Technische Universität München
Zentrum Mathematik

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Sheet 11

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# Bordisms and Topological Field Theories [MA5133]

**Exercise 1.** ***Example of a representation of quantum enveloping algebra of*** $\mathfrak{sl}(2)$

(a) Let $q$ be an invertible element of $\Bbbk$ different from 1 and $-1$ so that the fraction $\frac{1}{q-q^{-1}}$ is well-defined. Define $U_q = U_q(\mathfrak{sl}_2)$ as the algebra generated by the four variables $E, F, K, K^{-1}$ subject to the relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E,$$
$$KFK^{-1} = q^{-2}F \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Let the multiplication be defined by

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$
$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}.$$

Let the counit and antipode be defined by

$$\varepsilon(E) = 0 = \varepsilon(F), \quad \varepsilon(K) = 1 = \varepsilon(K^{-1}),$$
$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1} \quad \text{and} \quad S(K^{-1}) = K.$$

Prove that this defines a Hopf algebra.

(b) Let $V$ be a two-dimensional vector space with basis $\{v_{-1}, v_1\}$. Let $\rho$ be a map $U_q(\mathfrak{sl}_2) \to \textsc{End}(V)$ defined by

$$E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$K \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

Show that $(V, \rho)$ is a $U_q(\mathfrak{sl}_2)$-module. This is also called a representation of the $U_q(\mathfrak{sl}_2)$.

**Exercise 2.** ***Relation between the universal enveloping algebra*** $U(\mathfrak{sl}_2(\mathbb{C}))$ ***and the quantum enveloping algebra*** $U_q(\mathfrak{sl}_2(\mathbb{C}))$

(a) Using the basis $h, e$ and $f$ given below, show that the relations given below hold and define the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$[1].

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$$

(b) Let $U := U(\mathfrak{sl}_2(\mathbb{C}))$ be the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$. Convince yourself that there is an isomorphism of the following Lie algebras

$$U(\mathfrak{sl}_2(\mathbb{C})) \cong \frac{\mathbb{C} < x, y, z >}{(xy - yx + 2y, xz - zx - 2z, yz - zy + x)}$$

(c) This part of the exercise is optional. The solution can be found in *Quantum Groups - Christian Kassel - Proposition VI.2.2.*

**Definition.** Let $U'_q$ be an algebra generated by the five variables $E, F, K, K^{-1}, L$ and the relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F$$
$$[E, F] = L, \quad (q - q^{-1})L = K - K^{-1}, \quad [L, E] = q(EK + K^{-1}E)$$
$$[L, F] = -q^{-1}(FK + K^{-1}F),$$

where the parameter $q$ is allowed to take any value.

Prove the following relation:

$$U'_{q=1} \cong U[K]/(K^2 - 1).$$

(d) Using the following proposition conclude the relationship between the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ and the quantum enveloping algebra $U_q(\mathfrak{sl}_2(\mathbb{C}))$.

**Proposition.** The algebra $U_q := U_q(\mathfrak{sl}_2(\mathbb{C}))$[2] is isomorphic to $U'_q$.

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[1] This Lie algebra was defined in class as $\mathfrak{sl}(\mathbb{C}^2)$ or in Exercise 3 (a) on Sheet 11 as the $2 \times 2$ complex matrices with trace 0.

[2] This algebra was defined in Exercise 3 on Sheet 10.