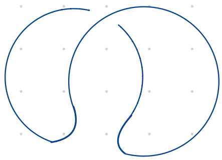
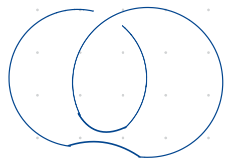
$$t^{-1} V(L_+) - t V(L_-) + (t^{-1/2} - t^{1/2}) V(L_0) = 0$$

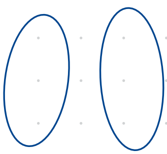
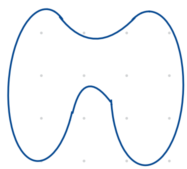
Made with Goodnotes

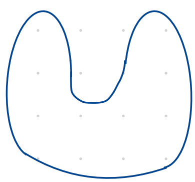
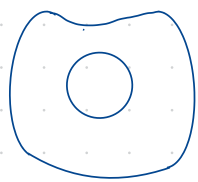
03 Jones polynomial of Hopf links

(a) | Start w/ Kaufmann bracket

$\langle$  $\rangle$

$= A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$= A\Big(A\langle$  $\rangle + A^{-1}\langle$  $\rangle\Big)$

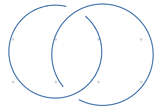
$+ A^{-1}\Big(A\langle$  $\rangle + A^{-1}\langle$  $\rangle\Big)$

$$= A\left(A\left(-A^2-A^{-2}\right)+A^{-1}\right)+A^{-1}\cdot\left(A+A^{-1}\left(-A^2-A^{-2}\right)\right)$$

$$= -A^4-\cancel{1}+\cancel{1}+\cancel{1}-\cancel{1}-A^{-4} = -A^4-A^{-4}$$

$\langle L \cup O\rangle = (-A^2-A^{-2})\langle L\rangle$, $\langle O\cup O\rangle = -A^2-A^{-2}$

Made with Goodnotes

$\chi(H^+) = (-A^3)^{-w(H^+)} \langle$  $\rangle$

$w(H^+) = \text{sign}($  $) + \text{sign}($  $) = 1 + 1 = 2$

$$\chi(H^+) = (-A^3)^{-2}(-A^4 - A^{-4})$$

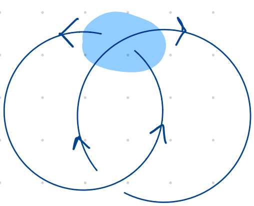
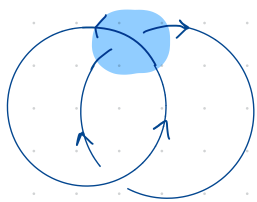
$$= -A^{-2} - A^{-10}$$

$$V(H^+) = -t^{1/2} - t^{5/2} = -\sqrt{t} - \sqrt{t^5}$$

$\uparrow$

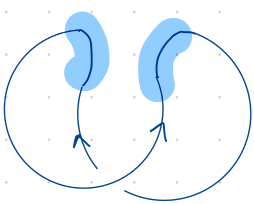
$A = t^{-1/4}$

II Using skein relations:

$L_+ =$  , $L_- =$  ,

$\parallel$

H^+

$L_0 =$ 

Made with Goodnotes

skein relation:

$$t^{-1} V(L_+) - t V(L_-) + (t^{-1/2} - t^{1/2}) V(L_0) = 0$$

$$L_+ = H^+$$

$$V(L_0) = V(\bigcirc) \quad \text{(Reidemeister I)}$$

$$V(L_0) = 1$$

$$V(L_-) = V(\bigcirc \bigcirc) \quad \text{(Reidemeister II)}$$

$$= -t^{1/2} - t^{-1/2}$$

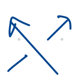
$$t^{-1} V(L_+) + \cancel{t^{1/2}} + t^{3/2} + t^{-1/2} - \cancel{t^{1/2}} = 0$$

$$V(t_+) = -t\left(t^{3/2} + t^{-1/2}\right) = -t^{5/2} - t^{1/2}$$

(b) $$V(H^-) = \chi(H^-)\Big|_{A = t^{-1/4}}$$

$$\chi(H^-) = (-A^3)^{-w(H^-)} \langle p(H^-) \rangle$$

$$p(H^-) = p(H^+) \Rightarrow \langle p(H^-) \rangle = -A^4 - A^{-4}$$

$w(H^-) = \text{sign}\big($  $\big) + \text{sign}\big($  $\big) = -2$

Made with Goodnotes

$$\Rightarrow \quad \chi(H^-) = (-A^3)^{+2}(-A^4 - A^{-4})$$

$$= -A^{10} - A^2$$

$$V(H^-) = -t^{-5/2} - t^{-1/2} \qquad \blacksquare$$

Made with Goodnotes