Technische Universität München
Zentrum Mathematik

Prof. Dr. Claudia Scheimbauer
Anja Švraka

Winter Semester 2023
Sheet 1

Bordisms and Topological Field Theories [MA5133]

Note: Remarks in parentheses at the beginning of an exercise refer to necessary prerequisites.

Exercise 1. ***Abelian structure of the cobordism group***
Show that the disjoint union induces an abelian group structure on the cobordism group Ω_n.

Exercise 2. ***Orientable Manifolds***

(a) Show that the circle S^1 is an orientable manifold.

(b) Show that the sphere S^2 is an orientable manifold.

(c) Show that the total space of the tangent bundle of a smooth n-manifold is an orientable manifold.

Exercise 3. ***Computation of*** Ω_0^{or}
Compute the oriented bordism group Ω_0^{or}.

Exercise 4. ***Computation of*** Ω_2^{or}

(a) Work through the argument in detail showing that Σ_g is cobordant to the empty set.

(b) Recall that the disjoint union is cobordant to the connected sum. Work through the details for an example that is different from what was shown in the lecture.

(c) Conclude that $\Omega_2^{\mathrm{or}} = 0$. (Here you may omit details about orientations of the 3-dimensional cobordisms.)

Exercise 5. ***Computation of Ω_2^{unor}:***

(a) Show that the Klein bottle K is cobordant to the empty set.

(b) (Poincaré duality and Euler characteristic) Show that $\mathbb{RP}^2$ is non-zero in Ω_2^{unor}, i.e. that there is no compact 3-manifold X with boundary $\partial X = \mathbb{RP}^2$.
Hint: Consider the double $D = X \underset{\mathbb{RP}^2}{\cup} X$. What does Poincaré duality imply about the Euler characteristic of 3-dimensional closed manifolds?

(c) Conclude from the above that $\Omega_2^{\text{unor}} = \mathbb{Z}/2\mathbb{Z}$.