Technische Universität München
Zentrum Mathematik

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Sheet 10

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Bordisms and Topological Field Theories [MA5133]

Exercise 1. $\mathrm{Tang}_1^{\mathrm{fr}}$ *is a ribbon category*

Definition 1. Let x, y be two dualizable objects in a symmetric monoidal category $\mathcal{C}$, and $f : x \to y$ be a morphism. Then the dual morphism $f^\vee$ is given by

$$f^\vee : y^\vee \xrightarrow{\mathrm{id}_{y^\vee} \otimes \mathrm{coev}_x} y^\vee \otimes x \otimes x^\vee \xrightarrow{\mathrm{id}_{y^\vee} \otimes f \otimes \mathrm{id}_{x^\vee}} y^\vee \otimes y \otimes x^\vee \xrightarrow{\mathrm{ev}_y \otimes \mathrm{id}_{x^\vee}} x^\vee .$$

Recollection. The category of framed tangles is right rigid and pivotal. Let k be any object in $\mathrm{Tang}_1^{\mathrm{fr}}$ category and let $\theta_k : k \to k$, where $\theta_k = (\mathrm{id}_k \otimes \mathrm{ev}_{k^\vee}) \circ (\beta_{k,k} \otimes \mathrm{id}_{k^\vee}) \circ (\mathrm{id}_k \otimes \mathrm{coev}_k)$.

(a) Draw a visual representation of the framed tangle corresponding to the twist for $k = 1$.

(b) Show that twists θ_k satisfy $(\theta_k)^\vee = \theta_{k^\vee}$.

Exercise 2. ***Lie algebras***

Definition 2. A **Lie algebra** is a vector space $\mathfrak{g}$ together with an alternating bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, satisfying the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \text{for all } x, y, z \in \mathfrak{g}.$$

(a) Let $\mathfrak{sl}_2(\mathbb{C})$ be the vector space of all two-by-two complex matrices with zero trace. Prove that the Lie bracket given by the commutator (i.e., $[A, B] = AB - BA$) gives $\mathfrak{sl}_2(\mathbb{C})$ the structure of a Lie algebra.

(b) Show that any associative $\mathbb{k}$-algebra A inherits a structure of a Lie algebra by using the commutator, i.e., $[a, b] = ab - ba$ for all $a, b \in A$.