

Diss. ETH No. 22130

Factorization Homology as a Fully Extended Topological Field Theory

A thesis submitted to attain the degree of

DOCTOR OF SCIENCES of ETH ZÜRICH

(Dr. sc. ETH Zürich)

presented by

Claudia Isabella Scheimbauer

Dipl. Ing., TU Wien

born 15.06.1986

citizen of Austria

accepted on the recommendation of

Prof. Dr. Damien Calaque

Prof. Dr. Giovanni Felder

Prof. Dr. Bertrand Toën

2014

Contents

Contents	iii
Abstract	vii
Zusammenfassung	ix
Résumé	xi
Introduction	xv
Motivation	xv
Topological field theories	xv
Factorization homology and factorization algebras	xvi
Overview of the thesis	xvi
Symmetric monoidal complete n -fold Segal spaces	xvii
Definition of a fully extended n -TFT	xvii
The target: E_n -algebras	xviii
Construction of the fully extended n -TFT	xix
Guide to the reader	xx
1 Preliminaries: symmetric monoidal (∞, n)-categories	1
1.1 The homotopy hypothesis and $(\infty, 0)$ -categories	1
1.2 Complete Segal spaces as models for $(\infty, 1)$ -categories	2
1.2.1 Segal spaces	2
1.2.2 The homotopy category of a Segal space	3
1.2.3 Complete Segal spaces	3
1.2.4 Segal categories	5
1.3 Complete n -fold Segal spaces as models for (∞, n) -categories	5
1.3.1 n -fold Segal spaces	5
1.3.2 Complete and hybrid n -fold Segal spaces	7
1.4 The homotopy bicategory of a 2-fold Segal space	9
1.5 Constructions of n -fold Segal spaces	9
1.5.1 Truncation	9
1.5.2 Extension	10
1.5.3 The higher category of morphisms and loopings	11
1.6 Symmetric monoidal n -fold Segal spaces	12

1.6.1	as a Γ -object	12
1.6.2	Symmetric monoidal n -fold Segal spaces as a tower of $(n + i)$ -fold Segal spaces	15
2	The (∞, n)-category of cobordisms	21
2.1	The n -fold Segal space of closed intervals in $(0, 1)$	21
2.1.1	The spatial structure of the levels Int_k	22
2.1.2	The simplicial set Int_\bullet	25
2.1.3	The Segal space Int_\bullet	25
2.2	A time-dependent Morse lemma	27
2.2.1	The classical Morse lemma	27
2.2.2	The time-dependent Morse lemma	27
2.3	The (∞, n) -category of bordisms Bord_n	31
2.3.1	The level sets $(\text{PBord}_n)_{k_1, \dots, k_n}$	32
2.3.2	The spaces $(\text{PBord}_n)_{k_1, \dots, k_n}$	33
2.3.3	The n -fold simplicial set $(\text{PBord}_n)_{\bullet, \dots, \bullet}$	40
2.3.4	The full structure of $(\text{PBord}_n)_{\bullet, \dots, \bullet}$ as an n -fold simplicial space	42
2.3.5	The complete n -fold Segal space Bord_n	43
2.3.6	Variants of Bord_n and comparison with Lurie's definition	45
2.4	The symmetric monoidal structure on Bord_n	49
2.4.1	The symmetric monoidal structure arising as a Γ -object	49
2.4.2	The monoidal structure and the tower	50
2.5	The homotopy (bi)category	53
2.5.1	The homotopy category $h_1(L_{n-1}(\text{Bord}_n))$	53
2.5.2	The homotopy bicategory $h_2(\text{Bord}_2)$ and comparison with 2Cob^{ext}	58
2.6	Cobordisms with additional structure: orientations and framings	62
2.6.1	Structured manifolds	62
2.6.2	The (∞, n) -category of structured cobordisms	63
2.6.3	Example: Objects in Bord_2^{fr} are 2-dualizable	64
2.7	Fully extended topological field theories	67
2.7.1	Definition	67
2.7.2	n -TFT yields k -TFT	68
2.7.3	Cobordism Hypothesis à la Baez-Dolan-Lurie and outlook	69
3	The Morita (∞, n)-category of E_n-algebras	71
3.1	The n -fold Segal space of closed covers in $(0, 1)$	71
3.1.1	Collapse-and-rescale maps	71
3.1.2	The sets Covers_k	75
3.1.3	The spatial structure of Covers_k	75
3.1.4	The simplicial set Covers_\bullet	77
3.1.5	The Segal space Covers_\bullet	78
3.2	The Morita (∞, n) -category of E_n -algebras Alg_n	79
3.2.1	Structured disks and E_n -algebras	79
3.2.2	Factorization algebras	80
3.2.3	Stratifications and locally constant factorization algebras	82
3.2.4	The level sets $(\text{Alg}_n)_{k_1, \dots, k_n}$	85
3.2.5	The spaces $(\text{Alg}_n)_{k_1, \dots, k_n}$	88
3.2.6	The n -fold simplicial set Alg_n	90
3.2.7	The full structure of Alg_n as an n -fold simplicial space	92
3.2.8	The n -fold Segal space Alg_n	93

3.2.9	Completeness of Alg_n and the Morita (∞, n) -category of E_n -algebras	95
3.3	The symmetric monoidal structure on Alg_n	96
3.3.1	The symmetric monoidal structure arising as a Γ -object	96
3.3.2	The monoidal structure and the tower	97
3.4	The homotopy category of Alg_1 and the Morita category	102
3.4.1	The $(\infty, n + 1)$ -category of E_n -algebras	102
3.4.2	An unpointed version	103
4	Factorization homology as a fully extended TFT	105
4.1	Factorization Homology	105
4.2	The auxiliary (∞, n) -category Fact_n	107
4.2.1	The spaces $(\text{Fact}_n)_{k_1, \dots, k_n}$	107
4.2.2	The n -fold Segal space Fact_n	108
4.2.3	Completeness of Fact_n	110
4.2.4	The symmetric monoidal structure on Fact_n	111
4.3	The map of n -fold simplicial sets $\int_{(-)} A$	111
4.4	Collapsing the factorization algebra and FAlg_n	113
4.4.1	The collapse-and-rescale map $\mathfrak{v} : \text{Int}_\bullet \rightarrow \text{Covers}_\bullet$	114
4.4.2	The “faux” Alg_n , the n -fold Segal space FAlg_n	116
4.4.3	The collapsing map $\mathfrak{V} : \text{Fact}_n \rightarrow \text{FAlg}_n$	118
4.5	The functor of (∞, n) -categories \mathcal{FH}_n	119
4.6	The fully extended topological field theory \mathcal{FH}_n	120
4.6.1	Symmetric monoidality via Γ -objects	120
4.6.2	Symmetric monoidality via the tower	122
4.7	The simplest example: $n = 1$	123
	Bibliography	125
	Curriculum Vitae	129

Abstract

Given an E_n -algebra A we explicitly construct a fully extended n -dimensional topological field theory which is essentially given by factorization homology. Under the cobordism hypothesis, this is the fully extended n -TFT corresponding to the E_n -algebra A , considered as an object in a suitable Morita- (∞, n) -category Alg_n . We first give a precise definition of a fully extended n -dimensional topological field theory using complete n -fold Segal spaces as a model for (∞, n) -categories. This involves developing an n -fold Segal space Bord_n of n -dimensional bordisms and endowing it with a symmetric monoidal structure. Exploiting the equivalence between E_n -algebras and locally constant factorization algebras proven by Lurie we use locally constant factorization algebras on stratified spaces to construct an (∞, n) -category with E_n -algebras as objects, (pointed) bimodules as 1-morphisms, (pointed) bimodules between bimodules as 2-morphisms, etc. and endow it with a symmetric monoidal structure. Finally, given an E_n -algebra we construct a morphism of n -fold Segal spaces from Bord_n to Alg_n given by a suitable pushforward of the factorization algebra obtained by taking factorization homology. We show that this map respects the symmetric monoidal structure.

Zusammenfassung

Für eine E_n -Algebra A geben wir eine explizite Konstruktion einer vollständig erweiterten n -dimensionalen topologischen Feldtheorie, die im Wesentlichen durch Faktorisierungshomologie gegeben ist. Unter Verwendung der Kobordismus-Hypothese entspricht diese der vollständig erweiterten n -TFT, die durch die E_n -Algebra A , als Objekt einer geeigneten Morita- (∞, n) -Kategorie Alg_n betrachtet, bestimmt ist. Als Modell für (∞, n) -Kategorien benutzen wir vollständige n -fache Segalräume und geben zunächst eine präzise Definition einer vollständig erweiterten n -dimensionalen topologischen Feldtheorie. Diese benötigt die Konstruktion eines n -fachen Segalraumes n -dimensionaler Bordismen Bord_n und einer symmetrisch monoidalen Struktur darauf. Motiviert durch die Äquivalenz zwischen E_n -Algebren und lokal konstanten Faktorisierungsalgebren, die von Lurie bewiesen wurde, verwenden wir lokal konstante Faktorisierungsalgebren auf stratifizierten Räumen um eine (∞, n) -Kategorie, deren Objekte E_n -Algebren, 1-Morphismen (punktierte) Bimoduln, 2-Morphismen (punktierte) Bimoduln zwischen Bimoduln, etc. sind, und eine symmetrisch monoidale Struktur darauf zu definieren. Schließlich konstruieren wir, in Abhängigkeit einer E_n -Algebra, einen Morphismus n -facher Segalräume von Bord_n nach Alg_n , der durch einen gewissen Pushout der Faktorisierungsalgebra, die mittels Faktorisierungshomologie erhalten wird, gegeben ist. Wir zeigen, dass diese Abbildung die symmetrisch monoidale Struktur respektiert.

Résumé

Étant donné une algèbre E_n , nous construisons explicitement une théorie des champs topologiques pleinement étendue de dimension n , essentiellement donnée par l'homologie de factorisation. D'après l'Hypothèse du Cobordisme il s'agit de la n -TFT pleinement étendue qui correspond à l'algèbre $E_n A$, considérée comme un objet dans une (∞, n) -catégorie appropriée de Morita Alg_n . Nous donnons dans un premier temps une définition précise d'une théorie des champs topologiques pleinement étendue de dimension n en utilisant les espaces de Segal complets n -uples comme un modèle pour les (∞, n) -catégories. Pour cela nous construisons un espace de Segal complet n -uple $Bord_n$ de bordismes de dimension n et lui donnons une structure monoïdale symétrique. En exploitant ensuite l'équivalence, démontrée par Lurie, entre les algèbres E_n et les algèbres de factorisation localement constantes, nous utilisons des algèbres de factorisation localement constantes sur des espaces stratifiés pour construire une (∞, n) -catégorie ayant les algèbres E_n pour objets, les bimodules (pointés) pour 1-morphismes, les bimodules entre bimodules pour 2-morphismes, etc... lui donnons une structure monoïdale symétrique. Finalement, étant donné une algèbre E_n , nous construisons un morphisme entre espaces de Segal n -uples depuis $Bord_n$ vers Alg_n , donné par un pushforward de l'algèbre de factorisation obtenue par l'homologie de factorisation. Nous montrons que cette construction préserve la structure monoïdale symétrique.

Acknowledgements

First and foremost my gratitude goes to Damien Calaque. Thank you for taking me on as a student and having faith in my mathematical abilities, for answering all of my questions, however basic they were, and for introducing me to such beautiful mathematics. You always were a source of inspiration and encouragement and I deeply appreciate that our discussions while doing mathematics together, while giving new insights and creating ideas, also were filled with positive energy and fun times.

I would like to express my gratitude to Giovanni Felder for his support throughout the years. Your genuine interest in mathematics has been very inspiring.

Many thanks go to Bertrand Toën. A lot of mathematics I have learned these past years are closely related to his work and I feel honored that he agreed to be my coreferee.

In Zürich I was very fortunate to be part of a great research group. At ETH Zurich Mat(t)hieu Anel taught me about toposes, derived algebraic geometry, and Iceland. Damien (Jr.) explained a bit of French culture to me and Claudio and Giorgia brought some Italian spirit into the group. I was first introduced to many abbreviations such as BV, BRST, AKSZ, and WRT throughout several courses at the University of Zürich by Alberto Cattaneo and his group. Special thanks go to Ivàn Contreras. Our discussions have provided many insights about the interplay of mathematics and physics and I look forward to many more. Thomas Willwacher, thank you for introducing me to mathematical physics in Zürich.

In Montpellier I was welcomed very warmly by the math department, in particular by Marco, Benjamin, Samuel, Mickael, and Claudia. Thanks for enduring my French with patience.

Throughout the past years I had the opportunity to meet many inspiring people in Switzerland and elsewhere. Among them, I crossed paths several times with Hiro Lee Tanaka, from whom I first learned what a TFT and factorization homology is. Who knew then that it would turn into my thesis. Thank you for pointing out the importance of pointings and for many conversations about math and non-math. Two years ago I had the honor to learn about factorization algebras and QFT when writing the notes for Kevin Costello's lectures on this topic, thanks. I would also like to thank Owen Gwilliam and Ryan Grady for enlightening conversations about these topics. I am very grateful to Gre-

gory Ginot and Domenico Fiorenza for their interest in my work, their support, and fun times at the Warhol museum. Theo Johnson-Freyd, I thoroughly enjoyed (and still do) our conversations about lax functors and more in Vienna, where we spent a week at the ESI. There I also had the pleasure to meet Chris Schommer-Pries, whose thesis helped me to understand higher bordisms and the cobordism hypothesis. Also, I was very kindly invited by Michel Vaquié, David Jordan, and Adrien Brochier. I would also like to thank the participants of the Winter School in Derived Algebraic Geometry who made the week a great one.

I am very grateful to have met many very good friends (some of whom have already been mentioned above) through math during these past years. I would like to mention my outstanding office mates throughout the years and my fellow alumni from Vienna. And of course Lisa, Beverly, Maria, and especially Paul.

Finally, but most importantly, my parents and for their constant support and encouragement to pursue my dreams.

This thesis was supported by by a grant from the Swiss National Science Foundation (project number 200021_137778).

Introduction

Motivation

Topological field theories

Topological field theories (TFTs) arose as toy models for physical quantum field theories, and have proven to be of mathematical interest, notably because they are a fruitful tool for studying topology. Inspired by Witten's paper [Wit82] relating supersymmetry and Morse theory, they were first axiomatized by Atiyah in [Ati88]. An n -dimensional TFT is a symmetric monoidal functor from the category of bordisms, which has closed $(n - 1)$ -dimensional manifolds as objects and n -dimensional bordisms as morphisms, to any other symmetric monoidal category, which classically is taken to be the category of vector spaces or chain complexes. In particular it assigns topological invariants to closed n -dimensional manifolds, which has turned out to be very useful in the study of low-dimensional topology. Early results by Witten in [Wit89] showed that the Jones polynomial of knot theory arises from the 3-dimensional Chern-Simons theory, which is a TFT. Interesting 4-dimensional examples are Donaldson invariants of 4-dimensional manifolds which arise from a twisted 4-dimensional supersymmetric gauge theory, [Wit88], and the related Seiberg-Witten invariants [Wit94, SW94a, SW94b].

A classification of 1- and 2-dimensional TFTs follows from classification theorems for 1- and 2-dimensional compact manifolds with boundary. In the 1-dimensional case, a 1-TFT is fully determined by its value at a point, which is a dualizable object in the target category and conversely, every dualizable object in the target gives rise to a 1-TFT. In the 2-dimensional case, a classification, given by the value at a circle, was proven by Abrams in [Abr96]. The question of a classification result for larger values of n appears naturally and raises the question of a suitable replacement of the classification of compact n -manifolds with boundary used in the low-dimensional cases. In [BD95], Baez and Dolan explain the need for higher categories of cobordisms for a classification of n -dimensional extended topological field theories. Here extended means that we need to be able to evaluate the n -TFT not only at n - and $(n - 1)$ -dimensional manifolds, but also at $(n - 2)$ -, ..., 1-, and 0-dimensional manifolds. In light of the hope of computability of the invariants determined by an n -TFT, e.g. by a triangulation, it is natural to include this data. They conjectured that extended n -TFTs are fully determined by their value at a point, calling this the cobordism hypothesis. A proof of a classification theorem of extended TFTs for dimension 2 and in particular a definition of a suitable bicategory of 2-cobordisms was

given in [SP09].

In his expository manuscript [Lur09c], Lurie explained the need for (∞, n) -categories for a proof of the cobordism hypothesis in arbitrary dimension n and gave a detailed sketch of such a proof using a suitable (∞, n) -category of cobordisms, which, informally speaking, has zero-dimensional manifolds as objects, bordisms between objects as 1-morphisms, bordisms between bordisms as 2-morphisms, etc., and for $k > n$ there are only invertible k -morphisms. Finding an explicit model for such a higher category poses one of the difficulties in rigorously defining these n -dimensional TFTs, which are called “fully extended”. His result shows that evaluation at a point gives a bijection, or more precisely an equivalence of ∞ -groupoids, between (isomorphism classes of) fully extended n -TFTs with values in a target symmetric monoidal (∞, n) -category \mathcal{C} and (isomorphism classes of) “fully dualizable” objects in \mathcal{C} . Thus any fully dualizable object in the target category determines a fully extended n -TFT. Full dualizability is a finiteness condition generalizing the condition of being a dualizable object in the 1-dimensional case.

Factorization homology and factorization algebras

Inspired by Segal’s approach to conformal field theories in [Seg04] and Atiyah’s axioms for TFTs mentioned above, there have been several approaches to describe (topological) quantum field theories in an axiomatic way. Factorization homology and factorization algebras are two such approaches which were developed and studied by many people, among them Beilinson-Drinfeld, Costello-Gwilliam, Francis, Lurie.

Factorization homology, also called topological chiral homology, was first defined by Jacob Lurie in [Lur]. It is a homology theory for topological manifolds satisfying a generalization of the Eilenberg-Steenrod axioms for ordinary homology, see [Fra12, AFT12]. The construction depends on the data of an E_n -algebra in a suitable symmetric monoidal $(\infty, 1)$ -category \mathcal{S} , which is an algebra in \mathcal{S} for the operad E_n , which in turn is equivalent to the little cubes operad in dimension n . In the case $n = 1$, E_1 -algebras are equivalent to associative algebras up to homotopy, i.e. A_∞ -algebras, and in the case of $n = 2$, E_2 -algebras in the category of categories are braided monoidal categories. In the special case that \mathcal{S} is the $(\infty, 1)$ -category of chain complexes, any commutative differential graded algebra A is in particular also an E_n -algebra and it was shown in [GTZ10] that factorization homology recovers the (higher) Hochschild homology of A . Factorization homology for n -dimensional manifolds with boundary yields an n -TFT, as was shown by Ayala-Francis-Tanaka in [AFT12] and, with different techniques, by Horel in [Hor14b]. Moreover, Lurie stated in [Lur09c] that it should lead to a fully extended n -TFT.

Factorization algebras are algebraic structures encoding the structure of the observables of a quantum field theory (henceforth QFT), as was shown in [CG] for perturbative QFTs. One can think of them as a multiplicative, non-commutative version of cosheaves and they turn out to be a tool useful for describing well-known algebraic structures such as E_n -algebras ([Lur09c]) and bimodules between algebras ([Gin]). Factorization algebras and factorization homology are related in a local-to-global way: in [GTZ10] it was shown that considering factorization homology locally on a given manifold M yields a factorization algebra on M whose global sections are the factorization homology of M .

Overview of the thesis

Lurie’s cobordism hypothesis gives a “recipe” for producing a fully extended n -TFT. Namely one first needs to find a suitable target, which is a symmetric monoidal (∞, n) -category, and then one needs to pick a fully dualizable object. However, this construction is not explicit in the sense that one might like to be able to actually compute the values of the n -TFT. The goal of this thesis was to, avoiding the use of the cobordism hypothesis, explicitly construct a family of examples of fully extended n -dimensional TFTs, which is essentially given by factorization homology with coefficients in a given E_n -algebra A . Under the cobordism hypothesis this fully extended n -TFT corresponds to the E_n -algebra A , which is a fully dualizable object in a suitable Morita- (∞, n) -category Alg_n . Informally it can be thought of as a higher category with E_n -algebras as objects, bimodules in E_{n-1} -algebras as 1-morphisms, bimodules between bimodules as 2-morphisms, etc. In fact, this (∞, n) -category is the truncation of an $(\infty, n+1)$ -category $\widetilde{\text{Alg}}_n$ whose $(n+1)$ -morphisms are morphisms in \mathcal{S} . Our construction allows to compute the topological invariants given by the TFT by taking global sections of a factorization algebra, and the gluing condition (locality) of the factorization algebra allows this to be computed locally. This extends the excision property of factorization homology proved by Ayala, Francis, and Tanaka in [AFT12].

The first two chapters aim to give a precise definition of a fully extended n -dimensional topological field theory. In the third chapter we define the target category of E_n -algebras and the final chapter contains the construction of the fully extended n -TFT as a morphism of n -fold Segal spaces. We now give a more detailed overview of the chapters.

Symmetric monoidal complete n -fold Segal spaces

First, in chapter 1 we recall the necessary tools from higher category theory needed to define fully extended TFTs. We explain the model for (∞, n) -categories given by complete n -fold Segal spaces. Moreover, we give two possible definitions of symmetric monoidal structures on complete n -fold Segal spaces, once as a Γ -object in complete n -fold Segal spaces following [TV09] and once as a tower of suitable $(n+k)$ -fold Segal spaces with one object, 1-morphism, ..., $(k-1)$ -morphism for $k \geq 0$ following the Stabilization Hypothesis.

Definition of a fully extended n -TFT

Chapter 2 deals with the symmetric monoidal (∞, n) -category of bordisms. Lurie gives a formal definition of this (∞, n) -category using complete n -fold Segal spaces, however, as we explain in section 2.3.6, this actually is not an n -fold Segal space. In our definition 2.3.1, we propose a stronger condition on elements in the levels of the Segal space and show that this indeed yields a n -fold Segal space PBord_n . Its completion Bord_n defines an (∞, n) -category of n -cobordisms and thus is a corrigendum to Lurie’s n -fold simplicial space of bordisms from [Lur09c].

Instead of using manifolds with corners and gluing them, Lurie’s idea was to conversely use embedded closed (not necessarily compact) manifolds and to specify points where they are cut into bordisms of which the embedded manifold is a composition. Whitney’s embedding theorem ensures that every n -dimensional manifold M can be embedded into some large enough vector space and suitable versions for manifolds with boundary can be adapted to obtain an embedding theorem for bordisms, see 2.5.1. Moreover, the rough

idea behind the definition of the levels of PBord_n is that the (k_1, \dots, k_n) -level of our n -fold Segal space PBord_n should be a classifying space for k_i -fold composable n -bordisms in the i th direction. Lurie’s idea was to use the fact that the space of embeddings of M into \mathbb{R}^∞ is contractible to justify the construction.

We base our construction of PBord_n on a simpler complete Segal space Int of closed intervals, which is defined in section 2.1. The closed intervals correspond to places where we are allowed to cut the manifold into the bordisms it composes. The fact that we prescribe closed intervals instead of just a point corresponds to fixing collars of the bordisms.

In section 2.2 we study a version of a time-dependent Morse lemma which serves as a motivation for our definition of the spatial structure of the levels of PBord_n . As we explain in 2.3.2, the spatial structure we define is almost obtained by taking differentiable simplices of the space of embeddings, but we add the data of a semi-group of diffeomorphisms between bordisms along a simplex. The time-dependent Morse lemma shows that this yields the same paths.

Section 2.3 is the central part of this chapter and consists of the construction of the complete n -fold Segal space Bord_n of cobordisms. It is endowed with a symmetric monoidal structure in section 2.4, both as a Γ -object and as a tower.

In section 2.5 we show that its homotopy (bi)category is what one should expect, namely the homotopy category of its $(n - 1)$ -fold looping $L_{n-1}(\text{Bord}_n)$ gives back the classical cobordism category $n\text{Cob}$ and the homotopy bicategory of Bord_2 is Schommer-Pries’ bicategory 2Cob^{ext} from [SP09].

Finally, in section 2.6 we consider bordism categories with additional structure such as orientations, denoted by $\text{Bord}_n^{\text{or}}$, and framings, denoted by $\text{Bord}_n^{\text{fr}}$, which allows us to define fully extended n -dimensional topological field theories in section 2.7.

The target: E_n -algebras

In chapter 3 we define the target of our fully extended n -TFT, namely a symmetric monoidal Morita- (∞, n) -category $\text{Alg}_n = \text{Alg}_n(\mathcal{S})$ of E_n -algebras. By an E_n -algebra, we mean an E_n -algebra object in a suitable symmetric monoidal $(\infty, 1)$ -category \mathcal{S} . Main examples we will be interested in are the category of chain complexes over a ring R , $\mathcal{S} = \text{Ch}_R$, or the category of (Lagrangian) correspondences $\mathcal{S} = (\text{Lag})\text{Corr}$.

To define this as a complete n -fold Segal space, we exploit the equivalence of $(\infty, 1)$ -categories between E_n -algebras and locally constant factorization algebras on $\mathbb{R}^n \cong (0, 1)^n$ (proven by Lurie in [Lur09c]) and define the objects of the n -fold Segal space to be locally constant factorization algebras on $(0, 1)^n$. Furthermore, following the observation that the data of a factorization algebra on $(0, 1)$ which is locally constant with respect to a stratification of the form $(0, 1) \supset \{p\}$ for any $p \in (0, 1)$ are equivalent to the data of a pointed (homotopy) bimodule, we model the “levels” of the n -fold Segal space as factorization algebras on $(0, 1)^n$ which are locally constant with respect to certain stratifications. For the existence of the factorization algebras we need the following assumption on \mathcal{S} .

Assumption 1. *Let \mathcal{S} be a symmetric monoidal $(\infty, 1)$ -category which admits all small colimits.*

As with Bord_n , we base the construction on a simpler complete Segal space Covers which we construct in section 3.1. The data given by Covers determine the stratification with respect to which the factorization algebras are locally constant.

Section 3.2 contains the main construction of the (∞, n) -category, i.e. the n -fold Segal space, Alg_n . In fact, it is the truncation of an $(\infty, n+1)$ -category $\widehat{\text{Alg}}_n$ given by an n -fold Segal object in Segal spaces. These Segal spaces, i.e. the levels, are $(\infty, 1)$ -categories of locally constant factorization algebras on $(0, 1)$ which are locally constant with respect to a stratification of a particular form. The simplicial structure of Alg_n essentially comes from the simplicial structure of the Segal space Covers and is given by the pushforward of the factorization algebra along a suitable collapse-and-rescale map. With this definition composition in the homotopy category corresponds to sending two bimodules ${}_A M_B$ and ${}_B N_C$ to their tensor product $({}_A M_B) \otimes_B ({}_B N_C)$.

The fact that factorization algebras naturally lead to pointed objects has an important consequence. Namely, it implies that the n -fold Segal space Alg_n is complete. This is shown in section 3.2.9.

In section 3.3 we endow Alg_n with a symmetric monoidal structure, both as a Γ -object and as a tower.

Finally we show in section 3.4 that the homotopy category of Alg_1 is the Morita category, whose objects are (homotopy) algebras and whose morphisms are isomorphism classes of pointed (homotopy) bimodules.

Construction of the fully extended n -dimensional topological field theory

The final chapter, chapter 4 connects the two previous chapters. It contains the construction of the fully extended n -TFT as a morphism of n -fold Segal spaces. As we want this to essentially be given by factorization homology, we need an extra assumption on \mathcal{S} :

Assumption 2. *Let \mathcal{S} be a symmetric monoidal $(\infty, 1)$ -category satisfying the conditions such that factorization homology exists, i.e. following [AFT14], it should be \otimes -sifted co-complete.*

The construction of the functor proceeds in two steps: we first define an auxiliary symmetric monoidal complete n -fold Segal space Fact_n of factorization algebras on $(0, 1)^n$ in section 4.2, which, like Bord_n is based on the Segal space Int . It translates the properties of PBord_n^{fr} via a map given by factorization homology with coefficients in a fixed E_n -algebra A ,

$$\int_{(-)} A : \quad \text{Bord}_n^{fr} \quad \longrightarrow \quad \text{Fact}_n,$$

$$\begin{array}{ccc} M & \xrightarrow{\iota} & V \times (0, 1)^n \\ & \searrow \pi & \downarrow \\ & & (0, 1)^n \end{array} \quad \longmapsto \quad \pi_* \left(\int_M A \right),$$

which is defined in section 4.3. However, this map is just a morphism of the underlying n -fold simplicial sets as it fails to extend to the spatial structure of the levels.

In a second step, in section 4.4, we define a map to an n -fold Segal space $\mathbf{FAlg}_n \supseteq \mathbf{Alg}_n$ of factorization algebras on $(0, 1)^n$ which have certain locally constancy properties, but do not lead to bimodules,

$$\mathbb{V} : \mathbf{Fact}_n \longrightarrow \mathbf{FAlg}_n .$$

This map can be understood as ‘‘collapsing’’ parts of the factorization algebra and then rescaling. It arises from a map $\mathfrak{v} : \mathbf{Int} \rightarrow \mathbf{Covers}$ of the simpler Segal spaces on which \mathbf{Fact}_n and \mathbf{FAlg}_n are based, which determines a collapse-and-rescale map $\varrho : (0, 1)^n \rightarrow (0, 1)^n$. Then the map \mathbb{V} is given by the pushforward of the factorization algebra along ϱ .

One should think of this process as collapsing the part of the factorization algebra in which the factorization algebra might change along a path, or an even higher simplex in \mathbf{Bord}_n^{fr} . The global sections of this part do not change, as the data of a higher simplex in \mathbf{Bord}_n include diffeomorphisms between bordisms along this simplex. Following this argument we show in section 4.5 that the composition of the two constructed maps $\mathbb{V} \circ \int_{(-)} A$ is a morphism of n -fold Segal spaces and its image in fact lands in \mathbf{Alg}_n ,

$$\mathcal{FH}_n(A) : \mathbf{PBord}_n^{fr} \longrightarrow \mathbf{Alg}_n .$$

By the universal property of the completion, this map extends to a map of complete n -fold Segal spaces,

$$\mathcal{FH}_n(A) : \mathbf{Bord}_n^{fr} \longrightarrow \mathbf{Alg}_n .$$

To conclude that $\mathcal{FH}_n(A)$ is the desired fully extended topological field theory we show in 4.6 that it extends to the symmetric monoidal structure for both structures.

$$\begin{array}{ccccc} \mathbf{PBord}_n^{fr} & \xrightarrow{\int_{(-)}} & \mathbf{Fact}_n & \xrightarrow{\mathbb{V}} & \mathbf{FAlg}_n \\ \uparrow & & \searrow & & \uparrow \\ \mathbf{Bord}_n & \xrightarrow{\mathcal{FH}_n(A)} & & & \mathbf{Alg}_n \end{array}$$

Guide to the reader

Parts of this thesis contain rather technical constructions of suitable (n -fold) Segal spaces, so let us explain which parts can be left aside on a first reading.

The first chapter mostly contains a recollection on complete n -fold Segal spaces as a model for (∞, n) -categories. The only original part in this section is that of the definition of a symmetric monoidal structure on an n -fold Segal space following the Delooping Hypothesis in subsection 1.6.2. We use the notion of k -hybrid n -fold Segal space, which is a suitable interpolation between complete n -fold Segal spaces and Segal n -categories. The second and third chapters are mostly independent of each other. In both, one can first brush over the rather technical constructions of the underlying simpler Segal spaces \mathbf{Int} and \mathbf{Covers} in sections 2.1 and 3.1 and go straight to the main constructions of the (∞, n) -categories \mathbf{Bord}_n and \mathbf{Alg}_n in sections 2.3 and 3.2. The fourth chapter contains the heart of this thesis. The fully extended TFT is constructed within this chapter.

Warning. In chapter 1 we define an (∞, n) -category to be a complete n -fold Segal space. We try to be consistent with this definition throughout the thesis, but at times have to

switch to different models for (∞, n) -categories, usually for $(\infty, 1)$ -categories. We will usually state this explicitly where necessary.

Conventions. We will use the following conventions throughout this thesis.

- By *space*, we will mean a simplicial set. This is to distinguish the n simplicial “directions” of the n -fold Segal space from the simplicial set of the “levels”, which we call spatial direction. The $(\infty, 1)$ -category of spaces will be denoted by *Space*.
- We fix a diffeomorphism $(0, 1) \xrightarrow{\cong} \mathbb{R}$. This will endow $(0, 1)$ with the structure of a vector space. Whenever we write “ $(0, 1) \cong \mathbb{R}$ ” we will mean this fixed diffeomorphism.
- To simplify notation, if we write $[a, b] \subseteq (0, 1)$, we allow $a = 0$ or $b = 1$ and mean $[a, b] \cap (0, 1)$.
- We denote $\{0, 1, \dots, n\}$ by $[n]$. They form the objects of the simplex category Δ whose morphisms are (weakly) order-preserving morphisms.

Preliminaries: symmetric monoidal (∞, n) -categories

A higher category, say, an n -category for $n \geq 0$, has not only objects and (1-)morphisms, but also k -morphisms between $(k-1)$ -morphisms for $1 \leq k \leq n$. Strict higher categories can be rigorously defined, however, most higher categories which occur in nature are not strict. Thus, we need to weaken some axioms and coherence between the weakenings become rather involved to formulate explicitly. Things turn out to become somewhat easier when using a geometric definition, in particular when furthermore allowing to have k -morphisms for all $k \geq 1$, which for $k \geq n$ are invertible. Such a higher category is called an (∞, n) -category. There are several models for such (∞, n) -categories, e.g. Segal n -categories (cf. [HS98]), Θ_n -spaces (cf. [Rez10]), and complete n -fold Segal spaces, which all are equivalent in an appropriate sense (cf. [Toë05, BS11]). For our purposes, the latter model turns out to be well-suited and in this section we recall some basic facts about complete n -fold Segal spaces as higher categories. This is not at all exhaustive, and more details can be found in e.g. [BR13, Zha13].

1.1 The homotopy hypothesis and $(\infty, 0)$ -categories

The basic hypothesis upon which higher category theory is based is the following

Hypothesis 1.1.1 (Homotopy hypothesis). Topological spaces are models for ∞ -groupoids, also referred to as $(\infty, 0)$ -categories.

Given a topological space X , its points are thought of as objects of the $(\infty, 0)$ -category, 1-morphisms as paths between points, 2-morphisms as homotopies between paths, 3-morphisms as homotopies between homotopies, and so forth. With this interpretation, it is clear that all n -morphisms are invertible up to homotopies, which are higher morphisms.

We take this hypothesis as the basic definition.

Definition 1.1.2. An $(\infty, 0)$ -category is a space.

1.2 Complete Segal spaces as models for $(\infty, 1)$ -categories

A good overview on different models for $(\infty, 1)$ -categories can be found in [Ber10]. Here we would just like to mention one particularly simple and quite rigid model, namely that of topologically enriched categories.

Definition 1.2.1. A *topological category* is a category enriched in topological spaces (or simplicial sets, depending on the purpose).

Topological categories are discussed and used in [Lur09a, TV05]. However, for our applications, complete Segal spaces, first introduced by Rezk in [Rez01] as models for $(\infty, 1)$ -categories, turn out to be very well-suited. We recall the definition in this section.

1.2.1 Segal spaces

Definition 1.2.2. A (*1-fold*) *Segal space* is a simplicial space $X = X_\bullet$ which is level-wise fibrant and satisfies the *Segal condition*, i.e. for any $n, m \geq 0$,

$$\begin{array}{ccc} X_{m+n} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_0 \end{array}$$

induced by the maps $[m] \rightarrow [m+n]$, $(0 < \dots < m) \mapsto (0 < \dots < m)$, and $[n] \rightarrow [m+n]$, $(0 < \dots < n) \mapsto (m < \dots < m+n)$, is a homotopy pullback square. In other words,

$$X_{m+n} \longrightarrow X_m \times_{X_0}^h X_n,$$

is a weak equivalence.

Defining a *map of Segal spaces* to be a map of the underlying simplicial spaces gives a category of Segal spaces, $\mathbf{SSpaces} = \mathbf{SSpaces}_1$.

Remark 1.2.3. Following [Lur09c] we omit the Reedy fibrant condition which often appears in the literature. In particular, this condition would guarantee in particular that the canonical map

$$X_m \times_{X_0} X_n \longrightarrow X_m \times_{X_0}^h X_n$$

is a weak equivalence. This explains the different appearance of the Segal condition.

Remark 1.2.4. Rezk showed in [Rez01] that Reedy fibrant Segal spaces are the fibrant objects for a model structure on the category of simplicial spaces given by a suitable localization of the injective model structure. However, Horel showed in [Hor14a] that for a suitable localization of the projective model structure, fibrant objects are Segal spaces which are level-wise fibrant instead of Reedy fibrant. Moreover, these two models are Quillen equivalent. In the following text we will point out where these subtleties need to be taken into account. Moreover, we will sometimes mean this model category (or the ∞ -category it represents) when writing $\mathbf{SSpaces}$.

Example 1.2.5. Let \mathcal{C} be a small topological category, i.e. a small category enriched over topological spaces. Then its nerve $N(\mathcal{C})$ is a Segal space.

Segal spaces as $(\infty, 1)$ -categories

The above example motivates the following interpretation of Segal spaces as models for $(\infty, 1)$ -categories. If X_\bullet is a Segal space then we view the set of 0-simplices of the space X_0 as the set of objects. For $x, y \in X_0$ we view

$$\mathrm{Hom}_X(x, y) = \{x\} \times_{X_0}^h X_1 \times_{X_0}^h \{y\}$$

as the $(\infty, 0)$ -category, i.e. the space, of arrows from x to y . More generally, we view X_n as the $(\infty, 0)$ -category, i.e. the space, of n -tuples of composable arrows together with a composition. Note that given an n -tuple of composable arrows, there is a contractible space of compositions. Moreover, one can interpret paths in the space X_1 of 1-morphisms as 2-morphisms, which thus are invertible up to homotopies, which themselves are 3-morphisms, and so forth.

Definition 1.2.6. We will later refer to the spaces X_n as the *levels* of the Segal space.

1.2.2 The homotopy category of a Segal space

To a higher category one can intuitively associate an ordinary category, its *homotopy category*, having the same objects, with morphisms being 2-isomorphism classes of 1-morphisms. For Segal spaces, one can realize this idea as follows.

Definition 1.2.7. The *homotopy category* $h_1(X)$ of a Segal space $X = X_\bullet$ has as set of objects the set of vertices of the space X_0 and as morphisms between objects $x, y \in X_0$,

$$\begin{aligned} \mathrm{Hom}_{h_1(X)}(x, y) &= \pi_0(\mathrm{Hom}_X(x, y)) \\ &= \pi_0\left(\{x\} \times_{X_0}^h X_1 \times_{X_0}^h \{y\}\right). \end{aligned}$$

For $x, y, z \in X_0$, the following diagram induces the composition of morphisms, as weak equivalences induce bijections on π_0 .

$$\begin{aligned} \left(\{x\} \times_{X_0}^h X_1 \times_{X_0}^h \{y\}\right) \times \left(\{y\} \times_{X_0}^h X_1 \times_{X_0}^h \{z\}\right) &\longrightarrow \{x\} \times_{X_0}^h X_1 \times_{X_0}^h X_1 \times_{X_0}^h \{z\} \\ &\xleftarrow{\simeq} \{x\} \times_{X_0}^h X_2 \times_{X_0}^h \{z\} \\ &\longrightarrow \{x\} \times_{X_0}^h X_1 \times_{X_0}^h \{z\}. \end{aligned}$$

Example 1.2.8. Given a small (ordinary) category \mathcal{C} , the homotopy category of its nerve, viewed as a simplicial space with discrete levels, is equivalent to \mathcal{C} ,

$$h_1(N(\mathcal{C})) \simeq \mathcal{C}.$$

1.2.3 Complete Segal spaces

In our interpretation of a Segal space $X = X_\bullet$ as an $(\infty, 1)$ -category above several Segal spaces give rise to the same $(\infty, 1)$ -category: we can replace X_0 by the ∞ -groupoid, i.e. the space, obtained by discarding all non-invertible morphisms. To avoid this ambiguity, we impose an extra condition which ensures that the space X_0 is ∞ -groupoid, or $(\infty, 0)$ -category, obtained by discarding all non-invertible morphisms.

Definition 1.2.9. An element $f \in X_1$ with source and target x and y , i.e. the two faces of f are x and y , is *invertible* if its image under

$$\{x\} \times_{X_0} X_1 \times_{X_0} \{y\} \longrightarrow \{x\} \times_{X_0}^h X_1 \times_{X_0}^h \{y\} \longrightarrow \pi_0 \left(\{x\} \times_{X_0}^h X_1 \times_{X_0}^h \{y\} \right) = \text{Hom}_{h_1(X)}(x, y),$$

is an invertible morphism in $h_1(X)$.

Denote by X_1^{inv} the subspace of invertible arrows and observe that the map $X_0 \rightarrow X_1$ factors through X_1^{inv} , since the image of $x \in X_0$ under $X_0 \rightarrow X_1 \rightarrow \text{Hom}_{h_1(X)}(x, x)$ is id_x , which is invertible.

Definition 1.2.10. A Segal space X_\bullet is *complete* if the map $X_0 \rightarrow X_1^{inv}$ is a weak equivalence. We denote the full subcategory of **SSpaces** whose objects are complete Segal spaces by **CSSpaces** = **CSSpaces**₁.

Remark 1.2.11. Similarly to remark 1.2.4, Rezk showed in [Rez01] that complete Segal spaces are the fibrant objects for a model structure on the category of simplicial spaces, namely a suitable localization of **SSpaces**. Horel showed the analogous statement for the projective model structure in [Hor14b]. We will usually mean this model category when writing **CSSpaces**.

Complete Segal spaces are $(\infty, 1)$ -categories

Rezk explained in [Rez01] that complete Segal spaces are a good model for $(\infty, 1)$ -categories. This justifies the following definition.

Definition 1.2.12. An $(\infty, 1)$ -category is a complete Segal space.

Remark 1.2.13. The completeness condition says that all invertible morphisms essentially are just identities up to the choice of a path. So strictly speaking, complete Segal spaces should be called *skeletal*, or, according to [Joy], *reduced* $(\infty, 1)$ -categories.

Completion of Segal spaces

Rezk showed in [Rez01] that Segal spaces can always be completed. He showed that there is a completion functor which to every Segal space X associates a complete Segal space \widehat{X} together with a map $i_X : X \rightarrow \widehat{X}$, which is a Dwyer-Kan equivalence, which in turn is defined below.

Even though the inclusion **CSSpaces** \hookrightarrow **SSpaces** does not have a left adjoint, it does when passing to the homotopy categories (see remarks 1.2.4 and 1.2.11). Thus, \widehat{X} is universal (in the homotopy category) among complete Segal spaces Y together with a map $X \rightarrow Y$.

Definition 1.2.14. A map $f : X \rightarrow Y$ of Segal spaces is a *Dwyer-Kan equivalence* if

1. the induced map $h_1(f) : h_1(X) \rightarrow h_1(Y)$ on homotopy categories is an equivalence of categories, and
2. for each pair of objects $x, y \in X_0$ the induced function on mapping spaces $\text{Hom}_X(x, y) \rightarrow \text{Hom}_Y(f(x), f(y))$ is a weak equivalence.

Relative categories and the classification diagram

In this section we recall a construction due to Rezk [Rez01] which produces a complete Segal space from a simplicial closed model category. More generally, Barwick and Kan proved in [BK11] that this construction also gives a complete Segal space for so-called partial model categories.

Definition 1.2.15. Let $(\mathcal{C}, \mathcal{W})$ be a relative category, i.e. a category \mathcal{C} with a distinguished subcategory \mathcal{W} . Consider the simplicial object in categories \mathcal{C}_\bullet given by $\mathcal{C}_n := \text{Fun}([n], \mathcal{C})$. It has a subobject $\mathcal{C}_\bullet^{\mathcal{W}}$, where $\mathcal{C}_n^{\mathcal{W}} \subset \mathcal{C}_n$ is the subcategory having the same objects and morphisms consisting only of those from \mathcal{W} . Taking its nerve we obtain a simplicial space $N(\mathcal{C}, \mathcal{W})_\bullet$ with

$$N(\mathcal{C}, \mathcal{W})_n = N(\mathcal{C}_n^{\mathcal{W}}).$$

It satisfies the Segal condition, but will not be level-wise fibrant unless we started with an ∞ -groupoid. Its level-wise fibrant replacement is called the *relative/simplicial nerve* or the *classification diagram*, which, by abuse of notation, we again denote by $N(\mathcal{C}, \mathcal{W})$.

Example 1.2.16. Let \mathcal{C} be a small category. Then it is straightforward to see that $N(\mathcal{C}, \text{Iso}\mathcal{C})$ is a complete Segal space. Alternatively, if \mathcal{C} has finite limits and colimits, it can be made into a closed model category in which the weak equivalences are the isomorphisms and all maps are fibrations and cofibrations. Then the above result also shows that the classification diagram is a complete Segal space, cf. [Rez01].

1.2.4 Segal categories

A second way to avoid the problem that in a Segal space and its homotopy category we do not use the topology on X_0 is to impose that X_0 is discrete. By this we obtain the notion of Segal categories, which are another model for $(\infty, 1)$ -categories and briefly mention here. More details and references can be found in the above mentioned [Ber10].

Definition 1.2.17. A Segal (1-)category is a Segal space $X = X_\bullet$ such that X_0 is discrete.

Segal categories also are the fibrant objects of a model category which is Quillen equivalent to **CSSpaces**. For our purposes, complete Segal spaces turn out to be the more useful model.

1.3 Complete n -fold Segal spaces as models for (∞, n) -categories

As a model for (∞, n) -categories, we will use complete n -fold Segal spaces, which were first introduced by Barwick in his thesis and appeared prominently in Lurie's [Lur09c].

1.3.1 n -fold Segal spaces

An n -fold Segal space is an n -fold simplicial space with certain extra conditions.

Definition 1.3.1. An n -fold simplicial space $X_{\bullet, \dots, \bullet}$ is *essentially constant* if there is a weak homotopy equivalence of n -fold simplicial spaces $Y \rightarrow X$, where Y is constant.

Definition 1.3.2. An n -fold Segal space is an n -fold simplicial space $X = X_{\bullet, \dots, \bullet}$ such that

- (i) For every $1 \leq i \leq n$, and every $k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_n \geq 0$,

$$X_{k_1, \dots, k_{i-1}, \bullet, k_{i+1}, \dots, k_n}$$

is a Segal space.

- (ii) For every $1 \leq i \leq n$, and every $k_1, \dots, k_{i-1} \geq 0$,

$$X_{k_1, \dots, k_{i-1}, 0, \bullet, \dots, \bullet}$$

is essentially constant.

Defining a *map of n -fold Segal spaces* to be a map of the underlying n -fold simplicial spaces gives a category of n -fold Segal spaces, $\mathbf{SSpaces}_n$.

Remark 1.3.3. Alternatively, one can formulate the conditions iteratively. First, an *n -iterated Segal space* is a simplicial object Y_\bullet in $(n-1)$ -fold Segal spaces which satisfies the Segal condition. Then, an n -fold Segal space is an n -iterated Segal space such that Y_0 is essentially constant (as an $(n-1)$ -fold Segal space). To get back the above definition, the ordering of the indices is crucial: $X_{k_1, \dots, k_n} = (Y_{k_1})_{k_2, \dots, k_n}$.

Remark 1.3.4. Similarly to remark 1.3.4 there is a model structure on the category of n -fold simplicial spaces obtained as a localization of the injective model structure whose fibrant objects are Reedy fibrant n -fold Segal spaces, see [Lur09b] or [BS11]. Similarly to the aforementioned [Hor14a], there is a model structure obtained by localizing the projective model structure and whose fibrant objects are complete Segal spaces which are level-wise fibrant instead of Reedy fibrant, see remark 1.5.6 in [Lur09b]. We will sometimes mean this model category (or the ∞ -category it represents) when writing $\mathbf{SSpaces}_n$.

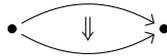
Interpretation as higher categories

An n -fold Segal space can be thought of as a higher category in the following way.

The first condition means that this is an n -fold category, i.e. there are n different “directions” in which we can “compose”. An element of X_{k_1, \dots, k_n} should be thought of as a composition consisting of k_i composed morphisms in the i th direction.

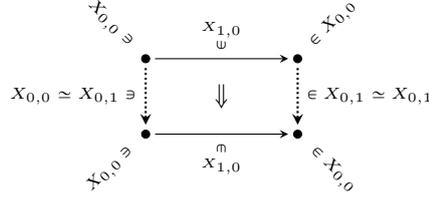
The second condition imposes that we indeed have a higher n -category, i.e. an n -morphism has as source and target two $(n-1)$ -morphisms which themselves have the “same” (in the sense that they are homotopic) source and target.

For $n = 2$ one can think of this second condition as “fattening” the objects in a bicategory. A 2-morphism in a bicategory can be depicted as



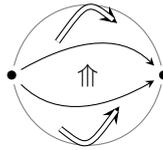
The top and bottom arrows are the source and target, which are 1-morphisms between the same objects.

In a 2-fold Segal space $X_{\bullet, \bullet}$, an element in $X_{1,1}$ can be depicted as

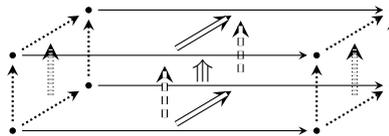


The images under the source and target maps in the first direction $X_{1,1} \rightrightarrows X_{1,0}$ are 1-morphisms which are depicted by the horizontal arrows. The images under the source and target maps in the second direction $X_{1,1} \rightrightarrows X_{0,1}$ are 1-morphisms, depicted by the dashed vertical arrows, which are essentially just identity maps, up to homotopy, since $X_{0,1} \simeq X_{0,0}$. Thus, here the source and target 1-morphisms (the horizontal ones) themselves do not have the same source and target anymore, but up to homotopy they do.

The same idea works with higher morphisms, in particular one can imagine the corresponding diagrams for $n = 3$. A 3-morphism in a tricategory can be depicted as



whereas a 3-morphism, i.e. an element in $X_{1,1,1}$ in a 3-fold Segal space X can be depicted as



Here the dotted arrows are those in $X_{0,1,1} \simeq X_{0,0,1} \simeq X_{0,0,0}$ and the dashed ones are those in $X_{1,0,1} \simeq X_{1,0,0}$.

Thus, we should think of the set of 0-simplices of the space $X_{0,\dots,0}$ as the objects of our category, and elements of $X_{1,\dots,1,0,\dots,0}$ as i -morphisms, where $0 < i \leq n$ is the number of 1's. Pictorially, they are the i -th “horizontal” arrows. Moreover, the other “vertical” arrows are essentially just identities of lower morphisms. Similarly to before, paths in $X_{1,\dots,1}$ should be thought of as $(n + 1)$ -morphisms, which therefore are invertible up to a homotopy, which itself is an $(n + 2)$ -morphism, and so forth.

1.3.2 Complete and hybrid n -fold Segal spaces

As with (1-fold) Segal spaces, we need to impose an extra condition to ensure that invertible k -morphisms are paths in the space of $(k - 1)$ -morphisms. Again, there are several ways to include its information.

Definition 1.3.5. Let X be an n -fold Segal space and $1 \leq i, j \leq n$. It is said to satisfy

CSS^i if for every $k_1, \dots, k_{i-1} \geq 0$,

$$X_{k_1, \dots, k_{i-1}, \bullet, 0, \dots, 0}$$

is a complete Segal space.

SC^j if for every $k_1, \dots, k_{j-1} \geq 0$,

$$X_{k_1, \dots, k_{j-1}, 0, \bullet, \dots, \bullet}$$

is discrete, i.e. a discrete space viewed as a constant $(n - j + 1)$ -fold Segal space.

Definition 1.3.6. An n -fold Segal space is

1. *complete*, if for every $1 \leq i \leq n$, X satisfies (CSS^i) .
2. a *Segal n -category* if for every $1 \leq j \leq n$, X satisfies (SC^j) .
3. *m -hybrid* for $m \geq 0$ if condition (CSS^i) is satisfied for $i > m$ and condition (SC^j) is satisfied for $j \leq m$.

Denote the full subcategory of $\mathbf{SSpaces}_n$ of complete n -fold Segal spaces by $\mathbf{CSSpaces}_n$.

Remark 1.3.7. Note that an n -hybrid n -fold Segal space is a Segal n -category, while an n -fold Segal space is 0-hybrid if and only if it is complete.

For our purposes, the model of complete n -fold Segal spaces is well-suited, so we define

Definition 1.3.8. An (∞, n) -category is an n -fold complete Segal space.

Model structure and weak equivalences

As in remarks 1.2.4, 1.2.11, and 1.3.4, there is a model structure on the category of n -fold simplicial spaces $\mathbf{sSpaces}_n$ whose fibrant objects are complete n -fold Segal spaces, see [Lur09b] or [BS11]. Moreover, iterating the construction of Segal categories enriched in a suitable model category and the construction of complete Segal objects for so-called absolute distributors in [Lur09b] should give a model in which m -hybrid n -fold Segal spaces are the fibrant objects.

Since $\mathbf{SSpaces}_n$ and $\mathbf{CSSpaces}_n$ are localizations of $\mathbf{sSpaces}_n$, they inherit a subcategory of weak equivalences. One can prove that they are exactly the *Dwyer-Kan equivalences*, the analogous notion to definition 1.2.14 for $n = 1$. More details can be found e.g. in [Zha13].

Completion

In light of the iterative definition of an n -fold Segal space, i.e. viewing an n -fold Segal space as an $(n - 1)$ -fold Segal space, condition (CSS^i) above means that the i th iteration is a complete Segal space object. Thus, given an n -fold Segal space $X_{\bullet, \dots, \bullet}$, one can apply the completion functor iteratively to obtain a complete n -fold Segal space $\widehat{X}_{\bullet, \dots, \bullet}$, its (n -fold)

completion. This yields a map $X \rightarrow \hat{X}$, the *completion map*, which is universal among all maps (in the homotopy category) to complete n -fold Segal spaces. It is a left adjoint to the inclusion of the homotopy category of $\mathbf{CSSpaces}_n$ into the homotopy category of $\mathbf{SSpaces}_n$.

If an n -fold Segal space $X_{\bullet, \dots, \bullet}$ satisfies (SC^j) for $j \leq m$, we can apply the completion functor just to the last $(n - m)$ indices to obtain an m -hybrid n -fold Segal space $\hat{X}_{\bullet, \dots, \bullet}^m$, its *m -hybrid completion*.

1.4 The homotopy bicategory of a 2-fold Segal space

To any higher category one can intuitively associate a bicategory having the same objects and 1-morphisms, and with 2-morphisms being 3-isomorphism classes of the original 2-morphisms.

Definition 1.4.1. The *homotopy bicategory* $\mathfrak{h}_2(X)$ of a 2-fold Segal space $X = X_{\bullet, \bullet}$ is defined as follows: objects are the points of the space $X_{0,0}$ and

$$\mathrm{Hom}_{\mathfrak{h}_2(X)}(x, y) = \mathfrak{h}_1(\mathrm{Hom}_X(x, y)) = \mathfrak{h}_1\left(\{x\} \times_{X_{0,\bullet}}^h X_{1,\bullet} \times_{X_{0,\bullet}}^h \{y\}\right)$$

as Hom categories. Horizontal composition is defined as follows:

$$\begin{aligned} \left(\{x\} \times_{X_{0,\bullet}}^h X_{1,\bullet} \times_{X_{0,\bullet}}^h \{y\}\right) \times \left(\{y\} \times_{X_{0,\bullet}}^h X_{1,\bullet} \times_{X_{0,\bullet}}^h \{z\}\right) &\longrightarrow \{x\} \times_{X_{0,\bullet}}^h X_{1,\bullet} \times_{X_{0,\bullet}}^h X_{1,\bullet} \times_{X_{0,\bullet}}^h \{z\} \\ &\xleftarrow{\sim} \{x\} \times_{X_{0,\bullet}}^h X_{2,\bullet} \times_{X_{0,\bullet}}^h \{z\} \\ &\longrightarrow \{x\} \times_{X_{0,\bullet}}^h X_{1,\bullet} \times_{X_{0,\bullet}}^h \{z\}. \end{aligned}$$

The second arrow happens to go in the wrong way but it is a weak equivalence. Therefore after taking \mathfrak{h}_1 it turns out to be an equivalence of categories, and thus to have an inverse (assuming the axiom of choice).

1.5 Constructions of n -fold Segal spaces

We describe several intuitive constructions of (∞, n) -categories in terms of (complete) n -fold Segal spaces.

1.5.1 Truncation

Given an (∞, n) -category, for $k \leq n$ its (∞, k) -truncation is the (∞, k) -category obtained by discarding the non-invertible m -morphisms for $k < m \leq n$.

In terms of n -fold Segal spaces, there is a functor of n -fold Segal spaces sending $X = X_{\bullet, \dots, \bullet}$ to its k -truncation, the k -fold Segal space

$$\tau_k X = X_{\underbrace{\bullet, \dots, \bullet}_{k \text{ times}}, \underbrace{0, \dots, 0}_{n-k \text{ times}}}.$$

Remark 1.5.1. Note that if X is m -hybrid then so is $\tau_k X$ by the definition of the conditions (CSS^i) and (SC^j) .

Warning. Truncation does not behave well with completion, i.e. the truncation of the completion is not the completion of the truncation. However, we get a map in one direction.

$$\begin{array}{ccc} \tau_k(X) & \longrightarrow & \tau_k(\widehat{X}) \\ \downarrow & \nearrow & \\ \widehat{\tau_k(X)} & & \end{array}$$

In general, we do not expect this map to be an equivalence.

Thus in general one should always complete an n -fold Segal space before truncating it, as

$$X_{\underbrace{1, \dots, 1}_k, 0, \dots, 0} \hookrightarrow X_{\underbrace{1, \dots, 1}_m, 0, \dots, 0}$$

are the invertible m -morphisms for $k < m \leq n$ if and only if X satisfies (1.3.5) for $k < i \leq n$. For example, if $X = X_\bullet$ is a (1-fold) Segal space then X_0 is the underlying ∞ -groupoid of invertible morphisms if and only if X is complete.

1.5.2 Extension

Any (∞, n) -category can be viewed as an $(\infty, n+1)$ -category with only identities as $(n+1)$ -morphisms.

In terms of iterated Segal spaces, any n -fold Segal space can be viewed as a constant simplicial object in n -fold Segal spaces, i.e. an $(n+1)$ -fold Segal space which is constant in the first index. Explicitly, if $X_{\bullet, \dots, \bullet}$ is an n -fold Segal space, then $\varepsilon(X)_{\bullet, \dots, \bullet}$ is the constant $(n+1)$ -fold Segal space such that for every $k \geq 0$,

$$\varepsilon(X)_{k, \bullet, \dots, \bullet} = X_{\bullet, \dots, \bullet}$$

with identities as face and degeneracy maps.

Lemma 1.5.2. *If X is complete, then $\varepsilon(X)$ is complete.*

Proof. Since X is complete, it satisfies (CSS^i) for $i > 1$. For $i = 0$, we have to show that $\varepsilon(X)_{\bullet, 0, \dots, 0}$ is complete. This is satisfied because

$$(\varepsilon(X)_{1, 0, \dots, 0})^{inv} = \varepsilon(X)_{1, 0, \dots, 0} = X_{0, \dots, 0} = \varepsilon(X)_{0, 0, \dots, 0},$$

since morphisms between two elements x, y in the homotopy category of $\varepsilon(X)_{\bullet, k_2, \dots, k_n}$ are just connected components of the space of paths in X_{k_2, \dots, k_n} , and thus are always invertible. \square

We call ε the extension functor, which is left adjoint to τ_1 . Moreover, the unit $\text{id} \rightarrow \tau_1 \circ \varepsilon$ of the adjunction is the identity.

1.5.3 The higher category of morphisms and loopings

Given two objects x, y in an (∞, n) -category, morphisms from x to y should form an $(\infty, n-1)$ -category. This can be realized for n -fold Segal spaces, which is one of the main advantages of this model for (∞, n) -categories.

Definition 1.5.3. Let $X = X_{\bullet, \dots, \bullet}$ be an n -fold Segal space. As we have seen above one should think of objects as vertices of the space $X_{0, \dots, 0}$. Let $x, y \in X_{0, \dots, 0}$. The $(n-1)$ -fold Segal space of morphisms from x to y is

$$\mathrm{Hom}_X(x, y)_{\bullet, \dots, \bullet} = \{x\} \times_{X_{0, \dots, \bullet}}^h X_{1, \bullet, \dots, \bullet} \times_{X_{0, \bullet, \dots, \bullet}}^h \{y\}.$$

Remark 1.5.4. Note that if X is m -hybrid, then $\mathrm{Hom}_X(x, y)$ is $(m-1)$ -hybrid.

Example 1.5.5 (Compatibility with extension). Let X be an $(\infty, 0)$ -category, i.e. a space, viewed as an $(\infty, 1)$ -category, i.e. a constant (complete) Segal space $\varepsilon(X)_{\bullet}$, $\varepsilon(X)_k = X$. For any two objects $x, y \in \varepsilon(X)_0 = X$ the $(\infty, 0)$ -category, i.e. the topological space, of morphisms from x to y is

$$\mathrm{Hom}_{\varepsilon(X)}(x, y) = \{x\} \times_{\varepsilon(X)_0}^h \varepsilon(X)_1 \times_{\varepsilon(X)_0}^h \{y\} = \{x\} \times_X^h \{y\} = \mathrm{Path}_X(x, y),$$

the path space in X , which coincides with what one expects by the interpretation of paths, homotopies, homotopies between homotopies, etc. being higher invertible morphisms.

Definition 1.5.6. Let X be an n -fold Segal space, and $x \in X_0$ an object in X . Then the *looping of X at x* is the $(n-1)$ -fold Segal space

$$L(X, x)_{\bullet, \dots, \bullet} = \mathrm{Hom}_X(x, x)_{\bullet, \dots, \bullet} = \{x\} \times_{X_{0, \bullet, \dots, \bullet}}^h X_{1, \bullet, \dots, \bullet} \times_{X_{0, \bullet, \dots, \bullet}}^h \{x\}.$$

In the following, it will often be clear at which element we are looping, e.g. if there essentially only is one element, or at a unit for the monoidal structure. Then we omit the x from the notation and just write

$$LX = L(X) = L(X, x).$$

Note that even if there is not a unique unit, this will be independent of the choice of unit.

We can iterate this procedure as follows.

Definition 1.5.7. Let $L_0(X, x) = X$. For $1 \leq k \leq n$, let the k -fold iterated looping be the $(n-k)$ -fold Segal space

$$L_k(X, x) = L(L_{k-1}(X, x), x),$$

where we view x as a trivial k -morphism via the degeneracy maps, i.e. an element in $L_{k-1}(X, x)_{0, \dots, 0} \subset X_{1, \dots, 1, 0, \dots, 0}$, with k 1's.

Remark 1.5.8. We remark that looping commutes with taking the ordinary or the m -hybrid completion, since completion is taken index per index.

1.6 Symmetric monoidal n -fold Segal spaces

1.6.1 as a Γ -object

Following [Toe, TV09], we define a symmetric monoidal n -fold Segal space in analogy to so-called Γ -spaces.

Definition 1.6.1. Segal's category Γ is the category whose objects are the finite sets

$$\langle m \rangle = \{0, \dots, m\},$$

for $m \geq 0$ which are pointed at 0. Morphisms are pointed functions, i.e. for $k, m \geq 0$, functions

$$f : \langle m \rangle \longrightarrow \langle k \rangle, \quad f(0) = 0.$$

For every $m \geq 0$, there are m canonical morphisms

$$\gamma_\beta : \langle m \rangle \longrightarrow \langle 1 \rangle, \quad j \longmapsto \delta_{\beta j}$$

for $1 \leq \beta \leq m$, called the *Segal morphisms*.

Remark 1.6.2. Segal's category Γ is the skeleton of the category of finite pointed sets.

We would now like to define a symmetric monoidal (complete) n -fold Segal space to be an ∞ -functor from Γ to the $(\infty, 1)$ -category of n -fold complete Segal spaces which satisfies certain properties. Recall from remark 1.3.4 and section 1.3.2 that the $(\infty, 1)$ -category of n -fold (complete) Segal spaces is presented by a model category $(\mathbf{C})\mathbf{SSpace}_n$. Using the strictification theorem of Toën-Vezzosi from ?? every such functor can be represented by a strict functor from Γ to $(\mathbf{C})\mathbf{SSpace}_n$. Moreover, the $(\infty, 1)$ -category of $(\infty, 1)$ -functors can be computed using the model category $((\mathbf{C})\mathbf{SSpace}_n)^\Gamma$ of Γ -diagrams in $(\mathbf{C})\mathbf{SSpace}_n$. Thus the following definition suffices.

Definition 1.6.3. A *symmetric monoidal (complete) n -fold Segal space* is a (strict) functor from Γ to the (strict) category of complete n -fold Segal spaces $(\mathbf{C})\mathbf{SSpace}_n$,

$$A : \Gamma \longrightarrow (\mathbf{C})\mathbf{SSpace}_n$$

such that for every $m \geq 0$, the induced map

$$A\left(\prod_{1 \leq \beta \leq m} \gamma_\beta\right) : A\langle m \rangle \longrightarrow (A\langle 1 \rangle)^m$$

is an equivalence of n -fold (complete) Segal spaces.

The (complete) n -fold Segal space $X = A\langle 1 \rangle$ is called the (complete) n -fold Segal space *underlying* A , and by abuse of language we will sometimes call a (complete) n -fold Segal space X symmetric monoidal, if there is a symmetric monoidal (complete) n -fold Segal space A such that $A\langle 1 \rangle = X$.

Remark 1.6.4. Note that in particular, for $m = 0$, this implies that $A\langle 0 \rangle$ is a point, viewed as a constant n -fold Segal space.

Definition 1.6.5. The $(\infty, 1)$ -category, i.e. Segal space, of functors from Γ to $(\mathbf{C})\mathbf{SSpace}_n$, which as mentioned above can be computed using the model category of Γ -diagrams in $(\mathbf{C})\mathbf{SSpace}_n$, has a full sub- $(\infty, 1)$ -category of symmetric monoidal (complete) n -fold Segal spaces. A 1-morphism in this category is called a *symmetric monoidal functor of (∞, n) -categories*.

Since the completion map $X \rightarrow \widehat{X}$ is a weak equivalence and preserves finite products of Segal spaces up to weak equivalence, we obtain the following

Lemma 1.6.6. *If $A : \Gamma \rightarrow \mathbf{SSpace}_n$ is a symmetric monoidal n -fold Segal space, then*

$$\begin{aligned} \widehat{A} : \Gamma &\rightarrow \mathbf{CSSpace}_n, \\ \langle m \rangle &\mapsto \widehat{A\langle m \rangle} \end{aligned}$$

is a symmetric monoidal complete n -fold Segal space.

Remark 1.6.7. In the following, all our symmetric monoidal structures will arise from functors (of actual categories)

$$\Gamma \rightarrow \mathbf{SSpace}_n,$$

and our symmetric monoidal functors from (strict) natural transformations of such. However, in the homotopy theoretic setting, one should allow our more flexible definition above.

Remark 1.6.8. Similarly, one can define k -monoidal (complete) n -fold Segal spaces as certain $(\infty, 1)$ -functors from \mathbf{E}_k to $(\mathbf{C})\mathbf{SSpace}_n$, where \mathbf{E}_k denotes the $(\infty, 1)$ -category obtained from the topological category of little- k -disks, see [Zha13] for more details on this definition.

Example 1.6.9. Let $A : \Gamma \rightarrow \mathbf{SSpace}_1$ be a symmetric monoidal Segal space. Consider the product of maps $\gamma_1 \times \gamma_2$ and the map induced by the map $\gamma : \langle 2 \rangle \rightarrow \langle 1 \rangle; 1, 2 \mapsto 1$,

$$A\langle 1 \rangle \times A\langle 1 \rangle \xleftarrow[A(\gamma_1) \times A(\gamma_2)]{\simeq} A\langle 2 \rangle \xrightarrow{A(\gamma)} A\langle 1 \rangle.$$

Passing to the homotopy category, we obtain a map

$$h_1(A\langle 1 \rangle) \times h_1(A\langle 1 \rangle) \rightarrow h_1(A\langle 1 \rangle).$$

Toën and Vezzosi showed in [TV09] that this is a symmetric monoidal structure on the category $h_1(A\langle 1 \rangle)$. Roughly speaking, this uses functoriality of A . Associativity uses the Segal space $A\langle 3 \rangle$, $A\langle 0 \rangle$ corresponds to the unit, and the map $c : \langle 2 \rangle \rightarrow \langle 2 \rangle; 1 \mapsto 2, 2 \mapsto 1$ induces the commutativity constraint.

Example 1.6.10. Truncations and extensions of symmetric monoidal (∞, n) -categories again are symmetric monoidal. Let A be a symmetric monoidal n -fold Segal space. Then we can define

$$\tau_k(A)\langle m \rangle = \tau_k(A\langle m \rangle), \quad \varepsilon(A)\langle m \rangle = \varepsilon(A\langle m \rangle).$$

Note that τ_k and ε are functors of n -fold Segal spaces which preserves weak equivalences. Thus, these assignments can be extended to functors $\tau_k(A)$ and $\varepsilon(A)$, and the images of $A(\prod_{1 \leq \beta \leq m} \gamma_\beta)$ are again weak equivalence.

Example 1.6.11. For every $m \geq 0$ there is a unique map $\langle 0 \rangle \rightarrow \langle m \rangle$, and since $A\langle 0 \rangle$ is the point as a constant (complete) n -fold Segal space, this induces, for every $m \geq 0$, a distinguished object $\mathbb{1}_{\langle m \rangle} \in A\langle m \rangle$. The looping of a symmetric monoidal n -fold Segal space A with respect to this object also is symmetric monoidal, with

$$L(A)\langle m \rangle = L(A\langle m \rangle, \mathbb{1}_{\langle m \rangle}),$$

which extends to an appropriate functor similarly to in the previous example.

Example 1.6.12. Important examples come from the classification diagram construction. Let \mathcal{C} be a small symmetric monoidal category and let $\mathcal{W} = \text{Iso}\mathcal{C}$. As we saw in section 1.2.3, this gives a complete Segal space $\mathcal{C}_\bullet = N(\mathcal{C}, \mathcal{W})$. The symmetric monoidal structure of \mathcal{C} endows \mathcal{C}_\bullet with the structure of a symmetric monoidal complete Segal space:

First note that $\mathcal{W}^{\times m} = \text{Iso}(\mathcal{C}^{\times m})$ for every m . On objects, let $A : \Gamma \rightarrow \mathbf{CSSpace}_1$ be given by $A\langle m \rangle = N(\mathcal{C}^{\times m}, \mathcal{W}^{\times m})_\bullet$. We explain the image of the map $\langle 2 \rangle \rightarrow \langle 1 \rangle; 1, 2 \mapsto 1$, which should be a map $A\langle 2 \rangle \rightarrow A\langle 1 \rangle$. The image of an arbitrary map $\langle m \rangle \rightarrow \langle l \rangle$ can be defined analogously.

An l -simplex in $A\langle 2 \rangle_0 = N(\mathcal{C} \times \mathcal{C}, \mathcal{W} \times \mathcal{W})_0$ is a pair

$$C_0 \xrightarrow{w_1} \dots \xrightarrow{w_l} C_l, \quad D_0 \xrightarrow{w'_1} \dots \xrightarrow{w'_l} D_l,$$

and is sent to

$$C_0 \otimes D_0 \xrightarrow{w''_1} \dots \xrightarrow{w''_l} C_l \otimes D_l,$$

where $w''_i : C_{i-1} \otimes D_{i-1} \xrightarrow{w_i \otimes \text{id}_{D_{i-1}}} C_i \otimes D_{i-1} \xrightarrow{\text{id}_{C_i} \otimes w'_i} C_i \otimes D_i$ is in \mathcal{W} . More generally, an l -simplex in

$$A\langle 2 \rangle_k = N(\mathcal{C} \times \mathcal{C}, \mathcal{W} \times \mathcal{W})_k$$

is a pair of diagrams

$$\begin{array}{ccc} C_{0,0} \xrightarrow{f_{10}} C_{1,0} \xrightarrow{f_{20}} \dots \xrightarrow{f_{k0}} C_{k,0} & & D_{0,0} \xrightarrow{g_{10}} D_{1,0} \xrightarrow{g_{20}} \dots \xrightarrow{g_{k0}} D_{k,0} \\ \downarrow w_{01} & \downarrow w_{11} & \downarrow v_{01} & \downarrow v_{11} & \downarrow v_{k1} \\ C_{0,1} \xrightarrow{f_{11}} C_{1,1} \xrightarrow{f_{21}} \dots \xrightarrow{f_{k1}} C_{k,1} & & D_{0,1} \xrightarrow{g_{11}} D_{1,1} \xrightarrow{g_{21}} \dots \xrightarrow{g_{k1}} D_{k,1} \\ \downarrow w_{02} & \downarrow w_{21} & \downarrow v_{02} & \downarrow v_{21} & \downarrow v_{k2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \downarrow w_{0l} & \downarrow w_{1l} & \downarrow v_{0l} & \downarrow v_{1l} & \downarrow v_{kl} \\ C_{0,l} \xrightarrow{f_{1l}} C_{1,l} \xrightarrow{f_{2l}} \dots \xrightarrow{f_{kl}} C_{k,l} & & D_{0,l} \xrightarrow{g_{1l}} D_{1,l} \xrightarrow{g_{2l}} \dots \xrightarrow{g_{kl}} D_{k,l} \end{array}$$

which is sent to the diagram

$$\begin{array}{ccc} C_{0,0} \otimes D_{0,0} \xrightarrow{f_{10} \otimes g_{10}} C_{1,0} \otimes D_{1,0} \xrightarrow{f_{20} \otimes g_{20}} \dots & & \\ \downarrow & \downarrow & \\ C_{0,1} \otimes D_{0,1} \xrightarrow{f_{11} \otimes g_{11}} C_{1,1} \otimes D_{1,1} \xrightarrow{f_{21} \otimes g_{21}} \dots & & \\ \downarrow & \downarrow & \\ \vdots & \vdots & \end{array}$$

where the vertical maps are defined as for the objects.

Finally, we need to check that $A(\prod_{1 \leq \beta \leq m} \gamma_\beta)$ is a weak equivalence. This follows from the fact that

$$(A\langle m \rangle)_k = N(\mathcal{C}^{\times m}, \mathcal{W}^{\times m})_k = (N(\mathcal{C}, \mathcal{W})_k)^{\times m} = (A\langle 1 \rangle_k)^m.$$

Remark 1.6.13. If we start with a symmetric monoidal relative category $(\mathcal{C}, \mathcal{W})$ (a definition can e.g. be found in [Cam14]) such that all $N(\mathcal{C}^{\times m}, \mathcal{W}^{\times m})$ are (complete) Segal spaces, then the above construction for $(\mathcal{C}, \mathcal{W})$ yields a symmetric monoidal (complete) Segal space $N(\mathcal{C}, \mathcal{W})$.

1.6.2 Symmetric monoidal n -fold Segal spaces as a tower of $(n + i)$ -fold Segal spaces

Our motivation for the following definition of a (k) -monoidal complete n -fold Segal space comes from the Delooping Hypothesis, which is inspired by the fact that a monoidal category can be seen as a bicategory with just one object. Similarly, a k -monoidal n -category should be a $(k + n)$ -category (whatever that is) with only one object, one 1-morphism, one 2-morphism, and so on up to one $(k - 1)$ -morphism.

Hypothesis 1.6.14 (Delooping Hypothesis). k -monoidal (∞, n) -categories can be identified with $(k - j)$ -monoidal, $(j - 1)$ -simply connected $(\infty, n + j)$ -categories for any $0 \leq j \leq k$, where $(j - 1)$ -simply connected means that any two parallel i -morphisms are equivalent for $i < j$. In particular, monoidal (∞, n) -categories can be identified with $(\infty, n + 1)$ -categories with (essentially) one object.

Monoidal n -fold complete Segal spaces

We use the last statement in the delooping hypothesis as the motivation for the following definition. However, first we need to explain what “having (essentially) one object” means.

Definition 1.6.15. An n -fold Segal space X is called *pointed* or *0-connected*, if

$$X_{0, \bullet, \dots, \bullet},$$

is weakly equivalent to the point viewed as a constant n -fold Segal space.

Definition 1.6.16. A *monoidal* complete n -fold Segal space is a 1-hybrid $(n + 1)$ -fold Segal space $X^{(1)}$ which is pointed. We say that this endows the n -fold complete Segal space

$$X = L(X^{(1)}, *)$$

with a monoidal structure and that $X^{(1)}$ is a *delooping* of X .

Remark 1.6.17. Note that as $X^{(1)}$ is 1-hybrid, $X_{0, \bullet, \dots, \bullet}^{(1)}$ is discrete. Thus, to be pointed implies that $X_{0, \bullet, \dots, \bullet}^{(1)}$ is equal to the point viewed as a constant n -fold Segal space.

Without the completeness condition, we could define a monoidal n -fold Segal space as an $(n + 1)$ -fold Segal space $X^{(1)}$ which is pointed. Then $L(X^{(1)}, *) = \text{Hom}_{X^{(1)}}(*, *)$ is independent of the choice of point $* \in X_{0, \dots, 0}$ and we can say that this endows the n -fold Segal space $X = L(X^{(1)}) = L(X^{(1)}, *)$ with a monoidal structure.

However, a complete Segal space will not have a contractible space as $X_{0, \dots, 0}$. Thus, we need to introduce a model for $(\infty, n+k)$ -categories which can have a point as the set of objects, 1-morphisms, et cetera, which motivates our use of hybrid Segal spaces.

Remark 1.6.18. Let X be an m -hybrid n -fold Segal space with $m > 0$ which is pointed. Then $X_{0, \bullet, \dots, \bullet} = *$, and the looping is

$$L(X)_{\bullet, \dots, \bullet} = \{*\} \times_*^h X_{1, \bullet, \dots, \bullet} \times_*^h \{*\} = X_{1, \bullet, \dots, \bullet}.$$

A similar definition works for hybrid Segal spaces.

Definition 1.6.19. A *monoidal* m -hybrid n -fold Segal space is an $(m+1)$ -hybrid $(n+1)$ -fold Segal space $X^{(1)}$ which is pointed. We say that this endows the m -hybrid n -fold Segal space

$$X = L(X^{(1)})$$

with a monoidal structure and that $X^{(1)}$ is a *delooping* of X .

Example 1.6.20. Let \mathcal{C} be a small monoidal category and let $\mathcal{W} = \text{Iso } \mathcal{C}$. As we saw in section 1.2.3, this gives a complete Segal space $\mathcal{C}_\bullet = N(\mathcal{C}, \mathcal{W})$. The monoidal structure of \mathcal{C} endows \mathcal{C}_\bullet with the structure of a monoidal complete Segal space:

Let $\mathcal{C}_{m,n} = \mathcal{C}_n^{\otimes m}$ be the category which has objects of the form

$$C_{01} \otimes \cdots \otimes C_{0m} \xrightarrow{c_1} \cdots \xrightarrow{c_n} C_{n0} \otimes \cdots \otimes C_{nm}$$

and morphisms of the form

$$\begin{array}{ccc} C_{01} \otimes \cdots \otimes C_{0m} & \xrightarrow{c_1} \cdots \xrightarrow{c_n} & C_{n0} \otimes \cdots \otimes C_{nm} \\ \downarrow f^0 & & \downarrow f^n \\ D_{01} \otimes \cdots \otimes D_{0m} & \xrightarrow{d_1} \cdots \xrightarrow{d_n} & D_{n0} \otimes \cdots \otimes D_{nm}, \end{array}$$

where $c_1, \dots, c_n, d_1, \dots, d_n$, and f^0, \dots, f^n are morphisms in \mathcal{C} .

Consider its subcategory $\mathcal{C}_{m,n}^{\mathcal{W}} \subset \mathcal{C}_{m,n}$ which has the same objects, and vertical morphisms involving only the ones in $\mathcal{W} = \text{Iso } \mathcal{C}$, i.e. f^0, \dots, f^n are morphisms in \mathcal{W} .

Now let

$$\mathcal{C}_{m,n}^{(1)} = N(\mathcal{C}_{m,n}^{\mathcal{W}}),$$

the (ordinary) nerve. By a direct verification one sees that the collection $\mathcal{C}_{\bullet, \bullet}^{(1)}$ is a 2-fold Segal space. Moreover,

1. $\mathcal{C}_{0,n}^{(1)} = N(\mathcal{C}_n^{\otimes 0}) = *$, so $\mathcal{C}_{0, \bullet}^{(1)}$ is discrete and equal to the point viewed as a constant Segal space, and
2. for every $m \geq 0$, $\mathcal{C}_{m, \bullet}^{(1)} = N(\mathcal{C}_{m, \bullet}^{\mathcal{W}}) = N((\mathcal{C}_{\bullet}^{\otimes m})^{\mathcal{W}})$, which is a complete Segal space.

Summarizing, $\mathcal{C}^{(1)}$ is a 1-hybrid 2-fold Segal space which is pointed and endows $L(\mathcal{C}^{(1)})_\bullet = \mathcal{C}_\bullet$ with the structure of a monoidal complete Segal space.

k -monoidal n -fold complete Segal spaces

To encode braided or symmetric monoidal structures, we can push this definition even further.

Definition 1.6.21. An n -fold Segal space X is called j -connected if for every $i < j$,

$$X_{\underbrace{1, \dots, 1}_i, 0, \bullet, \dots, \bullet}$$

is weakly equivalent to the point viewed as a constant n -fold Segal space.

Definition 1.6.22. A k -monoidal m -hybrid n -fold Segal space is an $(m+k)$ -hybrid $(n+k)$ -fold Segal space $X^{(k)}$ which is $(j-1)$ -connected for every $0 < j \leq k$.

Remark 1.6.23. Note that as $X^{(k)}$ is $(m+k)$ -hybrid, $X_{\underbrace{1, \dots, 1}_i, 0, \bullet, \dots, \bullet}^{(k)}$ is discrete. Thus, to be $(j-1)$ -connected implies that $X_{\underbrace{1, \dots, 1}_i, 0, \bullet, \dots, \bullet}^{(k)}$ is equal to the point viewed as a constant $(n-i+1)$ -fold Segal space.

By the following proposition this definition satisfies the delooping hypothesis. In practice this allows to define a k -monoidal n -fold complete Segal space step-by-step by defining a tower of monoidal i -hybrid $(n+i)$ -fold Segal spaces for $0 \leq i < k$.

Proposition 1.6.24. *The data of a k -monoidal n -fold complete Segal space is the same as a tower of monoidal i -hybrid $(n+i)$ -fold Segal spaces $X^{(i+1)}$ for $0 \leq i < k$ together with weak equivalences*

$$X^{(j)} \simeq L(X^{(j+1)})$$

for every $0 \leq j < k-1$.

Remark 1.6.25. We say that these equivalent data endow the complete n -fold Segal space

$$X = X^{(0)} \simeq L(X^{(1)})$$

with a k -monoidal structure. The $(n+i+1)$ -fold Segal space $X^{(i+1)}$ is called an i -fold delooping of X .

Before we prove this proposition, we need some lemmas:

Lemma 1.6.26. *If X is a k -monoidal m -hybrid n -fold Segal space, and $0 \leq l \leq k$, then X is also an l -monoidal $(m+k-l)$ -hybrid $(n+k-l)$ -fold Segal space.*

Proof. Since X is a k -monoidal m -hybrid n -fold Segal space, X is a $(m+k)$ -hybrid $(n+k)$ -fold Segal space such that for every $0 \leq i < k$,

$$X_{\underbrace{1, \dots, 1}_i, 0, \dots, 0} = *,$$

so in particular, this also holds for $0 \leq i < l$. □

Lemma 1.6.27. *Let X be a k -monoidal m -hybrid n -fold Segal space. Then $\mathrm{Hom}_X(*, *)$ is a $(k-1)$ -monoidal $(m-1)$ -hybrid n -fold Segal space.*

Proof. This follows from

$$(\mathrm{Hom}_X(*, *))_{\bullet, \dots, \bullet} = \{*\} \times_{X_{0, \bullet, \dots, \bullet}}^h X_{1, \bullet, \dots, \bullet} \times_{X_{0, \bullet, \dots, \bullet}}^h \{*\} = X_{1, \bullet, \dots, \bullet},$$

since $X_{0, \bullet, \dots, \bullet}$ is a point. \square

Proof of Proposition 1.6.24. Let X be a k -monoidal n -fold complete Segal space. By Lemma 1.6.26 $X^{(k)} = X$ is a monoidal $(k-1)$ -hybrid $(n+k-1)$ -fold Segal space.

Now let $X^{(k-1)} = L(X^{(k)})$. By Lemmas 1.6.27 and 1.6.26, this is a monoidal $(k-2)$ -hybrid $(n+k-2)$ -fold Segal space.

Inductively, define $X^{(i)} = L(X^{(i+1)})$ for $1 \leq i \leq k-1$. Similarly to above, by Lemmas 1.6.27 and 1.6.26, this is a monoidal $(i-1)$ -hybrid $(n+i-1)$ -fold Segal space.

Conversely, assume we are given a tower $X^{(i)}$ as in the proposition. Since $X = X^{(k)}$ is a monoidal $(k-1)$ -hybrid $(n+k-1)$ -fold Segal space,

$$X_{0, \bullet, \dots, \bullet} = X_{0, \bullet, \dots, \bullet}^{(k)} = *. \quad (1.1)$$

Since $X^{(k-1)}$ is a monoidal $(k-2)$ -hybrid $(n+k-2)$ -fold Segal space and by (1.1),

$$\begin{aligned} X_{1, 0, \bullet, \dots, \bullet} &= X_{1, 0, \bullet, \dots, \bullet}^{(k)} = \{*\} \times_{X_{0, 0, \bullet, \dots, \bullet}^{(k)}}^h X_{1, 0, \bullet, \dots, \bullet}^{(k)} \times_{X_{0, 0, \bullet, \dots, \bullet}^{(k)}}^h \{*\} \\ &= \left(\mathrm{Hom}_X^{(k)}(*, *) \right)_{0, \bullet, \dots, \bullet} \\ &\simeq X_{0, \bullet, \dots, \bullet}^{(k-1)} = *. \end{aligned} \quad (1.2)$$

Since $X^{(k)}$ is k -hybrid, $X_{1, 0, \bullet, \dots, \bullet}$ is discrete and so $X_{1, 0, \bullet, \dots, \bullet} = *$.

Inductively, for $0 \leq i < k$, since $X^{(k-i)}$ is a monoidal $(k-i-1)$ -hybrid $(n+k-i-1)$ -fold Segal space and by (1.1), (1.2), . . .

$$\begin{aligned} X_{\underbrace{1, \dots, 1}_i, 0, \bullet, \dots, \bullet} &= X_{\underbrace{1, \dots, 1}_i, 0, \bullet, \dots, \bullet}^{(k)} \\ &= \{*\} \times_{X_{0, \underbrace{1, \dots, 1}_{i-1}, 0, \bullet, \dots, \bullet}^{(k)}}^h X_{\underbrace{1, \dots, 1}_i, 0, \bullet, \dots, \bullet}^{(k)} \times_{X_{0, \underbrace{1, \dots, 1}_{i-1}, 0, \bullet, \dots, \bullet}^{(k)}}^h \{*\} \\ &= \left(\mathrm{Hom}_X^{(k)}(*, *) \right)_{\underbrace{1, \dots, 1}_{i-1}, 0, \bullet, \dots, \bullet} \\ &\simeq X_{\underbrace{1, \dots, 1}_{i-1}, 0, \bullet, \dots, \bullet}^{(k-1)} = \dots \simeq X_{0, \bullet, \dots, \bullet}^{(k-i)} = *. \end{aligned}$$

Again, since $X^{(k)}$ is k -hybrid, $X_{\underbrace{1, \dots, 1}_i, 0, \bullet, \dots, \bullet}$ is discrete and so

$$X_{\underbrace{1, \dots, 1}_i, 0, \bullet, \dots, \bullet} = *.$$

\square

Symmetric monoidal n -fold complete Segal spaces

The Stabilization Hypothesis, first formulated in [BD95], states that an n -category which is monoidal of a sufficiently high degree cannot be made “more monoidal”, and thus it makes sense to call it symmetric monoidal, see e.g. [Sim98] for a proof for Tamsamani’s weak n -categories.

Hypothesis 1.6.28 (Stabilization Hypothesis). For $k \geq n + 2$, a k -monoidal n -category is the same thing as an $(n + 2)$ -monoidal n -category.

In the world of (∞, n) -categories this statement must be false, otherwise, any E_n -algebra, which can be thought of as an n -monoidal, n -connected (∞, n) -category would already be symmetric monoidal, i.e. commutative.

However, in light of Proposition 1.6.24, we can require that a symmetric monoidal (∞, n) -category is k -monoidal for every $k \geq 0$ to encode that it is “monoidal enough”.

Definition 1.6.29. A *symmetric monoidal structure* on a complete n -fold Segal space X is a tower of monoidal i -hybrid $(n + i)$ -fold Segal spaces $X^{(i+1)}$ for $i \geq 0$ such that if we set $X = X^{(0)}$, for every $i \geq 0$,

$$X^{(i)} \simeq L(X^{(i+1)}).$$

Remark 1.6.30. This definition and definition 1.6.3 given in the previous section should be equivalent. Indeed, we should be able to switch between the definitions by setting

$$X\langle m \rangle_{\bullet, \dots, \bullet} = X_{m, \bullet, \dots, \bullet}^{(1)}.$$

Using the fact that $N(\Delta, \text{Iso } \Delta)$ and \mathbf{E}_1 are equivalent as $(\infty, 1)$ -categories and Dunn’s additivity theorem (“ \mathbf{E}_n is \mathbf{E}_1 in \mathbf{E}_1 in \dots in \mathbf{E}_1 , n times”), k -monoidal (complete) n -fold Segal spaces in the sense of this section should be equivalent to k -monoidal (complete) n -fold Segal spaces in the sense of remark 1.6.8.

Furthermore, since $N(\Gamma, \text{Iso } \Gamma)$ and \mathbf{E}_∞ are equivalent as $(\infty, 1)$ -categories, this leads to the desired equivalence. There are certainly details to be worked out about to make these statements precise.

The (∞, n) -category of cobordisms

To rigorously define fully extended topological field theories we need a suitable (∞, n) -category of cobordisms, which, informally speaking, has zero-dimensional manifolds as objects, bordisms between objects as 1-morphisms, bordisms between bordisms as 2-morphisms, etc., and for $k > n$ there are only invertible k -morphisms. Finding an explicit model for such a higher category, i.e. defining a complete n -fold Segal space of bordisms, is the main goal of this chapter. We endow it with a symmetric monoidal structure and also consider bordism categories with additional structure, e.g. orientations and framings, which allows us, in section 2.7, to rigorously define fully extended topological field theories.

We build a rather explicit model suitable for our purposes, which will allow us to later define a functor to the desired target category. We will construct a variant of this construction, which is perhaps more conceptual and less adhoc, in subsequent publications.

2.1 The n -fold Segal space of closed intervals in $(0, 1)$

In this section we define a Segal space Int_\bullet of closed intervals in $(0, 1)$ which will form the basis of the n -fold Segal space of cobordisms. First we define the sets of vertices, i.e. of 0-simplices, of the levels. Then we define the spatial structure of the levels. Next we endow the collection of sets $(\text{Int}_k)_k$ with a simplicial structure which we then extend to the l -simplices of the levels in a compatible way, giving the simplicial structure. Finally, we show that this construction yields a Segal space.

Definition 2.1.1. For an integer $k \geq 0$ let

$$\text{Int}_k = \{I_0 \leq \cdots \leq I_k\}$$

be the set consisting of ordered $(k+1)$ -tuples of intervals $I_j \subseteq (0, 1)$ with left endpoints a_j and right endpoints b_j such that I_j has non-empty interior, is closed in $(0, 1)$, and $a_0 = 0$, $b_k = 1$. By “ordered”, i.e. $I_j \leq I_{j'}$, we mean that the endpoints are ordered, i.e. $a_j \leq a_{j'}$ and $b_j \leq b_{j'}$ for $j \leq j'$.

2.1.1 The spatial structure of the levels Int_k

The l -simplices of the space Int_k

The l -simplices do not just consist of the data of a smooth family of intervals, but we add an extra rescaling datum which records how the intervals are deformed into each other along the simplex. This is given by a suitable family of order-preserving diffeomorphisms.

Definition 2.1.2. Let $(I_0(s) \leq \dots \leq I_k(s)) \in \text{Int}_k$ be a smooth family of intervals over $|\Delta^l|$, i.e. denoting the left endpoints by $a_j(s)$ and the right endpoints by $b_j(s)$, the maps $|\Delta^l| \rightarrow \mathbb{R}$, $s \mapsto a_j(s), b_j(s)$ are smooth maps. A smooth family of strictly monotonically increasing diffeomorphisms

$$(\varphi_{s,t} : (0, 1) \rightarrow (0, 1))_{s,t \in |\Delta^l|}$$

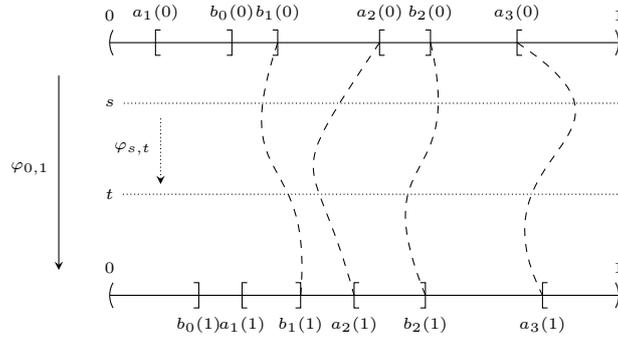
is said to *intertwine with the composed intervals* if the following condition is satisfied for every morphism $f : [m] \rightarrow [l]$ in the simplex category Δ .

Let $|f| : |\Delta^m| \rightarrow |\Delta^l|$ be the induced map between standard simplices. For every $0 \leq j < k$ such that

- either for every $s \in |f|(|\Delta^m|)$ the intersection $I_j(s) \cap I_{j+1}(s)$ is empty
- or for every $s \in |f|(|\Delta^m|)$ the intersection $I_j(s) \cap I_{j+1}(s)$ contains only one element,

we require that for every $s \in |f|(|\Delta^m|)$,

$$b_j(s) \xrightarrow{\varphi_{s,t}} b_j(t), \quad a_{j+1}(s) \xrightarrow{\varphi_{s,t}} a_{j+1}(t);$$



Remark 2.1.3. Note that it is enough to check this condition for $m \leq l$.

Definition 2.1.4. An l -simplex of Int_k consists of

1. a smooth family of *underlying 0-simplices*, i.e. for every $s \in |\Delta^l|$,

$$(I_0(s) \leq \dots \leq I_k(s)) \in \text{Int}_k,$$

depending smoothly on s ;

2. a *rescaling datum*, which is a smooth family of strictly monotonically increasing diffeomorphisms

$$(\varphi_{s,t} : (0, 1) \rightarrow (0, 1))_{s,t \in |\Delta^l|}$$

such that

- a) $\varphi_{s,s} = id$ for every $s \in |\Delta^l|$,
- b) $\varphi_{s,t} = \varphi_{t,s}^{-1}$ for every $s, t \in |\Delta^l|$, and
- c) $(\varphi_{s,t})_{s,t \in |\Delta^l|}$ intertwines with the composed intervals.

Remark 2.1.5. Note that in particular for $l = 0$ an l -simplex in this sense is an underlying 0-simplex together with $\varphi_{s,s} = id : (0, 1) \rightarrow (0, 1)$, so, by abuse of language we call both a 0-simplex.

Remark 2.1.6. The third condition will imply both that the simplicial structure is well defined (we can cut off intervals at the end) and that the spatial structure is well defined (the family of diffeomorphisms restricts well), as we will see in the next sections.

The space Int_k

The spatial structure arises similarly to that of the singular set of a topological space.

Fix $k \geq 0$ and let $f : [m] \rightarrow [l]$ be a morphism in the simplex category Δ , i.e. a (weakly) order-preserving map. Then let $|f| : |\Delta^m| \rightarrow |\Delta^l|$ be the induced map between standard simplices and let f^Δ be the map sending an l -simplex in Int_k given by

$$(I_0(s) \leq \dots \leq I_k(s))_{s \in |\Delta^l|}, \quad (\varphi_{s,t} : (0, 1) \rightarrow (0, 1))_{s,t \in |\Delta^l|},$$

to

$$I_0(|f|(s)) \leq \dots \leq I_k(|f|(s))_{s \in |\Delta^m|}, \quad (\varphi_{|f|(s),|f|(t)} : (0, 1) \rightarrow (0, 1))_{s,t \in |\Delta^m|}.$$

Lemma 2.1.7. *This assignment gives a functor $\Delta^{op} \rightarrow \text{Set}$ and thus Int_k is a space, i.e. a simplicial set.*

Proof. We need to verify that

$$I_0(|f|(s)) \leq \dots \leq I_k(|f|(s))_{s \in |\Delta^m|}, \quad (\varphi_{|f|(s),|f|(t)} : (0, 1) \rightarrow (0, 1))_{s,t \in |\Delta^m|}$$

is an m -simplex in Int_k . To see this, we need to check that the smooth family of diffeomorphisms $(\varphi_{|f|(s),|f|(t)})_{s,t \in |\Delta^m|}$ intertwines with the composed intervals. Let $g : [m'] \rightarrow [m]$, and $|g| : |\Delta^{m'}| \rightarrow |\Delta^m|$. Let $0 \leq j < k$ such that one of the two conditions on $I_j(|f|(s)) \cap I_{j+1}(|f|(s))$ in definition 2.1.2 is satisfied for every $s \in |g|(|\Delta^{m'}|)$. Then $f \circ g : [m'] \rightarrow [l]$ and the same condition is satisfied on $I_j(\tilde{s}) \cap I_{j+1}(\tilde{s})$ for every $\tilde{s} \in |f \circ g|(|\Delta^{m'}|)$. Since $(\varphi_{s,t})_{s,t \in |\Delta^l|}$ intertwines with the composed intervals, for every $\tilde{s} \in |f \circ g|(|\Delta^{m'}|)$,

$$b_j(\tilde{s}) \xrightarrow{\varphi_{s,t}} b_j(\tilde{t}), \quad a_{j+1}(\tilde{s}) \xrightarrow{\varphi_{s,t}} a_{j+1}(\tilde{t}),$$

so for every $s \in |g|(|\Delta^{m'}|)$, since $\tilde{s} = |f|(s) \in |f \circ g|(|\Delta^{m'}|)$,

$$b_j(|f|(s)) \xrightarrow{\varphi_{|f|(s),|f|(t)}} b_j(|f|(t)), \quad a_{j+1}(|f|(s)) \xrightarrow{\varphi_{|f|(s),|f|(t)}} a_{j+1}(|f|(t)).$$

Functoriality follows from the functoriality of the geometric realization. \square

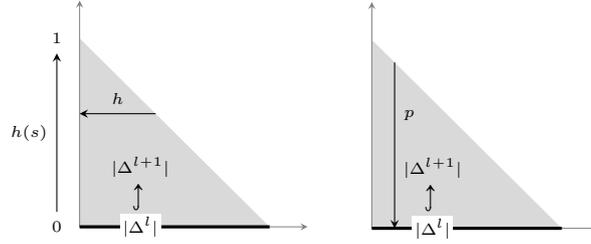
Notation 2.1.8. We denote the spatial face and degeneracy maps of Int_k by d_j^Δ and s_j^Δ for $0 \leq j \leq l$.

We will need the following lemma later for the Segal condition.

Proposition 2.1.9. *Each level Int_k is a contractible Kan complex. Moreover, the inclusion $* \hookrightarrow \text{Int}_\bullet$ given by degeneracies, where $*$ is seen as a constant complete Segal space, is a weak equivalence of complete Segal spaces.*

Proof. We first prove the contractibility. For every $k \geq 0$, consider the composition of degeneracy maps, which is the inclusion of the point $((0, 1) \leq \cdots \leq (0, 1)) \in \text{Int}_k$, which we will by abuse of notation again denote by $(0, 1)$. Given an l -simplex, we need to find an $(l + 1)$ -simplex with one l -dimensional face the given l -simplex and additional vertex $(0, 1)$ lying opposite that face.

Let $h : |\Delta^{l+1}| \rightarrow \mathbb{R}$ be the coordinate giving the distance from the given l -dimensional face and let $p : |\Delta^{l+1}| \rightarrow |\Delta^l|$ be the projection forgetting that coordinate,



For $s \in |\Delta^{l+1}|$ let

$$a_j(s) = (1 - h(s))a_j(p(s)), \quad b_j(s) = (1 - h(s))b_j(p(s)) + h(s).$$

This is a smooth family of intervals and this construction commutes with the simplicial structure of Int_\bullet .

To obtain the desired $(l + 1)$ -simplex we need to add a rescaling datum which restricts to the given rescaling datum on the given l -dimensional face. The condition for it to intertwine with the composed bordisms is non-trivial only on that face since every other face and the whole l -simplex itself contain the point $(0, 1)$ and thus all pairs of intervals will start to intersect along the face or the whole l -simplex.

Thus it is enough to find a smooth family of order-preserving diffeomorphisms $(\varphi_{s,t} : (0, 1) \rightarrow (0, 1))_{s,t \in |\Delta^{l+1}|}$ which restricts to the given one on $|\Delta^l| \hookrightarrow |\Delta^{l+1}|$ and satisfies $\varphi_{s,s} = id$ and $\varphi_{s,t} = \varphi_{t,s}^{-1}$, and in a way commuting with the simplicial structure. This follows from the fact that $\text{Diff}^+(\mathbb{R})$ is contractible.

Then Kan condition follows similarly from the contractibility of \mathbb{R} and $\text{Diff}^+(\mathbb{R})$. \square

2.1.2 The simplicial set Int_\bullet

In this subsection, the collection of sets Int_k is endowed with a simplicial structure by extending the assignment

$$[k] \longmapsto \text{Int}_k$$

to a functor from Δ^{op} .

Let $f : [m] \rightarrow [k]$ be a morphism in Δ . Then, let

$$\begin{aligned} \text{Int}_k &\xrightarrow{f^*} \text{Int}_m, \\ I_0 \leq \cdots \leq I_k &\longmapsto \rho_f(I_{f(0)} \leq \cdots \leq I_{f(m)}), \end{aligned}$$

where the rescaling map ρ_f is the unique affine transformation $\mathbb{R} \rightarrow \mathbb{R}$ sending $a_{f(0)}$ to 0 and $b_{f(m)}$ to 1.

Lemma 2.1.10. *The collection of sets $(\text{Int}_k)_k$ is a simplicial set.*

Proof. Given two maps $[m] \xrightarrow{f} [k] \xrightarrow{g} [p]$, and $I_0 \leq \cdots \leq I_p$, the rescaling map $\rho_{g \circ f}$ and the composition of the rescaling maps $\rho_g \circ \rho_f$ both send $a_{g \circ f(0)}$ to 0 and $b_{g \circ f(m)}$ to 1 and, since affine transformations $\mathbb{R} \rightarrow \mathbb{R}$ are uniquely determined by the image of two points, this implies that they coincide. Thus, this gives a functor $\Delta^{op} \rightarrow \text{Set}$. \square

Notation 2.1.11. We denote the (*simplicial*) *face and degeneracy maps* by $d_j : \text{Int}_k \rightarrow \text{Int}_{k-1}$ and $s_j : \text{Int}_k \rightarrow \text{Int}_{k+1}$ for $0 \leq j \leq k$.

Explicitly, they are given by the following formulas. The j th degeneracy map is given by inserting the j th interval twice,

$$\begin{aligned} \text{Int}_k &\xrightarrow{s_j} \text{Int}_{k+1}, \\ I_0 \leq \cdots \leq I_k &\longmapsto I_0 \leq \cdots \leq I_j \leq I_j \leq \cdots \leq I_k. \end{aligned}$$

The j th face map is given by deleting the j th interval and, for $j = 0, k$, by rescaling the rest linearly to $(0, 1)$. For $j = 0$, the rescaling map is the affine map ρ_0 sending $(a_1, 1)$ to $(0, 1)$, $\rho_0(x) = \frac{x-a_1}{1-a_1}$ and for $j = k$, it is the affine map $\rho_k : (0, b_{k-1}) \rightarrow (0, 1)$, $\rho_k(x) = \frac{x}{b_{k-1}}$. Explicitly,

$$\begin{aligned} \text{Int}_k &\xrightarrow{d_j} \text{Int}_{k-1}, \\ I_0 \leq \cdots \leq I_k &\longmapsto \begin{cases} I_0 \leq \cdots \leq \hat{I}_j \leq \cdots \leq I_k, & j \neq 0, k, \\ (0, \frac{b_1-a_1}{1-a_1}] \leq \cdots \leq [\frac{a_k-a_1}{1-a_1}, 1), & j = 0, \\ (0, \frac{b_0}{b_{k-1}}] \leq \cdots \leq [\frac{a_{k-1}}{b_{k-1}}, 1), & j = k. \end{cases} \end{aligned}$$

2.1.3 The Segal space Int_\bullet

The simplicial space Int_\bullet

We first extend the assignment $f \mapsto (f^* : \text{Int}_k \rightarrow \text{Int}_m)$ to l -simplices in a compatible way. Essentially, f^* arises from applying f^* to each of 0-simplices underlying the l -simplex.

Let $f : [m] \rightarrow [k]$ be a morphism in Δ . Recall that given $(I_0 \leq \cdots \leq I_k) \in \text{Int}_k$ we have an affine rescaling map $\rho_f : \mathbb{R} \rightarrow \mathbb{R}$ which sends $a_{f(0)}$ to 0 and $b_{f(m)}$ to 1.

Given a smooth family $(I_0(s) \leq \cdots \leq I_k(s))_{s \in |\Delta^l|}$, denote by $\rho_f(s)$ the rescaling map associated to the s th underlying 0-simplex $(I_0(s) \leq \cdots \leq I_k(s))$. Moreover, denote by $D_j(s) = (a_{f(0)}(s), b_{f(m)}(s))$.

Let f^* send an l -simplex of Int_k

$$(I_0(s) \leq \cdots \leq I_k(s))_{s \in |\Delta^l|} \quad (\varphi_{s,t})_{s,t \in |\Delta^l|}$$

to the following l -simplex of Int_m .

1. The underlying 0-simplices of the image are the images of the underlying 0-simplices under f^* , i.e. for $s \in |\Delta^l|$,

$$f^*(I_0(s) \leq \cdots \leq I_k(s));$$

2. its rescaling datum is

$$f^*(\varphi_{s,t}) = \rho_f(t) \circ \varphi_{s,t}|_{D_j(s)} \circ \rho_f(s)^{-1} : (0, 1)^n \rightarrow (0, 1)^n.$$

Note that $(f^*(\varphi_{s,t}))_{s,t \in |\Delta^l|}$ again intertwines well with the composed intervals since it is a restriction of a rescaling datum.

Using the fact that the rescaling maps behave functorially, we obtain the following lemma.

Lemma 2.1.12. *The collection of spaces $(\text{Int}_k)_k$ is a simplicial space.*

The complete Segal space Int_\bullet .

Proposition 2.1.13. *Int_\bullet is a complete Segal space.*

Proof. We have seen in lemma 2.1.9 that every Int_k is contractible. This ensures the Segal condition, namely that

$$\text{Int}_k \xrightarrow{\simeq} \text{Int}_1 \times_{\text{Int}_0}^h \cdots \times_{\text{Int}_0}^h \text{Int}_1,$$

and completeness. □

Definition 2.1.14. Let

$$\text{Int}_{\bullet, \dots, \bullet}^n = (\text{Int}_\bullet)^{\times n}.$$

Lemma 2.1.15. *The n -fold simplicial space $\text{Int}_{\bullet, \dots, \bullet}^n$ is a complete n -fold Segal space.*

Proof. The Segal condition and completeness follow from the Segal condition and completeness for Int_\bullet . Since every Int_k is contractible by lemma 2.1.9, $(\text{Int}_\bullet)^{\times n}$ satisfies essential constancy, so Int^n is an n -fold Segal space. □

2.2 A time-dependent Morse lemma

2.2.1 The classical Morse lemma

The following theorem is classical Morse lemma, as can be found e.g. in [Mil63].

Theorem 2.2.1 (Morse lemma). *Let f be a smooth proper real-valued function on a manifold M . Let $a < b$ and suppose that the interval $[a, b]$ contains no critical values of f . Then $M^a = f^{-1}((-\infty, a])$ is diffeomorphic to $M^b = f^{-1}((-\infty, b])$.*

We repeat the proof here since later on in this section we will adapt it to the situation we need.

Proof. Choose a metric on M , and consider the vector field

$$V = \frac{\nabla_y f}{|\nabla_y f|^2},$$

where ∇_y is the gradient on M . Since f has no critical value in $[a, b]$, V is defined in $f^{-1}((a - \epsilon, b + \epsilon))$, for suitable ϵ . Choose a smooth function $g : \mathbb{R} \rightarrow \mathbb{R}$ which is 1 on $(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$ and compactly supported in $(a - \epsilon, b + \epsilon)$. Extend g to a function $g : M \rightarrow \mathbb{R}$ by setting $g(y) = g(f(y))$. Then

$$\mathcal{V} = g \frac{\nabla_y f}{|\nabla_y f|^2}$$

is a compactly supported vector field on M and hence generates a 1-parameter group of diffeomorphisms

$$\psi_t : M \longrightarrow M.$$

Viewing $f - (a + t)$ as a function on $\mathbb{R} \times M$, $(t, y) \mapsto f(y) - (a + t)$, we find that in $f^{-1}((a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}))$,

$$\partial_t(f - (a + t)) = 1 = \frac{\nabla_y f}{|\nabla_y f|^2} \cdot (f - (a + t)) = V \cdot (f - (a + t)),$$

and so the flow preserves the set

$$\{(t, y) : f(y) = a + t\}.$$

Thus, the diffeomorphism ψ_{b-a} restricts to a diffeomorphism

$$\psi_{b-a}|_{M^a} : M^a \longrightarrow M^b.$$

□

2.2.2 The time-dependent Morse lemma

In Lemma 3.1 in [GWW] Gay, Wehrheim, and Woodward prove a time-dependent Morse lemma which shows that a smooth family of composed cobordisms in their (ordinary) category of (connected) cobordisms gives rise to a diffeomorphism which intertwines with the cobordisms. We adapt this lemma to a variant which will be suitable for our situation in the higher categorical setting.

Proposition 2.2.2. *Let M be a smooth manifold and let $(f_s : M \rightarrow (0, 1))_{s \in [0, 1]}$ be a smooth family of smooth functions which give rise to a smooth proper function $f : N = [0, 1] \times M \rightarrow (0, 1)$. Let $(I_0(s) \leq \dots \leq I_k(s))_{s \in [0, 1]}$ be a smooth family of closed intervals in $(0, 1)$ such that for every $s \in [0, 1]$, the function f_s has no critical value in $I_0(s) \cup \dots \cup I_k(s)$. Then there is a rescaling datum $(\varphi_{s,t} : (0, 1) \rightarrow (0, 1))_{s,t \in [0, 1]}$ which makes $(I_0(s) \leq \dots \leq I_k(s))_{s \in [0, 1]}$ into a 1-simplex in Int_k , and a smooth family of diffeomorphisms $(\psi_{s,t} : M \rightarrow M)_{s,t \in [0, 1]}$ such that for*

$$\begin{aligned} t_j(s) \in I_j(s) : \quad & \varphi_{s,t}(t_j(s)) \in I_j(t), \text{ and} \\ t_l(s) \in I_l(s) : \quad & \varphi_{s,t}(t_l(s)) \in I_l(t), \end{aligned}$$

$\psi_{s,t}$ restricts to diffeomorphisms

$$\psi_{s,t}|_{f_s^{-1}([t_j, t_l])} : f_s^{-1}([t_j, t_l]) \longrightarrow f_t^{-1}([\varphi_{s,t}(t_j), \varphi_{s,t}(t_l)]).$$

Proof. The main strategy of the proof is the same as for the classical Morse lemma. Namely, we will construct a suitable vector field whose flow gives the desired diffeomorphisms.

Step 1: disjoint intervals

First assume that for all $0 \leq j \leq k$ and for every $s \in [0, 1]$ we have

$$I_j(s) \cap I_{j+1}(s) = \emptyset.$$

Fix a metric on M . Denote the endpoints of the intervals by $a_j(s), b_j(s)$ as before, which yield smooth functions $a_j, b_j : [0, 1] \rightarrow (0, 1)$, and let

$$A_j = \bigcup_{s \in [0, 1]} \{s\} \times f_s^{-1}(a_j(s)), \quad B_j = \bigcup_{s \in [0, 1]} \{s\} \times f_s^{-1}(b_j(s)).$$

Now for $0 \leq j \leq k$ consider the vector fields

$$V_j = \left(\partial_s, \partial_s(a_j(s) - f_s) \frac{\nabla_y f_s}{|\nabla_y f_s|^2} \right), \quad W_j = \left(\partial_s, \partial_s(b_j(s) - f_s) \frac{\nabla_y f_s}{|\nabla_y f_s|^2} \right),$$

where ∇_y is the gradient on M . Since f_s has no critical value in $I_j(s)$, the vector fields V_j and W_j are defined on $f^{-1}(U_j)$, where U_j is a neighborhood of $\bigcup_{s \in [0, 1]} \{s\} \times I_j(s)$. Moreover, viewing $a_j : (s, y) \mapsto a_j(s)$ as a function on N ,

$$V_j(f - a_j) = \partial_s(f - a_j) + \partial_s(a_j - f) \frac{\nabla_y f}{|\nabla_y f|^2} (f - a_j) = \partial_s(f - a_j) + \partial_s(a_j - f) = 0,$$

So the vector field V_j is tangent to A_j and similarly, W_j is tangent to B_j .

We would now like to construct a vector field \mathcal{V} on N which for every $0 \leq j \leq k$, at A_j restricts to V_j and at B_j restricts to W_j , and such that there exists a family of functions $(c_x : [0, 1] \rightarrow (0, 1))_{x \in I_j(0)}$ such that

$$- c_x(0) = x, c_x(s) \in I_j(s),$$

- the graphs of c_x for varying x partition $\bigcup_{s \in [0,1]} \{s\} \times [a_j(s), b_j(s)]$, and
- \mathcal{V} is tangent to $C_x = \bigcup_{s \in [0,1]} \{s\} \times f_s^{-1}(c_x(s))$.

We will use c_x to define $\varphi_{0,s}(x) = c_x(s)$ and $\varphi_{s,t} = \varphi_{0,t} \circ \varphi_{0,s}^{-1}$. Moreover, the diffeomorphisms $\psi_{s,t}$ will arise as the flow along \mathcal{V} .

Fix smooth functions $g_j, h_j : [0, 1] \times (0, 1) \rightarrow \mathbb{R}$ which satisfy the following conditions:

1. g_j, h_j are compactly supported in U_j ,
2. $g_j = 1$ in a neighborhood of graph $a_j = \{(s, a_j(s)) : s \in [0, 1]\}$,
 $h_j = 1$ in a neighborhood of graph b_j
3. $g_j + h_j = 1$ in $\bigcup_{s \in [0,1]} \{s\} \times I_j(s)$, and the supports of the $g_j + h_j$ are disjoint.

By abuse of notation, extend the functions g_j, h_j to functions $g_j, h_j : N = [0, 1] \times M \rightarrow \mathbb{R}$ by setting $g_j(s, y) := g_j(s, f_s(y))$. Then consider the following vector field on N :

$$\mathcal{V}_j = \left(\partial_s, (g_j \partial_s(a_j) + h_j \partial_s(b_j) - \partial_s(f)) \frac{\nabla_y f}{|\nabla_y f|^2} \right)$$

This vector field is supported on the support of $g_j + h_j$ and thus extends to a vector field on N . Note that for $(s, y) \in A_j$, $\mathcal{V}_j(s, y) = V_j(s, y)$, and for $(s, y) \in B_j$, $\mathcal{V}_j(s, y) = W_j(s, y)$.

Now let \mathcal{V} be the vector field on N constructed by combining the above vector fields as follows:

$$\mathcal{V} = \left(\partial_s, \sum_{0 \leq j \leq k} (g_j \partial_s(a_j) + h_j \partial_s(b_j) - \partial_s(f)) \frac{\nabla_y f_s}{|\nabla_y f_s|^2} \right).$$

Note that in $\bigcup_{s \in [0,1]} \{s\} \times f_s^{-1}(I_j(s))$, it restricts to \mathcal{V}_j .

In order for \mathcal{V} to be tangent to C_x , the functions c_x must satisfy the following equation at points in C_x .

$$\begin{aligned} 0 &\stackrel{!}{=} \mathcal{V}_j(f - c_x) \\ &= \partial_s(f - c_x) + (g_j \partial_s(a_j) + h_j \partial_s(b_j) - \partial_s(f)) \frac{\nabla f}{|\nabla f|^2} (f - c_x) \\ &= -\partial_s(c_x) + g_j \partial_s(a_j) + h_j \partial_s(b_j). \end{aligned}$$

This leads to the ordinary differential equation with smooth coefficients on $[0, 1]$,

$$\begin{aligned} \partial_s(c_x)(s) &= g_j(s, c_x(s)) \partial_s(a_j)(s) + h_j(s, c_x(s)) \partial_s(b_j)(s), \\ c_x(0) &= x. \end{aligned}$$

By Picard-Lindelöf, it has a unique a priori local solution. To see that it extends to $s \in [0, 1]$, consider the smooth function $F : N \rightarrow [0, 1] \times (0, 1)$, $F(s, y) = (s, f(s, y)) = (s, f_s(y))$. Since f is proper, so is F . Moreover, $C_x = F^{-1}(\text{graph } c_x)$. For fixed x , we can show that C_x lies in a compact part of $N = [0, 1] \times M$ similarly to the argument given in example 2.3.2, and thus the local solution exists for all $s \in [0, 1]$.

We now define our rescaling data essentially by following the curve c_x . Explicitly, let $\varphi_{0,s} : (0, 1) \rightarrow (0, 1)$ be defined on $[a_j(0), b_j(0)]$ by sending x_0 to $c_{x_0}(s)$. Note that by construction, it sends $a_j(0), b_j(0)$ to $a_j(s), b_j(s)$. Since the solution c_x of the ODE varies smoothly with respect to the initial value x this map is a diffeomorphism. So we can define $\varphi_{s,t} : (0, 1) \rightarrow (0, 1)$ on $[a_j(s), b_j(s)]$ by sending $x_s = c_{x_0}(s)$ to $c_{x_0}(t)$. We extend $\varphi_{s,t}$ to a diffeomorphism in between these intervals in the following way. Let $\tilde{g}_j, \tilde{h}_j : [b_j(0), a_{j+1}(0)] \rightarrow \mathbb{R}$ be a partition of unity such that \tilde{g}_j is strictly decreasing, $\tilde{g}_j(b_j(s)) = 1$, and $\tilde{h}_j(a_{j+1}(s)) = 1$. Then, for $x_0 \in [b_j(0), a_{j+1}(0)]$ set

$$c_{x_0}(s) = \tilde{g}_j(x_0)c_{b_j(0)}(s) + \tilde{h}_j(x_0)c_{a_{j+1}(0)}(s) \quad \text{and} \quad \varphi_{s,t}(c_{x_0}(s)) = c_{x_0}(t).$$

As mentioned above, we obtain the diffeomorphisms $\psi_{s,t}$ by flowing along the vector field \mathcal{V} . Since \mathcal{V} is tangent to the sets $C_x = \bigcup_{s \in [0,1]} \{s\} \times f_s^{-1}(c_x(s))$ for $x \in I_0(0) \cup \dots \cup I_k(0)$, the flow preserves C_x , and $\bigcup_{s \in [0,1]} \{s\} \times f_s^{-1}([b_j(s), a_{j+1}(s)])$ in between. Again, this implies that the flow exists for all $s \in [0, 1]$. It is of the form $\Psi(t-s, (s, y)) = (t, \psi_{s,t}(y))$ for $0 \leq s \leq t \leq 1$, where $(\psi_{s,t})_{s,t \in [0,1]}$ is a family of diffeomorphisms and intertwines with the composed bordisms with respect to the rescaling data $\varphi_{s,t}$.

Step 2: common endpoints

Now consider the case that for $0 \leq j \leq k$ we have that either for every $s \in [0, 1]$, $I_j(s) \cap I_{j+1}(s) = \emptyset$ as in the previous case or for every $s \in [0, 1]$ we have

$$|I_j(s) \cap I_{j+1}(s)| = 1.$$

In this case, one can modify the above argument. We explain for the case of two intervals with one common endpoint, i.e. $b_j(s) = a_{j+1}(s)$.

Instead of choosing smooth functions $g_j, h_j, g_{j+1}, h_{j+1} : [0, 1] \times (0, 1) \rightarrow \mathbb{R}$ such that the supports of $g_j + h_j$ and $g_{j+1} + h_{j+1}$ are disjoint (which now is not possible), we fix three smooth functions $f_j, g_j, h_j : [0, 1] \times (0, 1) \rightarrow \mathbb{R}$ which satisfy the following conditions:

1. f_j, g_j, h_j are compactly supported in $U_j \cup U_{j+1}$,
2. $f_j = 1$ in a neighborhood of graph $a_j = \{(s, a_j(s)) : s \in [0, 1]\}$,
 $g_j = 1$ in a neighborhood of graph $b_j = \text{graph } a_{j+1}$,
 $h_j = 1$ in a neighborhood of graph b_{j+1} ,
3. $f_j + g_j + h_j = 1$ in $\bigcup_{s \in [0,1]} \{s\} \times (I_j(s) \cup I_{j+1}(s))$, and the support of the $f_j + g_j + h_j$ is disjoint to the sums associated to the other intervals.

Now continue the proof similarly to above.

Step 3: overlapping intervals

It remains to consider the case when for some $0 \leq j \leq k$ and some $s \in [0, 1]$,

$$I_j(s) \cap I_{j+1}(s)$$

has non-empty interior.

this process is that by Whitney's embedding theorem, every manifold can be embedded in some large enough vector space, so in the limit, we include representatives of every n -dimensional manifold. We use $V \times (0, 1)^n$ instead of $V \times \mathbb{R}^n$ as in this case the spatial structure is easier to write down explicitly.

2.3.1 The level sets $(\text{PBord}_n)_{k_1, \dots, k_n}$

For $S \subseteq \{1, \dots, n\}$ denote the projection from $(0, 1)^n$ onto the coordinates indexed by S by $\pi_S : (0, 1)^n \rightarrow (0, 1)^S$.

Definition 2.3.1. Let V be a finite dimensional vector space. For every n -tuple $k_1, \dots, k_n \geq 0$, let $(\text{PBord}_n^V)_{k_1, \dots, k_n}$ be the collection of tuples $(M, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n})$, satisfying the following conditions:

1. M is a closed n -dimensional submanifold of $V \times (0, 1)^n$ and the composition $\pi : M \hookrightarrow V \times (0, 1)^n \rightarrow (0, 1)^n$ is a proper map.
2. For $1 \leq i \leq n$,

$$(I_0^i \leq \dots \leq I_{k_i}^i) \in \text{Int}_{k_i}.$$
3. For every $S \subseteq \{1, \dots, n\}$, let $p_S : M \xrightarrow{\pi} (0, 1)^n \xrightarrow{\pi_S} (0, 1)^S$ be the composition of π with the projection π_S onto the S -coordinates. Then for every $1 \leq i \leq n$ and $0 \leq j_i \leq k_i$, at every $x \in p_{\{i\}}^{-1}(I_{j_i}^i)$, the map $p_{\{i, \dots, n\}}$ is submersive.

Remark 2.3.2. For $k_1, \dots, k_n \geq 0$, one should think of an element in $(\text{PBord}_n)_{k_1, \dots, k_n}$ as a collection of $k_1 \cdots k_n$ composed bordisms, with k_i composed bordisms with collars in the i th direction. They can be understood as follows.

- Condition 3 in particular implies that at every $x \in p_{\{n\}}^{-1}(I_j^n)$, the map $p_{\{n\}}$ is submersive, so if we choose $t_j^n \in I_j^n$, it is a regular value of $p_{\{n\}}$, and so $p_n^{-1}(t_j^n)$ is an $(n-1)$ -dimensional manifold. The embedded manifold M should be thought of as a composition of n -bordisms and $p_n^{-1}(t_j^n)$ is one of the $(n-1)$ -bordisms in the composition.
- At $x \in p_{\{n-1\}}^{-1}(I_l^{n-1})$, the map $p_{\{n-1, n\}}$ is submersive, so for $t_l^{n-1} \in I_l^{n-1}$, the preimage

$$p_{\{n-1, n\}}^{-1}\left(\left(t_l^{n-1}, t_j^n\right)\right)$$

is an $(n-2)$ -dimensional manifold, which should be thought of as one of the $(n-2)$ -bordisms which are connected by the composition of n -bordisms M . Moreover, again since $p_{\{n-1, n\}}$ is submersive at $p_{\{n-1\}}^{-1}(I_l^{n-1})$, the preimage $p_{\{n-1\}}^{-1}(t_l^{n-1})$ is a trivial $(n-1)$ -bordism between the $(n-2)$ -bordisms it connects.

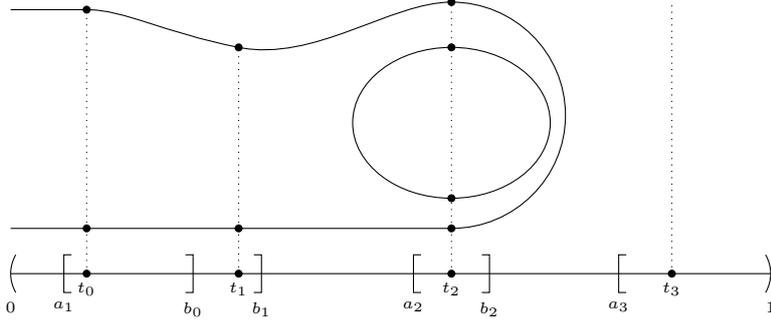
- Similarly, for $(t_{j_k}^k, \dots, t_{j_n}^n) \in I_{j_k}^k \times \dots \times I_{j_n}^n$, the preimage

$$p_{\{k, \dots, n\}}^{-1}\left(\left(t_{j_k}^k, \dots, t_{j_n}^n\right)\right)$$

is a $(k-1)$ -dimensional manifold, which should be thought of as one of the $(k-1)$ -bordisms which is connected by the composition of n -bordisms M .

- Moreover, the following proposition shows that different choices of “cutting points” $t_j^i \in I_j^i$ lead to diffeomorphic bordisms. One should thus think of the n -bordisms we compose as $\pi^{-1}(\prod_{i=1}^n [b_j^i, a_{j+1}^i])$, and the preimages of the specified intervals as collars of the bordisms along which they are composed.

We will come back to this interpretation in section 2.5 when we compute homotopy (bi)categories.



Proposition 2.3.3. For $1 \leq i \leq n$ let $u_j^i, v_j^i \in I_j^i$ and $u_{j+1}^i, v_{j+1}^i \in I_{j+1}^i$. Then there is a diffeomorphism

$$p_{\{i\}}^{-1}([u_j^i, u_{j+1}^i]) \longrightarrow p_{\{i\}}^{-1}([v_j^i, v_{j+1}^i]).$$

Proof. Since the map $p_{\{i\}}$ is submersive in I_j^i and I_{j+1}^i , we can apply the Morse lemma 2.2.1 to $p_{\{i\}}$ twice to obtain diffeomorphisms

$$p_{\{i\}}^{-1}([u_j^i, u_{j+1}^i]) \longrightarrow p_{\{i\}}^{-1}([v_j^i, u_{j+1}^i]) \longrightarrow p_{\{i\}}^{-1}([v_j^i, v_{j+1}^i]).$$

□

Applying the proposition successively for $i = 1, \dots, n$ yields

Corollary 2.3.4. Let $B_1, B_2 \subseteq (0, 1)^n$ be products of closed intervals with endpoints lying in the same I_j^i 's. Then there is a diffeomorphism

$$\pi^{-1}(B_1) \longrightarrow \pi^{-1}(B_2).$$

2.3.2 The spaces $(\text{PBord}_n)_{k_1, \dots, k_n}$

The level sets $(\text{PBord}_n^V)_{k_1, \dots, k_n}$ form the underlying set of 0-simplices of a space which we construct in this subsection.

The l -simplices of the space $(\text{PBord}_n^V)_{k_1, \dots, k_n}$

Let $|\Delta^l|$ denote the standard geometric l -simplex.

Definition 2.3.5. An l -simplex of $(\text{PBord}_n^V)_{k_1, \dots, k_n}$ consists of the following data:

1. A smooth family of *underlying 0-simplices*, which is a smooth family of elements

$$(M_s \subseteq V \times (0, 1)^n, (I_0^i(s) \leq \dots \leq I_{k_i}^i(s))_{i=1, \dots, n}) \in (\text{PBord}_n^V)_{k_1, \dots, k_n}$$

indexed by $s \in |\Delta^l|$. By this we mean that $\bigcup_{s \in |\Delta^l|} \{s\} \times M_s \subseteq |\Delta^l| \times V \times (0, 1)^n$ is a smooth submanifold with corners, and that the endpoint maps a_j^i, b_j^i of the intervals are smooth;

2. For every $1 \leq i \leq n$, a *rescaling datum* $(\varphi_{s,t}^i : (0, 1) \rightarrow (0, 1))_{s,t \in |\Delta^l|}$ which together with

$$(I_0^i(s) \leq \dots \leq I_{k_i}^i(s))_{s \in |\Delta^l|}$$

is an l -simplex in Int_{k_i} ;

3. A smooth family of *diffeomorphisms*

$$(\psi_{s,t} : M_s \longrightarrow M_t)_{s,t \in |\Delta^l|},$$

such that $\psi_{s,s} = \text{id}_{M_s}$ and $\psi_{s,t} = \psi_{t,s}^{-1}$, which *intertwine with the composed bordisms* with respect to the product of the rescaling data $\varphi_{s,t} = (\varphi_{s,t}^i)_{i=1}^n : (0, 1)^n \rightarrow (0, 1)^n$. By this we mean the following. Denoting by π_s the composition $M_s \hookrightarrow V \times (0, 1)^n \rightarrow (0, 1)^n$, for $1 \leq i \leq n$ and $0 \leq j_i, l_i \leq k_i$ let

$$\begin{aligned} t_{j_i}^i(s) &\in I_{j_i}^i(s) \text{ such that } \varphi_{s,t}(t_{j_i}^i(s)) \in I_{j_i}^i(t), \text{ and} \\ t_{l_i}^i(s) &\in I_{l_i}^i(s) \text{ such that } \varphi_{s,t}(t_{l_i}^i(s)) \in I_{l_i}^i(t). \end{aligned}$$

Then $\psi_{s,t}$ restricts to a diffeomorphism

$$\pi_s^{-1} \left(\prod_{i=1}^n [t_{j_i}^i(s), t_{l_i}^i(s)] \right) \xrightarrow{\psi_{s,t}} \pi_t^{-1} \left(\prod_{i=1}^n [\varphi_{s,t}(t_{j_i}^i(s)), \varphi_{s,t}(t_{l_i}^i(s))] \right),$$

i.e. denoting $B = \prod_{i=1}^n [t_{j_i}^i(s), t_{l_i}^i(s)]$,

$$\begin{array}{ccccc} & & M_t & \longleftrightarrow & \pi_t^{-1}(\varphi_{s,t}(B)) \\ & \nearrow \psi_{s,t} & & & \downarrow \pi_t \\ M_s & \longleftrightarrow & \pi_s^{-1}(B) & \xrightarrow{\psi_{s,t}} & \varphi_{s,t}(B) \\ \downarrow \pi_s & & \downarrow \pi_s & \nearrow \varphi_{s,t} & \\ (0, 1)^n & \longleftrightarrow & B & & \end{array}$$

Remark 2.3.6. The condition that the diffeomorphisms $\psi_{s,t}$ intertwine with the composed bordisms in the elements of the family means that $\psi_{s,t}$ induces diffeomorphisms of the composed bordisms in the family and the rescaling data remembers to which choice of cutoffs the specified diffeomorphism restricts.

Remark 2.3.7. In the above definition we let the intervals vary as $s \in |\Delta^l|$ varies. In practice, when dealing with a fixed element of an l -simplex, we can assume that these intervals are fixed as s varies by choosing a fixed vertex $t_0 \in |\Delta^l|_0$ and composing each ι_s with $\varphi_{s,t_0} : (0, 1)^n \rightarrow (0, 1)^n$ and keeping the intervals constant at $I_j^i(t_0)$. This new path is connected by a homotopy to the original one.

The space $(\text{PBord}_n)_{k_1, \dots, k_n}$

We now lift the spatial structure of $\text{Int}_{k_1, \dots, k_n}^n$ to $(\text{PBord}_n)_{k_1, \dots, k_n}$.

Fix $k \geq 0$ and let $f : [m] \rightarrow [l]$ be a morphism in the simplex category Δ , i.e. a (weakly) order-preserving map. Then let $|f| : |\Delta^m| \rightarrow |\Delta^l|$ be the induced map between standard simplices.

Let f^Δ be the map sending an l -simplex in $(\text{PBord}_n^V)_{k_1, \dots, k_n}$ to the m -simplex which consists of

1. for $s \in |\Delta^m|$,

$$M_{|f|(s)} \subseteq V \times (0, 1)^n;$$

2. for $1 \leq i \leq n$, the m -simplex in Int_{k_i} obtained by applying f^Δ ,

$$f^\Delta \left(I_0^i(s) \leq \dots \leq I_{k_i}^i(s), \varphi_{s,t}^i \right);$$

3. for $s, t \in |\Delta^m|$,

$$\psi_{|f|(s), |f|(t)} : M_{|f|(s)} \longrightarrow M_{|f|(t)}.$$

Proposition 2.3.8. $(\text{PBord}_n^V)_{k_1, \dots, k_n}$ is a space. Moreover, it is a Kan complex, i.e. fibrant in the category of simplicial sets with Quillen model structure.

Proof. The above assignment indeed is well-defined since the underlying assignment for the underlying intervals is well-defined and the conditions on $\psi_{|f|(s), |f|(t)}$ are a special case of those on $\psi_{s,t}$. Moreover, since this structure essentially comes from the spatial structure of Int_{k_i} and the simplicial structure of $N(\Delta)$, the assignment is functorial.

It remains to show that this space is a Kan complex. A morphism $\Lambda_k^l \rightarrow \text{PBord}_n^V$ is the data of, for $s, t \in |\Lambda_k^l|$,

$$\begin{aligned} (M_s \subseteq V \times (0, 1)^n, (I_0^i(s) \leq \dots \leq I_{k_i}^i(s))_{i=1, \dots, n}) \in (\text{PBord}_n^V)_{k_1, \dots, k_n}, \\ (\varphi_{s,t}^i : (0, 1) \rightarrow (0, 1)), \text{ and } (\psi_{s,t} : M_s \longrightarrow M_t). \end{aligned}$$

We have seen in 2.1.9 in particular that every Int_k is a Kan complex. The proof is completed by the fact that the inclusions of spaces $\text{Sub}^{\text{sm}}(M, V \times (0, 1)^n) \hookrightarrow \text{Sub}(M, V \times (0, 1)^n)$ and $\text{Diff}^{\text{sm},+}(M) \hookrightarrow \text{Diff}^+(M)$ are weak equivalences (see e.g.), and both $\text{Sub}(M, V \times (0, 1)^n)$ and $\text{Diff}^+(M)$ are Kan complexes by an argument similar to that of showing that the singular complex of a manifold is a Kan complex. \square

Notation 2.3.9. We denote the *spatial face and degeneracy maps* of $(\text{PBord}_n^V)_{k_1, \dots, k_n}$ by d_j^Δ and s_j^Δ for $0 \leq j \leq l$.

So far the definition depends on the choice of the vector space V . However, in the bordism category we need to consider all (not necessarily compact) n -dimensional manifolds. By Whitney's embedding theorem any such manifold can be embedded into some $V \times (0, 1)^n$ for some finite dimensional vector space V , so we need to allow big enough vector spaces.

Definition 2.3.10. Fix some (countably) infinite dimensional vector space, e.g. \mathbb{R}^∞ . Then

$$\text{PBord}_n = \varinjlim_{V \subset \mathbb{R}^\infty} \text{PBord}_n^V.$$

Example: Cutoff path

We now construct an example of a path which will be used several times later on. It shows that cutting off part of the collar of a bordism yields an element which is connected to the original one by a path.

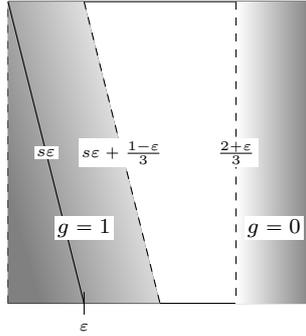
Let $(M) = (M \subseteq V \times (0, 1)^n, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n}) \in (\text{PBord}_n)_{k_1, \dots, k_n}$. We show that cutting off a short enough piece at an end of an element of $(\text{PBord}_n)_{k_1, \dots, k_n}$ leads to an element which is connected by a path to the original one. Explicitly, for ε small enough, we show that there is a 1-simplex with underlying 0-simplices

$$(\iota_s : M_s \hookrightarrow V \times (0, 1)^n, (I_0^i(s) \leq \dots \leq I_{k_i}^i(s))_{i=1}^n) \in (\text{PBord}_n^V)_{k_1, \dots, k_n}$$

such that $M_s = p_i^{-1}((s\varepsilon, 1))$ and $I_j^i(s) = \rho_s(I_j^i)$, where $\rho_s : (s\varepsilon, 1) \rightarrow (0, 1)$ is the affine rescaling map $x \mapsto \frac{x-s\varepsilon}{1-s\varepsilon}$, and

$$\iota_s : M_s \subseteq V \times (0, 1)^{n-1} \times (s\varepsilon, 1) \xrightarrow{id \times \rho_s} V \times (0, 1)^n.$$

Fix $1 \leq i \leq n$ and let $\varepsilon < b_0^i$. Let N be the manifold $[0, 1] \times M \subseteq [0, 1] \times V \times (0, 1)^n$ endowed with the induced metric, and view p_i as a function on N by setting $p_i(s, y) = p_i(y)$. Choose a smooth cutoff function $g : [0, 1] \times (0, 1) \rightarrow \mathbb{R}$ such that $g = 1$ in a neighborhood U_ε of $\{(s, z) : z \in [s\varepsilon, s\varepsilon + \frac{1-\varepsilon}{3}]\}$ and $g = 0$ on $U_1 = [0, 1] \times (\frac{2+\varepsilon}{3}, 1)$ and extend g to N by setting $g(s, y) = g(s, p_i(y))$.



Consider the vector field on N given by

$$V = (\partial_s, \varepsilon g \frac{\nabla_y p_i}{|\nabla_y p_i|^2}),$$

where ∇_y denotes the gradient on M . Note that over U_ε , $V = (\partial_s, \varepsilon \frac{\nabla_y p_i}{|\nabla_y p_i|^2})$ and over U_1 , $V = 0$. We now show that the flow along the vector field V exists for (s, y) such that $s\varepsilon < p_i(x) < 1$,

For $\xi < \frac{1-\varepsilon}{3}$, $(s, s\varepsilon + \xi) \in U_\varepsilon$, and, defining $p_i - s\varepsilon + \xi$ to be the function $(s, y) \mapsto p_i(y) - s\varepsilon + \xi$ on N ,

$$V \cdot (p_i - (s\varepsilon + \xi)) = -\varepsilon + \varepsilon \frac{\nabla_y p_i}{|\nabla_y p_i|^2} (p_i - (s\varepsilon + \xi)) = 0. \quad (2.1)$$

For $\alpha \neq i$ and $\xi_\alpha \in (0, 1)$, since all components of V except for the i th are 0,

$$V \cdot (p_\alpha - \xi_\alpha) = 0, \quad (2.2)$$

where again we view $p_\alpha - \xi_\alpha$ as a function on N . Let

$$\vec{\xi} : [0, 1] \rightarrow [0, 1] \times (0, 1)^n, \quad s \mapsto (s, \xi_1, \dots, \xi_{i-1}, s\varepsilon + \xi, \xi_{i+1}, \dots, \xi_n).$$

Equations 2.1 and 2.2 imply that the flow of V preserves the sets

$$\Xi_{\vec{\xi}} = \{(s, y) : \pi(y) = \vec{\xi}(s)\} = (id_{[0,1]} \times \pi)^{-1}(\text{graph } \vec{\xi}).$$

The graph of $\vec{\xi}$ is closed and therefore compact as it is a closed subset of $[0, 1] \times \{\xi_1\} \cdots \times [\xi, \varepsilon + \xi] \times \cdots \times \{\xi_n\}$. Since π is proper, $id \times \pi$ is proper, and thus $\Xi_{\vec{\xi}}$ is compact. Hence in

$$\{(s, y) : s\varepsilon < p_i(y) < \varepsilon + \frac{1-\varepsilon}{3}\}$$

the flow exists for all $s \in [0, 1]$.

In U_1 , the flow is of the form $\Psi(t - s, (s, y)) = (t, y)$ and so it also exists for $s \in [0, 1]$.

For points $(s_0, y) \in N$ such that $p_i(y) \in [s_0\varepsilon + \frac{1-\varepsilon}{3}, \frac{2+\varepsilon}{3}]$, the flow preserves the set

$$\Xi_{\vec{\xi}} = (id_{[0,1]} \times \pi)^{-1}(\text{graph } \vec{\xi}),$$

where $\vec{\xi} : s \mapsto (s, \xi_1, \dots, \xi_{i-1}, \xi_i(s), \xi_{i+1}, \dots, \xi_n)$, and $\vec{\xi}(s_0) = y$, and $\xi_i(s)$ is a solution at points in $\Xi_{\vec{\xi}}$ of the ordinary differential equation with smooth coefficients

$$\begin{aligned} 0 &\stackrel{!}{=} V \cdot (p_i - \xi_i) \\ &= -\partial_s \xi_i + \varepsilon g \partial_s \xi_i \frac{\nabla_y p_i}{|\nabla_y p_i|^2} (p_i - \xi_i) \\ &= -\partial_s \xi_i + \varepsilon g. \end{aligned}$$

By Picard-Lindelöf, this ordinary differential equation has a unique, a priori local, solution. Similarly, the flow exists locally. Furthermore, the preimage of the proper map $(id_{[0,1]} \times \pi)$ of the compact set $[0, 1] \times [\frac{1-\varepsilon}{3}, \frac{2+\varepsilon}{3}]$ is compact. Since $\Xi_{\vec{\xi}}$ is a subset of this preimage, we are looking for solutions of the above differential equation on this compact manifold. By compactness, they exist globally and therefore the flow exists for all $s \in [0, 1]$.

Piecing this together, the flow takes on the form

$$\Psi(t - s, y) = (t, \psi_{s,t}(y))$$

for $s\varepsilon < p_i(y)$ and exists for all $s \in [0, 1]$. This gives the desired family of diffeomorphisms $\psi_{s,t} : p_i^{-1}((s\varepsilon, 1)) \rightarrow p_i^{-1}((t\varepsilon, 1))$. The rescaling data $\varphi_{s,t} : (0, 1)^n \rightarrow (0, 1)^n$ is the identity on coordinates except for the i th, where it is given by

$$\varphi_{s,t}^i(x_s) = \begin{cases} \rho_s(x_s + (t-s)\varepsilon), & \text{for } x_s < s\varepsilon + \frac{1-\varepsilon}{3}, \\ \rho_s(x_s), & \text{for } x_s > \frac{2+\varepsilon}{3}, \\ \rho_s(\xi_i(t)), & \text{for } s\varepsilon + \frac{1-\varepsilon}{3} \leq x_s \leq \frac{2+\varepsilon}{3}, \end{cases}$$

where ξ_i is the integral curve through x_s , which is the solution to the differential equation above.

Remark 2.3.11. In the above example we constructed a path from an element in $(\text{PBord}_n^V)_{k_1, \dots, k_n}$ to its “cutoff”, where we cut off the preimage of $p_i^{-1}((0, \varepsilon])$ for suitably small ε . Note that the same argument holds for cutting off the preimage of $p_i^{-1}([1 - \delta, 1))$ for suitably small δ . Moreover, we can iterate the process and cut off ε_i, δ_i strips in all i directions. Choosing $\varepsilon_i = \frac{b_0^i}{2}, \delta_i = \frac{a_{k_i}^i}{2}$ yields a path to its “cutoff” with underlying submanifold

$$\text{cut}(M) = \pi^{-1}\left(\prod_{i=1}^n \left(\frac{b_0^i}{2}, \frac{a_{k_i}^i}{2}\right)\right).$$

The map $\pi : M \rightarrow (0, 1)^n$ is proper, which implies that $\pi^{-1}(\prod_{i=1}^n [\frac{b_0^i}{2}, \frac{a_{k_i}^i}{2}]) \supset \text{cut}(M)$ is compact and thus bounded in the V -direction. Thus, any element in $(\text{PBord}_n^V)_{k_1, \dots, k_n}$ is connected by a path to an element whose underlying submanifold is bounded in the V -direction.

Variants of the spatial structure

Following [Lur09c], one could define the spatial structure of $(\text{PBord}_n^V)_{k_1, \dots, k_n}$ as follows to obtain classifying spaces of bordisms:

1. One could make $(\text{PBord}_n^V)_{k_1, \dots, k_n}$ into a topological space (instead of a simplicial set) by endowing it with the following topology coming from the Whitney topology.

On the set $\text{Sub}(V \times (0, 1)^n)$ of closed (not necessarily compact) submanifolds $M \subseteq V \times (0, 1)^n$, a neighborhood basis at M is given by

$$\{N \hookrightarrow V \times (0, 1)^n : N \cap K = j(M) \cap K, j \in W\},$$

where $K \subseteq V \times (0, 1)^n$ is compact and $W \subseteq \text{Emb}(M, V \times (0, 1)^n)$ is a neighborhood of the inclusion $M \hookrightarrow V \times (0, 1)^n$ in the Whitney C^∞ -topology (see [Gal11]). Using the standard topology on \mathbb{R} and the product topology gives a topology on

$$\text{Sub}(V \times \mathbb{R}^n) \times \bigcup_{i=1}^n \bigcup_{j=1}^{k_i} \{a_j^i, b_{j-1}^i \in [0, 1] : a_j^i < b_j^i\}.$$

We take the quotient topology of this topology with respect to the relation identifying elements $(M_0, I_j^i(0)$'s), $(M_1, I_j^i(1)$'s) if the preimages of the boxes $B(l) = [b_0^1(l), a_{k_1}^1(l)] \times \dots \times [b_0^n(l), a_{k_n}^n(l)]$ for $l = 0, 1$, respectively, under their composition with the projection to $(0, 1)^n$ coincide, i.e.

$$\pi_1^{-1}(B(0)) = \pi_2^{-1}(B(1)),$$

where for $l = 0, 1$, $\pi_l : M \hookrightarrow V \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Finally, $(\text{PBord}_n^V)_{k_1, \dots, k_n}$ is a subspace thereof.

However, the reason for our choice of using simplicial sets instead of spaces is that we eventually want to construct a fully extended topological field theory and the levels of our target which we construct in the next chapter will be naturally modelled as simplicial sets. Thus it is more natural to also model the levels of our source category,

the bordism category, as simplicial sets. If one would rather have topological spaces as the spatial structure of the levels, one can apply geometric realization to the simplicial sets.

2. To model the levels of the bordism category as simplicial sets, we could start with the above version as a topological space and take singular or, even better, differentiable chains of this space to obtain a simplicial set. Then, the l -vertices would consist of smooth submanifolds

$$I : \Delta^l \times M \hookrightarrow \Delta^l \times V \times (0, 1)^n,$$

where I commutes with the projections to Δ^l , such that $\forall s \in |\Delta^l|$,

$$(M_s = \text{Im}(I(s, -)) \subseteq V \times (0, 1)^n, (I_0^i(s) \leq \dots \leq I_{k_i}^i(s))_{i=1, \dots, n}) \in (\text{PBord}_n^V)_{k_1, \dots, k_n}.$$

Note that as abstract manifolds, $M_s = M$, but as submanifolds, they are diffeomorphic images of the same abstract manifold along the path. Thus, there are diffeomorphisms

$$\psi_{s,t} : M_s \longrightarrow M_t$$

as in our definition. Moreover, for $l = 1$, proposition 2.3.12 below, which is a corollary of proposition 2.2.2, the time-dependent Morse lemma, implies that there exists such a family of diffeomorphisms and some rescaling data which intertwine. So paths in this simplicial set yield paths in ours.

It also implies that for $l > 0$, given any two fixed points $s, t \in |\Delta^l|$, we obtain a diffeomorphism $\psi_{s,t}$ and a rescaling function $\varphi_{s,t}$, by applying the lemma to any path between s and t and defining $\psi_{s,t} = \psi_{0,1}$ and $\varphi_{s,t} = \varphi_{0,1}$. Choosing the same (shape of) partitions of unity in this process we get a smooth family of such diffeomorphisms, so l -simplices in this simplicial set yield l -simplices in ours.

Moreover, the diffeomorphisms appearing are all isotopic to the identity and therefore arise as the flow of a (time-dependent) vector field [Thu74], which however may not be of the form of the ones we considered. We nevertheless believe that the two simplicial sets are weakly equivalent under the map simply forgetting the family of diffeomorphisms.

Proposition 2.3.12. *Consider a smooth one-parameter family of embeddings*

$$(I : [0, 1] \times M \hookrightarrow [0, 1] \times V \times (0, 1)^n, [0, 1] \ni s \mapsto (I_0^i(s) \leq \dots \leq I_{k_i}^i(s))_{i=1, \dots, n}),$$

which gives rise to

$$(M_s) = (M \xrightarrow{I(s, -)} V \times (0, 1)^n, (I_0^i(s) \leq \dots \leq I_{k_i}^i(s))_{i=1, \dots, n})$$

in $(\text{PBord}_n)_{k_1, \dots, k_n}$. Then there is a rescaling data $(\varphi_{s,t} : (0, 1)^n \rightarrow (0, 1)^n)_{s,t \in [0,1]}$ and a family of diffeomorphisms $(\psi_{s,t} : M \rightarrow M)_{s,t \in [0,1]}$ which intertwines with the rescaling data.

Proof. For $1 \leq i \leq n$, let $0 \leq j_i \leq k_i - 1$. Let

$$\pi_s : M \xrightarrow{I(s, -)} V \times (0, 1)^n \rightarrow (0, 1)^n$$

and denote by $(p_i)_s : M \rightarrow (0, 1)$ the composition of π_s with the projection to the i th coordinate. Note that by condition 3 in definition 2.3.1, the function $(p_i)_s$ does not have a critical point in $I_0^i(s) \cup \dots \cup I_{k_i}^i(s)$.

We cannot quite apply the the time-dependent Morse lemma 2.2.2 to $(p_i)_s$, because we only have properness of the individual π_s , and moreover, this would ensure intertwining only in the i th direction. However, we can adapt the proof to our situation.

Choosing the metric on M coming from $I(0, -)$, and following the proof of the proposition 2.2.2, for each i we get a vector field

$$\begin{aligned} \mathcal{V}^i &= \left(\partial_s, \sum_{0 \leq j \leq k} (g_j \partial_s(a_j) + h_j \partial_s(b_j) - \partial_s(p_i)) \frac{\nabla_y(p_i)_s}{|\nabla_y(p_i)_s|^2} \right) \\ &=: \left(\partial_s, \Pi_i(s, y) \frac{\nabla_y(p_i)_s}{|\nabla_y(p_i)_s|^2} \right). \end{aligned}$$

We combine them to obtain a new vector field on $[0, 1] \times M$,

$$\tilde{\mathcal{V}} = \left(\partial_s, \sum_{i=1}^n \Pi_i(s, y) \frac{\nabla_y(p_i)_s}{|\nabla_y(p_i)_s|^2} \right).$$

The projections $(p_i)_0$ and $(p_j)_0$ are orthogonal with respect to the metric on M induced by the embedding $I(0, -)$, and moreover, $(p_i)_s, (p_j)_s$ stay orthogonal along the path, because the change of metric on M induced by the change of embedding respects orthogonality on $(0, 1)^n$. This implies that

$$\frac{\nabla_y(p_i)_s}{|\nabla_y(p_i)_s|^2} p_j = \delta_{ij},$$

and so $\tilde{\mathcal{V}}$ still is tangent to the respective C_x^i in each direction and thus its flow, if it exists globally, will give rise to the desired diffeomorphisms and rescaling data.

The global existence follows from the special form of the vector field. Given a point $(t, y_t) \in N$, the flow will preserve a set of the form

$$\{(s, y) : \pi_s(y_s) = (c_{x_0}^1(s), \dots, c_{x_0}^n(s)) = (\xi_1(s), \dots, \xi_n(s))\},$$

where the right hand side is in the notation of example 2.3.2, and $\vec{c}_{x_0}(t) = \vec{\xi}(t) = y_t$. Similarly to in the example, one can show that this set lies in a compact part of N and thus the flow exists globally. \square

2.3.3 The n -fold simplicial set $(\text{PBord}_n)_{\bullet, \dots, \bullet}$

In the next two subsections we will make the collection of spaces $(\text{PBord}_n)_{\bullet, \dots, \bullet}$ into an n -fold simplicial space by lifting the simplicial structure of $\text{Int}_{\bullet, \dots, \bullet}^{\times n}$. In this section we define the structure on 0-simplices, which makes $(\text{PBord}_n)_{\bullet, \dots, \bullet}$ into an n -fold simplicial set. In the next subsection we extend the structure to l -vertices of the levels to obtain an n -fold simplicial space $(\text{PBord}_n)_{\bullet, \dots, \bullet}$.

Fixing $1 \leq i \leq n$, we first need to extend the assignment

$$[k_i] \longmapsto (\text{PBord}_n)_{k_1, \dots, k_n}$$

to a functor from Δ^{op} . Let $f : [m_i] \rightarrow [k_i]$ be a morphism in the simplex category Δ , i.e. a (weakly) order-preserving map. Then we need to define the map

$$(\text{PBord}_n)_{k_1, \dots, k_i, \dots, k_n} \xrightarrow{f^*} (\text{PBord}_n)_{k_1, \dots, m_i, \dots, k_n}.$$

Notation 2.3.13. Recall that the map $f^* : \text{Int}_{k_i} \rightarrow \text{Int}_{m_i}$ is defined using an affine rescaling map $\rho_f : \mathbb{R} \rightarrow \mathbb{R}$ which sends $a_{f(0)}^i$ to 0 and $b_{f(m)}^i$ to 1 and thus restricts to a diffeomorphism $\rho_f : D_f = (a_{f(0)}^i, b_{f(m)}^i) \rightarrow (0, 1)$. By abuse of notation, we again denote by ρ_f the map

$$\rho_f : V \times \prod_{\alpha \neq i} (0, 1) \times (a_{f(0)}^i, b_{f(m)}^i) \rightarrow V \times (0, 1)^n,$$

which is ρ_f in the i th component of $(0, 1)^n$ and the identity otherwise.

Definition 2.3.14. Let $f : [m_i] \rightarrow [k_i]$ be a morphism in Δ . Then

$$(\text{PBord}_n)_{k_1, \dots, k_i, \dots, k_n} \xrightarrow{f^*} (\text{PBord}_n)_{k_1, \dots, m_i, \dots, k_n}.$$

applies f^* to the i th tuple of intervals and perhaps cuts the manifold and rescales. Explicitly, it sends an element

$$(M) := (\iota : M \hookrightarrow V \times (0, 1)^n, (I_0^\alpha \leq \dots \leq I_{k_\alpha}^\alpha)_{\alpha=1}^n)$$

to

$$\left(\rho_f \circ \iota|_{p_i^{-1}(D_f)} : p_i^{-1}(D_f) \hookrightarrow V \times (0, 1)^n, (I_0^\alpha \leq \dots \leq I_{k_i}^\alpha)_{\alpha \neq i}, f^*(I_0^i \leq \dots \leq I_{k_i}^i) \right).$$

Remark 2.3.15. In the following, we will omit explicitly writing out the restriction of ι to $p_i^{-1}(D_f)$ for readability.

Notation 2.3.16. We denote the (*simplicial*) *face and degeneracy maps* by $d_j^i : (\text{PBord}_n)_{k_1, \dots, k_n} \rightarrow (\text{PBord}_n)_{k_1, \dots, k_i-1, \dots, k_n}$ and $s_j^i : (\text{PBord}_n)_{k_1, \dots, k_n} \rightarrow (\text{PBord}_n)_{k_1, \dots, k_i+1, \dots, k_n}$ for $0 \leq j \leq k_i$.

Proposition 2.3.17. $(\text{PBord}_n)_{\bullet, \dots, \bullet}$ is an n -fold simplicial set.

Proof. This follows from the fact that Int_\bullet is a simplicial set and rescalings behave functorially. \square

Remark 2.3.18. Recall from remark 2.3.2 that for $k_1, \dots, k_n \geq 0$, one should think of an element in the set $(\text{PBord}_n)_{k_1, \dots, k_n}$ as a collection of $k_1 \cdots k_n$ composed bordisms with k_i composed bordisms with collars in the i th direction. These composed collared bordisms are the images under the maps

$$D(j_1, \dots, j_k) : (\text{PBord}_n)_{k_1, \dots, k_n} \longrightarrow (\text{PBord}_n)_{1, \dots, 1}$$

for $(1 \leq j_i \leq k_i)_{1 \leq i \leq n}$ arising as compositions of face maps, i.e. $D(j_1, \dots, j_k)$ is the map determined by the maps

$$[1] \rightarrow [k_i], \quad (0 < 1) \mapsto (j_i - 1 < j_i)$$

in the category Δ of finite ordered sets. This should be thought of as sending an element to the (j_1, \dots, j_k) -th collared bordism in the composition.

2.3.4 The full structure of $(\text{PBord}_n)_{\bullet, \dots, \bullet}$ as an n -fold simplicial space

In this subsection, we show that the maps defined in the previous paragraph are compatible with the structure of the levels as simplicial sets, i.e. for a morphism $f : [m_i] \rightarrow [k_i]$ in the simplex category Δ , we will define compatible maps f^* for l -simplices of the simplicial set $(\text{PBord}_n)_{k_1, \dots, k_n}$. They will be defined similarly as on vertices, namely by applying the map f^* to each underlying 0-simplex and by perhaps restricting the rescaling data and the diffeomorphisms. For the face and degeneracy maps, this will amount to the following.

- Degeneracy maps arise from the degeneracy maps of $\text{Int}_{\bullet, \dots, \bullet}^n$ by repeating one of the families of intervals $I_j^i(s)$.

Fix $1 \leq i \leq n$.

- For $0 < j < k_i$ the j th face map d_j^i arises from the face map of $\text{Int}_{\bullet, \dots, \bullet}^n$ by deleting the j th family of intervals $I_j^i(s)$ in the i th direction.
- Face maps for $j = 0, k_i$ require cutting and rescaling:

Notation 2.3.19. Recall that for a morphism f of the simplex category Δ , we have a rescaling map $\rho_f : \mathbb{R} \rightarrow \mathbb{R}$ which restricts to a diffeomorphism $\rho_f : D_f \rightarrow (0, 1)$, with $D_f = (a_{f(0)}, b_{f(m)})$. By abuse of notation, we also denote by ρ_f the diffeomorphism $\rho_f : \prod_{\alpha \neq i} (0, 1) \times D_f \rightarrow (0, 1)^n$ which is ρ_f in the i th coordinate and the identity otherwise. Moreover, denote by ρ_f^s be the analog of the map ρ_f associated to the s th underlying 0-simplex $(I_0^i(s) \leq \dots \leq I_{k_i}^i(s)) \in \text{Int}_{k_i}$.

Definition 2.3.20. Let $f : [m_i] \rightarrow [k_i]$ be a morphism in the simplex category Δ . Consider an l -simplex of $(\text{PBord}_n)_{k_1, \dots, k_n}$ consisting of

$$\begin{aligned} & (\iota_s : M_s \hookrightarrow V \times (0, 1)^n, (I_0^i(s) \leq \dots \leq I_{k_i}^i(s))_{s \in |\Delta^l|}, \\ & (\varphi_{s,t} : (0, 1)^n \longrightarrow (0, 1)^n)_{s,t \in |\Delta^l|}, \quad \text{and} \quad (\psi_{s,t} : M_s \longrightarrow M_t)_{s,t \in |\Delta^l|}. \end{aligned}$$

Let f^* send it to the l -simplex of $(\text{Bord}_n)_{k_1, \dots, k_{i-1}, \dots, k_n}$ consisting of the following data.

1. The underlying 0-simplices of the image are the images of the underlying 0-simplices under f^* , i.e. for $s \in |\Delta^l|$,

$$\begin{aligned} f^*(M_s \subseteq V \times (0, 1)^n, (I_0^i(s) \leq \dots \leq I_{k_i}^i(s))) &= \\ &= \left(\rho_f(s) \circ \iota_s|_{N_s} : N_s \hookrightarrow V \times (0, 1)^n, \right. \\ & \quad \left. (I_0^\alpha(t) \leq \dots \leq I_{k_\alpha}^\alpha(t))_{\alpha \neq i}, f^*(I_0^i(t) \leq \dots \leq I_{k_i}^i(t)) \right), \end{aligned}$$

where $N_s = (p_s)_i^{-1}(D_f(s))$.

2. The underlying l -simplex in Int_{k_i} is sent to its image under f^* , i.e. its rescaling data is $f^*(\varphi_{s,t})$. Recall from section 2.1.3 that this is

$$f^*(\varphi_{s,t}) = \rho_f(s) \circ \varphi_{s,t}|_{D_f(s)} \circ \rho_f(s)^{-1} : (0, 1)^n \rightarrow (0, 1)^n.$$

3. Since the diffeomorphisms $\psi_{s,t}$ intertwine with the composed bordisms with respect to the rescaling data $\varphi_{s,t}$, for every $s, t \in \Delta^l$ we have diffeomorphisms

$$\psi_{s,t}|_{N_s} : N_s \rightarrow N_t,$$

which intertwine with the (new) composed bordisms with respect to with the (new) rescaling data.

Proposition 2.3.21. *The spatial and simplicial structures of $(\text{PBord}_n)_{\bullet, \dots, \bullet}$ are compatible, i.e. for $g : [l] \rightarrow [p]$, $f_\alpha : [m_\alpha] \rightarrow [k_\alpha]$, and $f_\beta : [m_\beta] \rightarrow [k_\beta]$, for $1 \leq \alpha < \beta \leq n$, the induced maps*

$$g^\Delta, f_\alpha^*, \text{ and } f_\beta^*$$

commute. We thus obtain an n -fold simplicial space $(\text{PBord}_n)_{\bullet, \dots, \bullet}$.

Proof. By construction, g^Δ commutes with the simplicial structure. Moreover, the maps f_α^*, f_β^* commute since they modify different parts of the structure. \square

2.3.5 The complete n -fold Segal space Bord_n

Proposition 2.3.22. *$(\text{PBord}_n)_{\bullet, \dots, \bullet}$ is an n -fold Segal space.*

Proof. We need to prove the following conditions:

1. *The Segal condition is satisfied.* For clarity, we explain the Segal condition in the following case. The general proof works similarly. We will show that

$$(\text{PBord}_n)_{k_1, \dots, 2, \dots, k_n} \xrightarrow{\sim} (\text{PBord}_n)_{k_1, \dots, 1, \dots, k_n} \times_{(\text{PBord}_n)_{k_1, \dots, 0, \dots, k_n}}^h (\text{PBord}_n)_{k_1, \dots, 1, \dots, k_n}.$$

We will omit the indices and corresponding intervals for $\alpha \neq i$ for clarity. Our goal is to construct a map $glue$ such that $glue \circ (d_0 \times d_2) \sim id$, $(d_0 \times d_2) \circ glue \sim id$,

$$(\text{PBord}_n)_1 \times_{(\text{PBord}_n)_0}^h (\text{PBord}_n)_1 \begin{array}{c} \xrightarrow{glue} \\ \xleftarrow{d_0 \times d_2} \end{array} (\text{PBord}_n)_2$$

Since every level set $(\text{PBord}_n)_{k_1, \dots, k_n}$ is a Kan complex by proposition 2.3.8, i.e. fibrant, the homotopy fiber product consists of triples consisting of two points and a path between them. Choose such an element given by

$$\begin{array}{l} (M) = (\iota : M \hookrightarrow V \times (0, 1)^n, (0, b] \leq [a, 1)), \\ (\tilde{M}) = (\tilde{\iota} : \tilde{M} \hookrightarrow V \times (0, 1)^n, (0, \tilde{b}] \leq [\tilde{a}, 1)) \end{array} \in (\text{PBord}_n)_1 \times_{(\text{PBord}_n)_0}^h (\text{PBord}_n)_1.$$

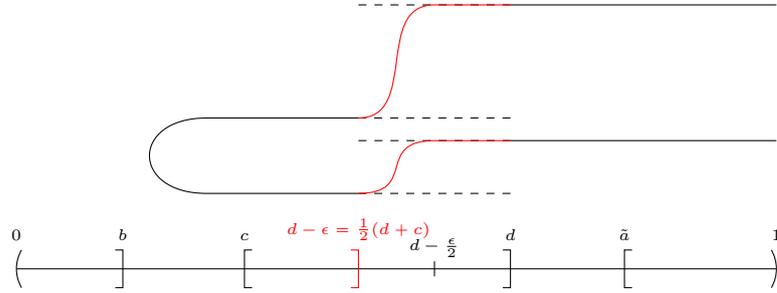
We will construct their image under $glue$, which is an element in $(\text{PBord}_n)_2$, essentially by gluing them.

We saw in example 2.3.2 that cutting off a short enough piece at an end of an element of $(\text{PBord}_n)_1$ leads to an element which is connected by a path to the original one,

i.e. $(\iota : M \hookrightarrow V \times (0, 1)^n, (0, b] \leq [a, 1)) \sim (\iota : p_i^{-1}((0, 1 - \epsilon)) \hookrightarrow V \times (0, 1)^n, (0, b] \leq [a, 1 - \epsilon))$, composed with suitable rescalings, for $0 < \epsilon < a$. So if the source of our glued element is such a “cutoff”, there is a path to the original (M) .

Since we started with an element in the homotopy fiber product, there is a path between the target of the first, $N = t((M))$, and the source of the second, $\tilde{N} = s(\tilde{M})$. Composing this path with the inverse of the path connecting $t((M))$ and its cutoff as described above gives a path between the “cutoff” and (N) .

Let us now assume that we have rescaled the embeddings and intervals such that they fit into $(0, d)$ respectively $(c, 1)$, and moreover, $(a, 1)$ and $(0, b)$ are sent to (c, d) . Now we glue the embeddings along $(d - \epsilon, d)$ for $\epsilon = \frac{1}{2}(d - c)$ using a partition of unity subordinate to the cover $\{(0, d - \frac{\epsilon}{2}), (d - \epsilon, 1)\}$. This gives a new embedded manifold $\tilde{\iota} : \tilde{M} \hookrightarrow V \times (0, 1)^n$ and together with the intervals $(0, b] \leq [c, \frac{1}{2}(d - c)] \leq [\tilde{a}, 1)$ they form an element in $(\text{PBord}_n)_2$.



This construction extends to l -simplices: by construction, this procedure depends smoothly on the parameter for a smooth family of manifolds and intervals. It remains to see that given an l -simplex in the homotopy fiber product we can obtain a family of diffeomorphisms intertwining with the composed bordisms. This follows because the data of a path between l -simplices includes the data of diffeomorphisms which we can use to patch together the given diffeomorphisms of the source and target.

2. For every i and every k_1, \dots, k_{i-1} , the $(n-i)$ -fold Segal space $(\text{PBord}_n)_{k_1, \dots, k_{i-1}, 0, \bullet, \dots, \bullet}$ is essentially constant.

We show that the degeneracy inclusion map

$$(\text{PBord}_n)_{k_1, \dots, k_{i-1}, 0, 0, \dots, 0} \hookrightarrow (\text{PBord}_n)_{k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n}$$

admits a deformation retraction and thus is a weak equivalence.

For $s \in [0, 1]$, consider the map γ_s sending an element in $(\text{PBord}_n)_{k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n}$ represented by

$$(M) := (M \subseteq V \times (0, 1)^n, (I_0^\beta \leq \dots \leq I_{k_\beta}^\beta)_{1 \leq \beta < i}, (0, 1), (I_0^\alpha \leq \dots \leq I_{k_\alpha}^\alpha)_{i < \alpha \leq n})$$

to

$$(M)_s := \left(M \subseteq V \times (0, 1)^n, (I_0^\beta \leq \dots \leq I_{k_\beta}^\beta)_{1 \leq \beta < i}, (0, 1), (I_0^\alpha(s) \leq \dots \leq I_{k_\alpha}^\alpha(s))_{i < \alpha \leq n} \right),$$

where for $\alpha > i$, $a_j^\alpha(s) = (1-s)a_j^\alpha$ and $b_j^\alpha(s) = (1-s)b_j^\alpha + s$. Note that for $s = 0$, $I_0^\alpha(0) = I_0^\alpha$, $I_j^\alpha(0) = I_j^\alpha$ and for $s = 1$, $I_j^\alpha(1) = (0, 1)$.

The maps γ_s form a homotopy between the degeneracy inclusion and the identity on $(\text{PBord}_n)_{k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n}$ provided that every γ_s indeed maps to this space. It suffices to check condition (3) in definition 2.3.1 for $(M)_s$. Since $(M) \in (\text{PBord}_n)_{k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n}$, this reduces to checking

For every $i < \alpha \leq n$ and $0 \leq j \leq k_\alpha$, at every $x \in p_{\{\alpha\}}^{-1}(I_j^\alpha(s))$, the map $p_{\{\alpha, \dots, n\}}$ is submersive.

Condition (3) on (M) for i implies that $p_{\{i, \dots, n\}}$ is a submersion in $p_{\{i\}}^{-1}((0, 1)) = M \supset p_{\{\alpha\}}^{-1}(I_j^\alpha(s))$, so $p_{\{\alpha, \dots, n\}}$ is submersive there as well.

Again, this construction extends to l -simplices since it only involves moving the intervals.

□

Remark 2.3.23. An interesting property of PBord_n is that it also satisfies the strict Segal condition,

$$(\text{PBord}_n)_k \xrightarrow{\cong} (\text{PBord}_n)_1 \times_{(\text{PBord}_n)_0} \cdots \times_{(\text{PBord}_n)_0} (\text{PBord}_n)_1,$$

where as above, we omit all indices except for the i th. This follows from the fact that we can glue the embedded manifolds along open sets. In fact, the maps

$$(\text{PBord}_n)_k \longrightarrow (\text{PBord}_n)_0$$

are fibrations and therefore the homotopy fiber product and the fiber product are weakly equivalent.

The last condition necessary to be a good model for the (∞, n) -category of bordisms is completeness, which PBord_n in general does not satisfy. However, we can pass to its completion Bord_n .

Definition 2.3.24. The (∞, n) -category of cobordisms Bord_n is the n -fold completion $\widehat{\text{PBord}_n}$ of PBord_n , which is a complete n -fold Segal space.

Remark 2.3.25. For $n \geq 6$, PBord_n is not complete, see the full explanation in [Lur09c], 2.2.8. For $n = 1$ and $n = 2$, by the classification theorems of one- and two-dimensional manifolds, PBord_n is complete, and therefore $\text{Bord}_n = \text{PBord}_n$.

2.3.6 Variants of Bord_n and comparison with Lurie's definition

Bounded submanifolds, cutting points, and \mathbb{R} as a parameter space

Bounded submanifolds Recall from 2.3.11 that for every element in $(\text{PBord}_n^V)_{k_1, \dots, k_n}$, we constructed a path to its cutoff, whose underlying submanifold

$$\text{cut}(M) = \pi^{-1}\left(\prod_{i=1}^n \left(\frac{b_0^i}{2}, \frac{a_{k_i}^i}{2}\right)\right)$$

is bounded in the V -direction. This construction extends to l -simplices and yields a map of n -fold simplicial spaces

$$\text{cut} : \text{PBord}_n \longrightarrow \text{PBord}_n,$$

sending an element to its “cutoff”. Its image lands in $\text{PBord}_n^{bd} \subseteq \text{PBord}_n$, the sub- n -fold Segal space of elements for which the underlying submanifold is bounded in the V -direction. Moreover, it induces a strong homotopy equivalence between PBord_n^{bd} and PBord_n .

Cutting points Another variant of an n -fold Segal space of cobordisms can be obtained by replacing the intervals I_j^i in definition 2.3.1 of PBord_n by specified “cutting points” $t_j^i \in (0, 1)$, which correspond to where we cut our composition of bordisms. Equivalently, we can say that in this case the intervals are replaced by intervals consisting of just one point, i.e. $a_j^i = b_j^i =: t_j^i$. The levels of this n -fold Segal space PBord_n^t can be made into spaces as we did for PBord_n , but we now need to impose the extra condition that elements of the levels are connected by a path if they coincide inside the “box” of t ’s, i.e. over $[t_0^1, t_{k_1}^1] \times \cdots \times [t_0^n, t_{k_n}^n]$. However, for PBord_n^t the Segal condition is more difficult to prove, as in this case we do specify the collar along which we glue. Since the space of collars is contractible, sending an interval $I = [a, b] \cap (0, 1)$ to its midpoint $t = \frac{1}{2}(a + b)$ induces a level-wise weak equivalence from PBord_n to PBord_n^t .

\mathbb{R} as a parameter space There also is a version of PBord_n replacing the closed intervals $I_j^i \subseteq (0, 1)$ by closed intervals in \mathbb{R} . We impose conditions on elements in this n -fold Segal space PBord_n^∞ which are analogous to (1)–(3) in definition 2.3.1 of PBord_n . This amounts to using the identification $(0, 1) \xrightarrow{\cong} \mathbb{R}$. However, in this case the face and degeneracy maps d_j^i, s_j^i for $j = 0, k_i$ are more complicated to write down since they require the use of rescaling maps $\rho_0 : (a_1^i, \infty) \rightarrow \mathbb{R}$, respectively $\rho_{k_i} : (-\infty, b_{k_i-1}^i) \rightarrow \mathbb{R}$. In this case, sending an interval to its midpoint as above leads to an variant with cutting points and \mathbb{R} as a parameter space $\text{PBord}_n^{t, \infty}$.

Comparison with Lurie’s definition of cobordisms

In [Lur09c], Lurie defined the n -fold Segal space of cobordisms as follows:

Definition 2.3.26. Let V be a finite dimensional vector space. For every n -tuple $k_1, \dots, k_n \geq 0$, let $(\text{PBord}_n^{V, L})_{k_1, \dots, k_n}$ be the collection of tuples $(M, (t_0^i \leq \dots \leq t_{k_i}^i)_{i=1, \dots, n})$, where

1. M is a closed n -dimensional submanifold of $V \times \mathbb{R}^n$ and the composition $\pi : M \hookrightarrow V \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ is a proper map.
2. For $1 \leq i \leq n$,

$$t_0^i \leq \dots \leq t_{k_i}^i$$

is an ordered $(k_i + 1)$ -tuple of elements in \mathbb{R} .

3. For every $S \subseteq \{1, \dots, n\}$ and for every collection $\{j_i\}_{i \in S}$, where $0 \leq j_i \leq k_i$, the composition $p_S : M \xrightarrow{\pi} \mathbb{R}^n \rightarrow \mathbb{R}^S$ does not have $(t_{j_i}^i)_{i \in S}$ as a critical value.
4. For every $x \in M$ such that $p_{\{i\}}(x) \in \{t_0^i, \dots, t_{k_i}^i\}$, the map $p_{\{i+1, \dots, n\}}$ is submersive at x .

It is endowed with a topology coming from the Whitney topology similar to what we described in remark 2.3.2, which we will not repeat here. Similarly to before, we define

$$\text{PBord}_n^L = \varinjlim_{V \subset \mathbb{R}^\infty} \text{PBord}_n^{V,L}$$

Comparing this definition with definition 2.3.1 and $\text{PBord}_n^{t,\infty}$ from above, note that our condition (3) on $\text{PBord}_n^{t,\infty}$ is replaced by the two strictly weaker conditions $(\tilde{3})$ and $(\tilde{4})$ on PBord_n^L , which are implied by (3):

Lemma 2.3.27. *Let M be a closed n -dimensional manifold and $\pi : M \rightarrow \mathbb{R}^n$. Moreover, for $1 \leq i \leq n$ let $(t_0^i \leq \dots \leq t_{k_i}^i)$ be an ordered $(k_i + 1)$ -tuple of elements in \mathbb{R} . Denote for $S \subseteq \{1, \dots, n\}$ the composition $M \xrightarrow{\pi} \mathbb{R}^n \rightarrow \mathbb{R}^S$ by p_S . Assume that condition (3) from definition 2.3.1 holds, i.e. for every $1 \leq i \leq n$ and $0 \leq j_i \leq k_i$ for $x \in M$ such that $p_{\{i\}}(x) = t_{j_i}^i$ the map $p_{\{i, \dots, n\}}$ is submersive at x . Then,*

- $\tilde{3}$. For every $S \subseteq \{1, \dots, n\}$ and for every collection $\{j_i\}_{i \in S}$, where $0 \leq j_i \leq k_i$, the composition $p_S : M \xrightarrow{\pi} \mathbb{R}^n \rightarrow \mathbb{R}^S$ does not have $(t_{j_i}^i)_{i \in S}$ as a critical value.
- $\tilde{4}$. For every $x \in M$ such that $p_{\{i\}}(x) \in \{t_0^i, \dots, t_{k_i}^i\}$, the map $p_{\{i+1, \dots, n\}}$ is submersive at x .

Proof. Let $i_0 = \inf S$. Consider the following diagram

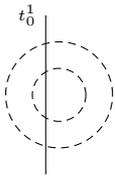
$$\begin{array}{ccc} M & \xrightarrow{p_{\{i_0, \dots, n\}}} & \mathbb{R}^{\{i_0, \dots, n\}} \\ & \searrow p_S & \downarrow \text{proj} \\ & & \mathbb{R}^S \end{array}$$

For $\tilde{3}$. let $x \in p_S^{-1}((t_{j_i}^i)_{i \in S})$. Then $p_{\{i_0\}}(x) = t_{j_{i_0}}^{i_0}$, so by assumption the map $p_{\{i_0, \dots, n\}}$ is submersive at x . Since proj is submersive, $p_S = \text{proj} \circ p_{\{i_0, \dots, n\}}$ also is submersive at x .

For $\tilde{4}$. note that $p_{\{i, \dots, n\}}$ being submersive at x implies that $p_{\{i+1, \dots, n\}}$ is submersive at x . \square

However, Lurie's n -fold simplicial space PBord_n^L is not an n -fold Segal space as we will see in the example below. Thus, our $\text{PBord}_n^{t,\infty}$ is a corrigendum of Lurie's PBord_n^L from [Lur09c].

Example 2.3.28.



Consider the 2 dimensional torus T in $\mathbb{R} \times \mathbb{R}^2$, and consider the tuple $(T \hookrightarrow \mathbb{R} \times \mathbb{R}^2, t_0^1, t_0^2 \leq \dots \leq t_{k_2}^2)$, where t_0^1 is indicated in the picture of the projection plane \mathbb{R}^2 below. Then, because of condition $(\tilde{3})$, $t_0^2 \leq \dots \leq t_{k_2}^2$ can be chosen everywhere such that any (t_0^1, t_j^2) is not a point where the vertical (t_0^1) -line intersects the two circles in the picture. Thus, the space of these choices is not contractible. However, it satisfies the conditions (1), (2), $(\tilde{3})$, and $(\tilde{4})$ in the definition of $(\text{PBord}_2^L)_{0, k_2}$, so $(\text{PBord}_2^L)_{0, \bullet}$ is not essentially constant.

The n -fold category Bord_n

Condition $(\tilde{3})$ in definition 2.3.26 ensures that the fibers $p_S^{-1}((t_{j_i})_{i \in S})$ are $(n - |S|)$ -dimensional smooth manifolds, i.e. that a k -morphisms, which is an k -dimensional cobordism, indeed goes from a $(k - 1)$ -dimensional cobordism to another one.

Our condition (3) ensures in addition the globularity condition, i.e. essential constancy, namely that we have an “ n -category” instead of an “ n -fold category” (unfortunately the use of “fold” here does not match up with the conventions for “fold” and “uple” for Segal spaces). This difference for $n = 2$ is the same as the difference between a “bicategory” and a “double category”.

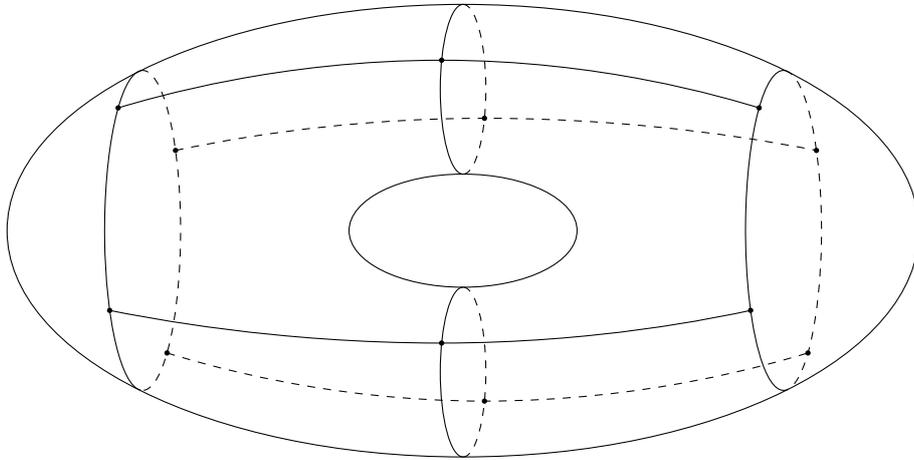
Consider the following interval version of condition $(\tilde{3})$

- $\tilde{3}$. For every $S \subseteq \{1, \dots, n\}$ and for every collection $\{j_i\}_{i \in S}$, where $0 \leq j_i \leq k_i$, the composition $p_S : M \xrightarrow{\pi} \mathbb{R}^n \rightarrow \mathbb{R}^S$ does not have any critical value in $(I_{j_i})_{i \in S}$.

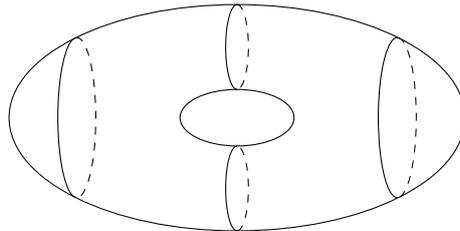
Relaxing our condition (3) in definition 2.3.1 to this $(\tilde{3})$ gives an n -uple Segal space $\text{PBord}_n^{\text{uple}}$. Completing gives a complete n -uple Segal space $\text{Bord}_n^{\text{uple}}$.

An example: the torus as a composition

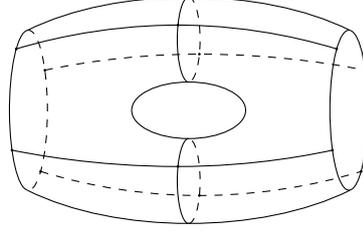
The torus as a 2-morphism in Bord_2 can be decomposed into elementary pieces as follows:



By this we mean that we first consider the decomposition in the first coordinate



which is an element in $(\text{Bord}_2)_{4,1}$ and then decompose the two middle pieces, the images under the face maps $d_1, d_2 : (\text{Bord}_2)_{4,1} \rightrightarrows (\text{Bord}_2)_{1,1}$, as



The essential constancy condition forces to do the decomposition in two steps, whereas in the 2-fold category Bord_2^{uple} we could have cut up the 2-morphism simultaneously in both directions.

2.4 The symmetric monoidal structure on Bord_n

The (∞, n) -category Bord_n is symmetric monoidal with its symmetric monoidal structure essentially arising from taking disjoint unions. In this section we endow Bord_n with a symmetric monoidal structure in two ways. In section 2.4.1 the symmetric monoidal structure arises from a Γ -object. In section 2.4.2 a symmetric monoidal structure is defined using a tower of monoidal i -hybrid $(n+i)$ -fold Segal spaces.

2.4.1 The symmetric monoidal structure arising as a Γ -object

We construct a sequence of n -fold Segal spaces $(\text{Bord}_n^V[m])_{\bullet, \dots, \bullet}$ which form a Γ -object which endows Bord_n with a symmetric monoidal structure as defined in section 1.6.

Definition 2.4.1. Let V be a finite dimensional vector space. For every k_1, \dots, k_n , let $(\text{PBord}_n^V[m])_{k_1, \dots, k_n}$ be the collection of tuples

$$(M_1, \dots, M_m, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n}),$$

where M_1, \dots, M_m are disjoint n -dimensional submanifolds of $V \times (0, 1)^n$, and each $(M_\beta, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n})$ is an element of $(\text{PBord}_n^V)_{k_1, \dots, k_n}$. It can be made into a simplicial set similarly to PBord_n^V . Moreover, similarly to the definition of Bord_n , we take the limit over all $V \subset \mathbb{R}^\infty$ and complete to get an n -fold complete Segal space $\text{Bord}_n[m]$.

Proposition 2.4.2. *The assignment*

$$\begin{aligned} \Gamma &\longrightarrow \mathbf{SSpace}_n, \\ [m] &\longmapsto \text{Bord}_n[m] \end{aligned}$$

extends to a functor and endows Bord_n with a symmetric monoidal structure.

Proof. By lemma 1.6.6 it is enough to show that the functor sending $[m]$ to $\text{PBord}_n[m]$ and a morphism $f : [m] \rightarrow [k]$ to the morphism

$$\begin{aligned} \text{PBord}_n[m] &\longrightarrow \text{PBord}_n[k], \\ (M_1, \dots, M_m, I's) &\longmapsto \left(\coprod_{\beta \in f^{-1}(1)} M_\beta, \dots, \coprod_{\beta \in f^{-1}(k)} M_\beta, I's \right), \end{aligned}$$

is a functor $\Gamma \rightarrow \mathbf{SSpace}_n$ with the property that

$$\prod_{1 \leq \beta \leq n} \gamma_\beta : \text{PBord}_n[m] \longrightarrow (\text{PBord}_n[1])^m$$

is an equivalence of n -fold Segal spaces.

The map $\prod_{1 \leq \beta \leq n} \gamma_\beta$ is an inclusion of n -fold Segal spaces and we show that level-wise it is a weak equivalence of spaces. Let $((M_1), \dots, (M_n)) \in (\text{PBord}_n[1])^m$. We construct a path to an element in the image of $\prod_{1 \leq i \leq n} \gamma_\beta$ which induces a strong homotopy equivalence between the above spaces. First, there is a path to an element for which all (M_α) have the same specified intervals by composing all except one with a suitable smooth rescaling. Secondly, there is a path with parameter $s \in [0, 1]$ given by composing the embedding $M_\alpha \hookrightarrow V \times (0, 1)^n$ with the embedding into $\mathbb{R} \times V \times (0, 1)^n$ given by the map $V \rightarrow \mathbb{R} \times V$, $v \mapsto (s\alpha, v)$. \square

2.4.2 The monoidal structure and the tower

Our goal for this section is to endow Bord_n with a symmetric monoidal structure arising from a tower of monoidal l -hybrid $(n + l)$ -fold Segal spaces $\text{Bord}_n^{(l)}$ for $l \geq 0$.

The $(\infty, n + l)$ -category of n -bordisms for $l \geq -n$

We now define an $(n + l)$ -fold Segal space whose $(n + l)$ -morphisms are n -bordisms for $l \geq -n$.

Definition 2.4.3. Let V be a finite dimensional vector space and let $n \geq 0, l \geq -n$. For every n -tuple $k_1, \dots, k_{n+l} \geq 0$, we let $(\text{PBord}_n^{l,V})_{k_1, \dots, k_{n+l}}$ be the collection of tuples $(M \hookrightarrow V \times (0, 1)^{n+l}, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n+l})$ satisfying conditions analogous to (1)-(3) in definition 2.3.1, i.e.

1. M is a closed n -dimensional submanifold of $V \times (0, 1)^{n+l}$, and $I_j^i \subseteq (0, 1)$ are closed intervals in $(0, 1)$ with endpoints $a_j^i < b_j^i$, $a_0^i = 0$, $b_{k_i}^i = 1$, and $I_j^i \leq I_l^i$ iff $a_j^i \leq a_l^i$, $b_j^i \leq b_l^i$,
2. the composition $\pi : M \hookrightarrow V \times (0, 1)^{n+l} \rightarrow (0, 1)^{n+l}$ is a proper map,
3. for every $S \subseteq \{1, \dots, n + l\}$ let p_S be the composition $p_S : M \xrightarrow{\pi} (0, 1)^{n+l} \rightarrow (0, 1)^S$. Then for every $1 \leq i \leq n + l$ and $0 \leq j_i \leq k_i$, at every $x \in p_{\{i\}}^{-1}(I_{j_i}^i)$, the map $p_{\{i, \dots, n+l\}}$ is submersive.

We make $(\text{PBord}_n^{l,V})_{k_1, \dots, k_{n+l}}$ into a space similarly to $(\text{PBord}_n^V)_{k_1, \dots, k_n}$, and again we take the limit over all finite dimensional vector spaces in a given infinite dimensional vector

space, say \mathbb{R}^∞ :

$$\text{PBord}_n^l = \varinjlim_{V \subset \mathbb{R}^\infty} \text{PBord}_n^{l,V}.$$

Proposition 2.4.4. $(\text{PBord}_n^l)_{\bullet, \dots, \bullet}$ is an $(n+l)$ -fold Segal space.

Proof. The proof is completely analogous to the proof of Proposition 2.3.22. \square

Definition 2.4.5. For $l \leq 0$ let Bord_n^l be the $(n+l)$ -fold completion of PBord_n^l , the $(\infty, n+l)$ -category of n -bordisms.

Remark 2.4.6. For $l > 0$, the underlying submanifold of objects of PBord_n^l , i.e. elements in $(\text{PBord}_n^l)_{0, \dots, 0}$, are n -dimensional manifolds M which have a submersion onto $(0, 1)^{n+l}$. This implies that $M = \emptyset$. Thus, the only object is $(\emptyset, (0, 1), \dots, (0, 1))$. Similarly, $(\text{PBord}_n^l)_{0, k_2, \dots, k_{n+l}}$ has only one element, which is the image of compositions of the degeneracy maps. Thus, $(\text{PBord}_n^l)_{0, \bullet, \dots, \bullet}$ is the point viewed as a constant $(n-1)$ -fold Segal space. Similarly, $(\text{PBord}_n^l)_{1, \dots, 1, 0, \bullet, \dots, \bullet}$, with $(l-1)$ 1's, is the point viewed as a constant $(n-l)$ -fold Segal space. Thus for $l > 0$ it makes sense and is more useful to use the l -hybrid completion of PBord_n^l .

Definition 2.4.7. For $l > 0$ let $\text{Bord}_n^{(l)}$ be the l -hybrid completion of PBord_n^l .

Loopings of PBord_n^l

In any PBord_n^l , there is the distinguished object $\emptyset = (\emptyset, (0, 1))$ in PBord_n^l , the unit for the monoidal structure. Recall from definition 1.5.7 the k -fold iterated loopings of PBord_n^l for $k \leq n+l$,

$$L_k(\text{PBord}_n^l) = L(L_{k-1}(\text{PBord}_n^l, \emptyset), \emptyset), \quad L_k(\text{Bord}_n^l) = L(L_{k-1}(\text{Bord}_n^l, \emptyset), \emptyset).$$

Proposition 2.4.8. For $n+l \geq k \geq 0$, there are weak equivalences

$$L_k(\text{PBord}_n^l) \begin{array}{c} \xrightarrow{u} \\ \xleftarrow{\ell} \end{array} \text{PBord}_n^{l-k}.$$

Proof. We show that $L(\text{PBord}_n^l) = \text{Hom}_{\text{PBord}_n^l}(\emptyset, \emptyset) \simeq \text{PBord}_n^{l-1}$. The statement for general k follows by induction.

We define a map

$$u : L(\text{PBord}_n^l) \xrightarrow{\simeq} \text{PBord}_n^{l-1}$$

by sending an element in $\text{Hom}_{\text{PBord}_n^l}(\emptyset, \emptyset)_{k_2, \dots, k_{n+l}}$,

$$(M_l) = (M \subseteq V \times (0, 1)^{n+l}, (0, b_0^1] \leq [a_1^1, 1), (I_0^i \leq \dots \leq I_{k_i}^i)_{i=2, \dots, n+l}) \in (\text{PBord}_n^l)_{1, k_2, \dots, k_{n+l}}$$

to

$$(M_{l-1}) = (M \subseteq \underbrace{(V \times (0, 1))}_{=\tilde{V}} \times (0, 1)^{n+l-1}, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=2, \dots, n+l}),$$

so it “forgets” the first specified intervals. First of all, we need to check that this map is well-defined, that is, that $(M_{l-1}) \in (\text{PBord}_n^{l-1})_{k_2, \dots, k_{n+l}}$. Note that in the above, we view

$\tilde{V} = V \times (0, 1)$ as a vector space using the identification $(0, 1) \stackrel{\times}{\cong} \mathbb{R}$. The condition we need to check is the second one, i.e. we need to check that $M \hookrightarrow \tilde{V} \times (0, 1)^{n+l-1} \rightarrow (0, 1)^{n+l-1}$ is proper. We know that $M \rightarrow (0, 1)^{n+l}$ is proper, and moreover, since $p_1^{-1}((0, b_0^1)) = p_1^{-1}((a_1^1, 1)) = \emptyset$, we know that M is bounded in the direction of the first coordinate, since $M = p_1^{-1}([b_0^1, a_1^1])$. Together this implies the statement. Note that the map u we just constructed actually is defined by a system of maps

$$u_V : L(\text{PBord}_n^{l,V}) \longrightarrow \text{PBord}_n^{l,\tilde{V}},$$

where $\tilde{V} = V \oplus \langle v \rangle$.

To construct a map in the other direction we will also need to change the vector space V , but this time we need to "delete" a direction. To make this procedure precise, we fix the following notations. In the definition of $\text{PBord}_n^{l,V}$ we let V vary within a fixed countably infinite dimensional space. Choose \mathbb{R}^∞ with a countable basis consisting of vectors v_1, v_2, \dots . In taking the limit is enough to consider the finite dimensional subspaces V_d spanned by the first d vectors v_1, \dots, v_d . Then the map u we constructed above was defined as an inductive system of maps

$$\begin{aligned} u_d : L(\text{PBord}_n^{l,V_d}) &\longrightarrow \text{PBord}_n^{l,V_{d+1}}, \\ (M \subseteq V_d \times (0, 1) \times (0, 1)^{n+l-1}) &\longmapsto \left(M \subseteq \underbrace{\langle v_1 \rangle}_{\cong (0,1)} \oplus \underbrace{\langle v_2, \dots, v_{d+1} \rangle}_{\cong V_d} \times (0, 1)^{n+l-1} \right), \end{aligned}$$

where we use the canonical morphisms $(0, 1) \cong \mathbb{R} \cong \langle v_1 \rangle$ and $\langle v_2, \dots, v_{d+1} \rangle \rightarrow V_d$, $v_\beta \mapsto v_{\beta-1}$.

In remark 2.3.11, we constructed a path from an element in PBord_n to its "cutoff", whose underlying submanifold is $\pi^{-1}(\prod_{i=1}^n (\frac{b_0^i}{2}, \frac{a_{k_i}^i}{2}))$, which is bounded in the V -direction. We saw in section 2.3.6 that this map gives rise to a strong homotopy equivalence

$$\text{cut} : \text{PBord}_n \longrightarrow \text{PBord}_n^{bd}.$$

Similarly, we obtain equivalences of n -fold Segal spaces

$$\text{cut} : \text{PBord}_n^{l-1} \longrightarrow \text{PBord}_n^{l-1,bd}, \quad \text{cut} : L(\text{PBord}_n^l) \longrightarrow L(\text{PBord}_n^{l,bd}).$$

Note that u_d restricts to a map between the bounded versions,

$$u_d^{bd} : L(\text{PBord}_n^{l,V_d,bd}) \longrightarrow \text{PBord}_n^{l,V_{d+1},bd}$$

It suffices to show that this map induces a strong homotopy equivalence, with homotopy inverse given by the following inductive system of maps

$$\ell_d^{bd} : \text{PBord}_n^{l-1,V_{d+1},bd} \longrightarrow L(\text{PBord}_n^{l,V_d}).$$

Start with an element $(M_{l-1}) = (M \subseteq V_{d+1} \times (0, 1)^{n+l-1}, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=2,\dots,n+l}) \in \text{PBord}_n^{l-1,V_{d+1},bd}$. Since it is bounded in the V -direction, there are A, B such that

$$B < \pi_{v_1}(M) < A,$$

where $\pi_{v_1} : M \subseteq (\langle v_1 \rangle \oplus \langle v_2, \dots, v_{d+1} \rangle) \times (0, 1)^{n+l-1} \rightarrow \langle v_1 \rangle = \mathbb{R}v_1$. Let \tilde{B} be the supremum of such B and let \tilde{A} be the infimum of such A . Let $\tilde{B} = \frac{\tilde{B}}{2}, \tilde{A} = \frac{\tilde{A}+1}{2}$. Now let

$b, a \in (0, 1) \cong \mathbb{R}$ correspond to \tilde{B}, \tilde{A}, B . Finally, we send (M_{l-1}) to

$$(M_l) = \left(M \subseteq \underbrace{\langle v_2, \dots, v_{d+1} \rangle}_{\cong V_d} \times \underbrace{(0, 1)}_{\cong \langle v_1 \rangle} \times (0, 1)^{n+l-1}, (0, b] \leq [a, 1), (I_0^i \leq \dots \leq I_{k_i}^i)_{i=2, \dots, n+l} \right).$$

By construction,

$$\ell_d^{bd} \circ u_d \sim id, \quad u_d \circ \ell_d^{bd} = id,$$

where $\ell_d^{bd} \circ u_d$ just changes the first two intervals $I_0^1 \leq I_1^1$ and thus is homotopy equivalent to the identity. \square

Definition 2.4.9. The map ℓ in the proof is called the *looping* and u the *delooping map*.

Recall from remark 1.5.8 that looping commutes with completion. Taking the appropriate completions, we obtain the following corollary.

Corollary 2.4.10. *Let $k \geq 0$.*

1. *If $l - k > 0$,*

$$L_k(\text{Bord}_n^{(l)}) \simeq \text{Bord}_n^{(l-k)}. \quad (2.3)$$

2. *If $k \geq l > 0$ and $n + l - k \geq 0$,*

$$L_k(\text{Bord}_n^{(l)}) \simeq \text{Bord}_n^{l-k}. \quad (2.4)$$

3. *If $l \leq 0$ and For $n + l \geq k \geq 0$, $n + l - k \geq 0$,*

$$L_k(\text{Bord}_n^l) \simeq \text{Bord}_n^{l-k}. \quad (2.5)$$

The tower and the symmetric monoidal structure

Recall from definition 2.4.7 that $\text{Bord}_n^{(l)}$ is the l -hybrid completion of PBord_n^l . By remark 2.4.6 and (2.3) in corollary 2.4.10, proposition 2.4.8 has an immediate corollary.

Corollary 2.4.11. *The $(n + l)$ -fold Segal spaces $\text{Bord}_n^{(l)}$ are l -hybrid and endow Bord_n with the structure of a symmetric monoidal n -fold Segal space.*

2.5 The homotopy (bi)category

2.5.1 The homotopy category $h_1(L_{n-1}(\text{Bord}_n))$

The symmetric monoidal structure on $h_1(L_{n-1}(\text{Bord}_n))$

The $(n - 1)$ -fold looping $L_{n-1}(\text{Bord}_n) \simeq \text{Bord}_n^{-(n-1)}$ is a $(\infty, 1)$ -category with a symmetric monoidal structure defined in two ways similarly to that of Bord_1 . Both induce a symmetric monoidal structure on the homotopy category $h_1(L_{n-1}(\text{Bord}_n)) \simeq h_1(\text{Bord}_n^{-(n+1)})$.

...coming from a Γ -object We can either obtain the symmetric monoidal structure as a Γ -object on $L_{n-1}(\text{Bord}_n) \simeq \text{Bord}_n^{-(n-1)}$ by iterating the construction of the symmetric monoidal structure on the looping from example 1.6.11 or by constructing a functor from an assignment $[m] \mapsto \text{Bord}_n^{-(n-1)}[m]$. In the second case, $\text{Bord}_n^{-(n-1)}[m]$ arises, similarly to $\text{Bord}_n[m]$, from the spaces $(\text{PBord}_n^{V, -(n-1)}[m])_{k_1, \dots, k_n}$, which as a set is the collection of tuples

$$(M_1, \dots, M_m, (I_0 \leq \dots \leq I_k)),$$

where M_1, \dots, M_m are disjoint n -dimensional submanifolds of $V \times (0, 1)^n$, and each $(M_\beta, (I_0 \leq \dots \leq I_k)) \in (\text{PBord}_n^{V, -(n-1)})_{k_1, \dots, k_n}$.

We saw in example 1.6.9 that a Γ -object endows the homotopy category of its underlying Segal space with a symmetric monoidal structure. Explicitly, in the second case, it comes from the following maps.

$$\begin{array}{ccc} \text{Bord}_n^{-(n+1)}[1] \times \text{Bord}_n^{-(n+1)}[1] & \xleftarrow[\gamma_1 \times \gamma_2]{\simeq} & \text{Bord}_n^{-(n+1)}[2] & \xrightarrow{\gamma} & \text{Bord}_n^{-(n+1)}[1], \\ (M_1, I\text{'s}), (M_2, I\text{'s}) & \xleftarrow{\quad} & (M_1, M_2, I\text{'s}) & \mapsto & (M_1 \amalg M_2, I\text{'s}) \end{array}$$

...coming from a tower To understand the symmetric monoidal structure on $h_1(L_{n-1}(\text{Bord}_n))$ coming from a symmetric monoidal structure as a tower, we use that $L_{n-1}(\text{Bord}_n) \simeq \text{Bord}_n^{-(n-1)}$ and that $\text{Bord}_n^{-(n-1)}$ has a symmetric monoidal structure coming from the collection of l -hybrid $(l+1)$ -fold Segal spaces given by the l -hybrid completion of PBord_n^{l-n+1} , the completion in the last index. This symmetric monoidal structure induces one on the homotopy category $h_1(\text{Bord}_n^{-(n-1)}) \simeq h_1(L_{n-1}(\text{Bord}_n))$, which we will explain explicitly. Since completion is a Dold-Kan equivalence, see 1.2.3, it is enough to understand the symmetric monoidal structure on $h_1(\text{PBord}_n^{-(n-1)})$.

Essentially, the monoidal structure is given by composition in $\text{PBord}_n^{1-(n-1)}$, the next layer of the tower $\text{PBord}_n^{2-(n-1)}$ gives a braiding and the higher layers show that it is symmetric monoidal. Consider the diagram

$$(\text{PBord}_n^{1-(n-1)})_{1, \bullet} \times (\text{PBord}_n^{1-(n-1)})_{1, \bullet} \xleftarrow[d_0^1 \times d_2^1]{\simeq} (\text{PBord}_n^{1-(n-1)})_{2, \bullet} \xrightarrow{s_1^1} (\text{PBord}_1^{n-(n-1)})_{1, \bullet}.$$

Using the fact from remark 1.6.18 that $L(\text{PBord}_n^{1-(n-1)})_{\bullet} = (\text{PBord}_n^{1-(n-1)})_{1, \bullet}$, we find that $(\text{PBord}_n^{1-(n-1)})_{1, \bullet} \simeq (\text{PBord}_n^{-(n-1)})_{\bullet}$, which induces a map

$$h_1(\text{PBord}_n^{-(n-1)}) \times h_1(\text{PBord}_n^{-(n-1)}) \longrightarrow h_1(\text{PBord}_n^{-(n-1)}).$$

This is a monoidal structure on $h_1(\text{PBord}_n^{-(n-1)})$. We can explicitly construct this map. Consider two objects or 1-morphisms (M) and (N) in $(\text{PBord}_n^{-(n-1)})_k$ for $k = 0$ or $k = 1$,

$$(M) = (M \subseteq V \times (0, 1), I_0 \leq \dots \leq I_k), \quad (N) = (N \subseteq \tilde{V} \times (0, 1), \tilde{I}_0 \leq \dots \leq \tilde{I}_k).$$

Without loss of generality $V = \tilde{V} = V_d$, and $(M), (N) \in (\text{Bord}_1^{bd})_k$.

Under the map $\ell_d^{bd} : \text{Bord}_1^{bd} \rightarrow L(\text{Bord}_1^{1, bd})$ from proposition 2.4.8, (M) and (N) are sent to

$$\begin{aligned} (M_1) &= (M \subseteq V_{d-1} \times (0, 1)^2, (0, b] \leq [a, 1), I_0 \leq \dots \leq I_k), \\ (N_1) &= (N \subseteq \tilde{V}_{d-1} \times (0, 1)^2, (0, \tilde{b}] \leq [\tilde{a}, 1), \tilde{I}_0 \leq \dots \leq \tilde{I}_k). \end{aligned}$$

In the proof of the Segal condition for PBord_n proposition 2.3.22 we explicitly constructed a homotopy inverse *glue* to $d_0^1 \times d_2^1$. Similarly one can obtain such a homotopy inverse for PBord_n^l , which applied to (M_1) and (N_1) gives

$$\left(M \amalg N \hookrightarrow \tilde{V}_{d-1} \times (0, 1)^2, (0, b_0^1) \leq [a_1^1, b_1^1] \leq [a_2^1, 1], \tilde{I}_0 \leq \cdots \leq \tilde{I}_k \right),$$

since $d_1^1((M_1)) = d_0^1((N_1)) = \emptyset$. The third face map sends it to

$$\left(M \amalg N \hookrightarrow \tilde{V}_{d-1} \times (0, 1)^2, (0, b_0^1) \leq [a_2^1, 1], \tilde{I}_0 \leq \cdots \leq \tilde{I}_k \right)$$

which by $u_d^{bd} : L(\text{Bord}_1^{1,bd}) \rightarrow \text{Bord}_1^{bd}$ is sent to

$$\left(M \amalg N \hookrightarrow \tilde{V}_d \times (0, 1), \tilde{I}_0 \leq \cdots \leq \tilde{I}_k \right).$$

The homotopy category and $n\text{Cob}$

The homotopy category of Bord_1 turns out to be what we expect, namely 1Cob . We can show even more, namely that our higher categories of cobordisms also give back the ordinary categories of n -cobordisms, as we see in the following proposition.

Proposition 2.5.1. *There is an equivalence of symmetric monoidal categories between the homotopy category of the $(n-1)$ -fold looping of Bord_n and the category of n -cobordisms,*

$$h_1(L_{n-1}(\text{Bord}_n)) \simeq n\text{Cob}.$$

Proof. We first show that there is an equivalence of categories $h_1(L_{n-1}(\text{Bord}_n)) \simeq n\text{Cob}$ and then show that it respects the symmetric monoidal structures.

Rezk's completion functor is a Dwyer-Kan equivalence of Segal spaces, and thus by definition induces an equivalence of the homotopy categories. Moreover, completion commutes with looping, so it is enough to show that

$$h_1(L_{n-1}(\text{PBord}_n)) \simeq n\text{Cob}.$$

We define a functor

$$F : h_1(L_{n-1}(\text{PBord}_n)) \longrightarrow n\text{Cob}$$

and show that it is essentially surjective and fully faithful.

Definition of the functor By definition,

$$(L_{n-1}(\text{PBord}_n))_k = \{\emptyset\} \underset{(L_{n-2}(\text{PBord}_n))_{0,k}}{\times} \underset{(L_{n-2}(\text{PBord}_n))_{0,k}}{\times} (L_{n-2}(\text{PBord}_n))_{1,k} \underset{(L_{n-2}(\text{PBord}_n))_{0,k}}{\times} \{\emptyset\},$$

and, iterating this process, we find that an element in $(L_{n-1}(\text{PBord}_n))_k$ is an element (M) of $(\text{PBord}_n)_{1,\dots,1,k}$ such that for every $i \neq n$, $d_j^i((M))$ has \emptyset as its underlying manifold, i.e. in every direction except for the n th direction, the source and target both have \emptyset as its underlying manifold.

So an object in $h_1(L_{n-1}(\text{PBord}_n))$ is an element $(M) \in (\text{PBord}_n)_{1,\dots,1,0}$ such that for $i \neq n$, the underlying manifold of $d_j^i((M))$ is \emptyset . We let the functor F send (M) to

$$\pi^{-1}([b_0^1, a_1^1] \times \cdots \times [b_0^{n-1}, a_1^{n-1}] \times \{\frac{1}{2}\}).$$

Since $\frac{1}{2}$ is a regular value of $p_{\{n\}}$, $F((M))$ is an $(n-1)$ -dimensional manifold, and since π is proper, it is compact. Moreover its boundary is empty. This follows from

$$F((M)) \hookrightarrow V \times [b_0^1, a_1^1] \times \cdots [b_0^{n-1}, a_1^{n-1}] \times \left\{ \frac{1}{2} \right\}$$

which implies that

$$\partial F((M)) = F((M)) \cap \partial \left(V \times [b_0^1, a_1^1] \times \cdots [b_0^{n-1}, a_1^{n-1}] \times \left\{ \frac{1}{2} \right\} \right)$$

and since for every $i \neq n$, the underlying manifold of $d_j^i((M))$ is \emptyset .

So, as an abstract manifold, $F((M))$ is a closed compact $(n-1)$ -dimensional manifold, i.e. an object in $n\text{Cob}$.

Similarly, the functor F sends a morphism in $h_1(L_{n-1}(\text{PBord}_n))$, which is an element in $\pi_0(L_{n-1}(\text{PBord}_n)_1)$ which is represented by an element $(M) \in (\text{PBord}_n)_{1, \dots, 1, 1}$ such that for $i \neq n$, the underlying manifold of $d_j^i((M))$ is \emptyset , to the isomorphism class of

$$\bar{M} = \pi^{-1}([b_0^1, a_1^1] \times \cdots \times [b_0^{n-1}, a_1^{n-1}] \times [b_0^n, a_1^n]).$$

This is an n -dimensional manifold with boundary

$$\pi^{-1}([b_0^1, a_1^1] \times \cdots \times [b_0^{n-1}, a_1^{n-1}] \times \{b_0^n\}) \amalg \pi^{-1}([b_0^1, a_1^1] \times \cdots \times [b_0^{n-1}, a_1^{n-1}] \times \{a_1^n\}).$$

This is well-defined, since a path in $L_{n-1}(\text{PBord}_n)_1$ by definition gives diffeomorphism $\psi_{0,1} : M_0 \rightarrow M_1$ which intertwines with the composed bordisms and thus restricts to diffeomorphisms of the images defined above.

The functor is an equivalence of categories Whitney's embedding theorem shows that F is essentially surjective. Moreover, it is injective on morphisms: Let $\iota_0 : M_0 \hookrightarrow V \times (0,1)^n$ and $\iota_1 : M_1 \hookrightarrow V \times (0,1)^n$ be representatives of two 1-morphisms which have diffeomorphic images. This means that there is a diffeomorphism $\psi : \bar{M}_0 \rightarrow \bar{M}_1$, which can be extended to their collars, i.e. we get a diffeomorphism $\psi : M_0 \rightarrow M_1$. Since $\text{Emb}(M_1, \mathbb{R}^\infty \times (0,1)^n)$ is contractible, the quotient $\text{Emb}(M_1, \mathbb{R}^\infty \times (0,1)^n)/\text{Diff}(M_1)$ is path-connected, so there is a path of embedded submanifolds $\tilde{\iota}_s : M_1 \hookrightarrow \mathbb{R}^\infty \times (0,1)^n$ such that $\tilde{\iota}_1 = \iota_1$ is the given one and $\tilde{\iota}_0 = \iota_0 \circ \psi$. Note that $\tilde{\iota}_0$ and ι_0 give the same submanifold. By lemma 2.3.12, this family ι_s determines a rescaling data and a family of diffeomorphisms $\psi_{s,t}$ which intertwine and thus a path in PBord_n , which by construction lies in $L_{n-1}(\text{Bord}_n)$. It remains to show that F is full.

In the case $n = 1, 2$ this is easy to show, as we have a classification theorem for 1- and 2-dimensional manifolds with boundary. In the 1-dimensional case it is enough to show that an open line, the circle and the half-circle, once as a bordism from 2 points to the empty set and once vice versa, lie in the image of the map, which is straightforward. In the two dimensional case, the pair-of-pants decomposition tells us how to embed the manifold.

For general n we first embed the manifold with boundary into $\mathbb{R}^+ \times \mathbb{R}^{2n}$ using a variant of Whitney's embedding theorem for manifolds with boundary, cf. [Lau00]. Then the boundary of the halfspace is $\partial(\mathbb{R}^+ \times \mathbb{R}^{2n}) = \mathbb{R}^{2n}$. We want to transform this embedding

into an embedding into $(0, 1) \times \mathbb{R}^{2n}$ such that the incoming boundary is sent into $\{\epsilon\} \times \mathbb{R}^{2n}$ and the outgoing boundary is sent into $\{1 - \epsilon\} \times \mathbb{R}^{2n}$.

We first show that the boundary components can be separated by a hyperplane in \mathbb{R}^{2n} . The boundary components are compact so they can be embedded into balls B^{2n} . By perhaps first applying a suitable “stretching” transformation, one can assume that these balls do not intersect. Now, since $2n > 1$, $\pi_0(\text{Conf}(B^{2n}, \mathbb{R}^{2n})) = *$, there is a transformation to a configuration in which the boundary components are separated by a hyperplane, without loss of generality given by the equation $\{x_1 = 0\} \subset \mathbb{R}^{2n}$.

Consider the (holomorphic) logarithm function on $(\mathbb{R}^+ \times \mathbb{R}) \setminus \{(0, 0)\} \cong \mathbb{H} \setminus 0 \subseteq \mathbb{C}$ with branch cut $-i\mathbb{R}^+$. It is a homeomorphism to $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \pi\}$. We can apply $\log \times id_{\mathbb{R}^{2n-1}}$ to $(\mathbb{R}^+ \times \mathbb{R}_{x_1}) \times \mathbb{R}^{2n-1}$ and, composing this with a suitable rescaling, obtain an embedding into $(\epsilon, 1 - \epsilon) \times \mathbb{R}^{2n}$. Now choose a collaring of the bordism to extend the embedding to $(0, 1) \times \mathbb{R}^{2n}$.

The functor is a symmetric monoidal equivalence Explicitly analyzing the two symmetric monoidal structures on $h_1(\text{Bord}_n^{-(n-1)})$, one sees that they both send two elements (represented by)

$$(M) = (M \subseteq V_d \times (0, 1), I_0 \leq \dots \leq I_k), \quad (N) = (N \subseteq V_d \times (0, 1), \tilde{I}_0 \leq \dots \leq \tilde{I}_k)$$

for $k = 0$ or $k = 1$ to an embedding of $M \amalg N$ into V_{d+1} , which sends M and N to different heights in the extra $(d + 1)$ st direction.

In the case of the structure coming from a Γ -object, one can similarly to in the previous paragraph define an equivalence of categories

$$F[m] : \text{Bord}_n^{-(n-1)}[m] \longrightarrow n\text{Cob}^m.$$

Then one can easily check that the following diagram commutes.

$$\begin{array}{ccccc} \text{Bord}_n^{-(n-1)}[1] \times \text{Bord}_n^{-(n-1)}[1] & \xleftarrow{\simeq} & \text{Bord}_n^{-(n-1)}[2] & \longrightarrow & \text{Bord}_n^{-(n-1)}[1] \\ \downarrow F \times F & & \downarrow F[2] & & \downarrow F \\ n\text{Cob} \times n\text{Cob} & \xlongequal{\quad} & n\text{Cob} \times n\text{Cob} & \xrightarrow{\amalg} & n\text{Cob} \end{array}$$

For the case of the structure coming from a tower, we explicitly saw that the symmetric structure on $h_1(\text{Bord}_n^{-(n-1)})$ sends two objects or 1-morphisms determined by

$$(M) = (M \subseteq V \times (0, 1), I_0 \leq \dots \leq I_k), \quad (N) = (N \subseteq \tilde{V} \times (0, 1), \tilde{I}_0 \leq \dots \leq \tilde{I}_k)$$

to

$$(M \amalg N) = (M \amalg N \hookrightarrow \tilde{V}_d \times (0, 1), \tilde{I}_0 \leq \dots \leq \tilde{I}_k),$$

where the embedding of M is changed by a rescaling. This change of rescaling is precisely such that under the functor F the element $(M \amalg N)$ is sent to $F((M)) \amalg F((N))$. \square

2.5.2 The homotopy bicategory $h_2(\text{Bord}_2)$ and comparison with 2Cob^{ext}

C. Schommer-Pries defined a symmetric monoidal bicategory $n\text{Cob}^{ext}$ of n -dimensional cobordisms in his thesis [SP09]. In this section we show that the homotopy bicategory of our $(\infty, 2)$ -category of 2-dimensional bordisms is symmetric monoidally equivalent to this bicategory.

The bicategory 2Cob^{ext}

We first briefly recall the definition of 2Cob^{ext} .

Definition 2.5.2. The bicategory 2Cob^{ext} has

- 0-dimensional manifolds as objects,
- 1-morphisms are 1-bordisms between objects, and
- 2-morphisms are isomorphism classes of 2-bordisms between 1-morphisms,

where

1. a *1-bordism* between two 0-dimensional manifolds Y_0, Y_1 is a smooth compact 1-dimensional manifold with boundary W with a decomposition and isomorphism

$$\partial W = \partial_{in} W \amalg \partial_{out} W \cong Y_0 \amalg Y_1;$$

2. a *2-bordism* between two 1-bordisms W_0, W_1 between objects Y_0, Y_1 is a compact 2-dimensional $\langle 2 \rangle$ -manifold S equipped with

- a decomposition and isomorphism

$$\partial_0 S = \partial_{0,in} S \amalg \partial_{0,out} S \xrightarrow{\sim} W_0 \amalg W_1,$$

- a decomposition and isomorphism

$$\partial_1 S = \partial_{1,in} S \amalg \partial_{1,out} S \xrightarrow{\sim} Y_0 \times [0, 1] \amalg Y_1 \times [0, 1].$$

Recall that a $\langle 2 \rangle$ -manifold is a manifold with faces X with a pair of faces $(\partial_0 X, \partial_1 X)$ such that

$$\partial_0 X \cup \partial_1 X = \partial X, \quad \partial_0 X \cap \partial_1 X \text{ is a face.}$$

3. Two 2-bordisms S, S' are *isomorphic* if there is a diffeomorphism $h : S \rightarrow S'$ compatible with the boundary data.

Vertical and horizontal compositions of 2-morphisms are defined by choosing collars and gluing. This is well-defined because 2-morphisms are isomorphism classes of 2-bordisms, and thus the composition doesn't depend on the choice of the collar. However, composition of 1-morphisms requires the use of a choice of a collar, which requires the axiom of choice, and then composition is defined by the unique gluing. However, this gluing is associative

only up to non-canonical isomorphism of 1-bordisms which gives a canonical isomorphism class of 2-bordisms realizing the associativity of horizontal composition in the axioms of a bicategory.

It is symmetric monoidal, with symmetric monoidal structure given by taking disjoint unions. For the exact details we refer to the above mentioned thesis [SP09].

The symmetric monoidal structure on $h_2(\text{Bord}_2)$

The symmetric monoidal structure on Bord_2 arising as a Γ -object gives us

$$\text{Bord}_2[1] \times \text{Bord}_2[1] \xleftarrow{\cong} \text{Bord}_2[2] \longrightarrow \text{Bord}_2[1]$$

which induces

$$h_2(\text{Bord}_2) \times h_2(\text{Bord}_2) \longrightarrow h_2(\text{Bord}_2).$$

This makes $h_2(\text{Bord}_2)$ into a symmetric monoidal bicategory, where the associativity follows from the equivalence $\text{Bord}_2[3] \xrightarrow{\sim} \text{Bord}_2[1] \times^3$.

The homotopy bicategory and 2Cob^{ext}

In this section we show that our $(\infty, 2)$ category of 2-cobordisms indeed gives back the bicategory 2Cob^{ext} as its homotopy bicategory.

Proposition 2.5.3. *There is an equivalence of symmetric monoidal bicategories between $h_2(\text{Bord}_2)$ and 2Cob^{ext} .*

Proof. By Whitehead's theorem for symmetric monoidal bicategories, see [SP09], theorem 2.21, it is enough to find a functor F which is

1. essentially surjective on objects, i.e. F induces an isomorphism $\pi_0(h_2(\text{Bord}_2)) \cong \pi_0(2\text{Cob}^{ext})$,
2. essentially full on 1-morphisms, i.e. for every $x, y \in \text{Ob } h_2(\text{Bord}_2)$, the induced functor $F_{x,y} : h_2(\text{Bord}_2)(x, y) \rightarrow 2\text{Cob}^{ext}(Fx, Fy)$ is essentially surjective, and
3. fully-faithful on 2-morphisms, i.e. for every $x, y \in \text{Ob } h_2(\text{Bord}_2)$, the induced functor $F_{x,y} : h_2(\text{Bord}_2)(x, y) \rightarrow 2\text{Cob}^{ext}(Fx, Fy)$ is fully-faithful.

First of all, recall from remark 2.3.25 that for $n = 2$, PBord_2 is a complete 2-fold Segal space, so $\text{Bord}_2 = \text{PBord}_2$.

Definition of the functor Let

$$F : h_2(\text{Bord}_2) \longrightarrow 2\text{Cob}^{ext}$$

be the functor defined as follows:

On objects,

$$(M \subseteq V \times \mathbb{R}^2, (0, 1), (0, 1)) \in (\text{Bord}_2)_{0,0} \xrightarrow{F} \pi^{-1} \left(\left(\frac{1}{2}, \frac{1}{2} \right) \right),$$

where the image is thought of as an abstract manifold. This is well-defined, because as π is proper and $(\frac{1}{2}, \frac{1}{2})$ is a regular value of π , the preimage $\pi^{-1}((\frac{1}{2}, \frac{1}{2}))$ is compact and 0-dimensional, so it is a finite disjoint union of points. Note that because of condition (3) in the definition of $\text{Bord}_2 = \text{PBord}_2$, we could have taken the fiber over any other point in $(0, 1)^2$ and would have gotten a diffeomorphic image.

On 1-morphisms,

$$(M \subseteq V \times \mathbb{R}^2, (0, b_0^1] \leq [a_1^1, 1), (0, 1)) \in (\text{Bord}_2)_{1,0} \xrightarrow{F} \pi^{-1} \left([b_0^1, a_1^1] \times \left\{ \frac{1}{2} \right\} \right).$$

The point $\frac{1}{2}$ is a regular value of the projection map $p_2 : M \hookrightarrow V \times (0, 1)^2 \rightarrow (0, 1)$, so $\pi^{-1}([b_0^1, a_1^1] \times \{\frac{1}{2}\})$ is a 1-dimensional manifold with boundary. Moreover, the decomposition of the boundary of the image is given by

$$\pi^{-1} \left((b_0^1, \frac{1}{2}) \right) \amalg \pi^{-1} \left((a_1^1, \frac{1}{2}) \right).$$

Note that again, we could have taken the preimage $\pi^{-1}([c, d] \times \{t\})$ for any $t \in [0, 1]$, $c \in (0, b_0^1]$, and $d \in [a_1^1, 1)$ and would have gotten a diffeomorphic image.

On 2-morphisms, the functor F comes from the assignment

$$(M \subseteq V \times \mathbb{R}^2, (0, b_0^1] \leq [a_1^1, 1), (0, b_0^2] \leq [a_1^2, 1)) \xrightarrow{F} \pi^{-1}([b_0^1, a_1^1] \times [b_0^2, a_1^2]) =: S.$$

As π is proper, S is a compact 2-dimensional manifold with corners and moreover has the structure of a $\langle 2 \rangle$ -manifold coming from the decomposition of the boundary coming from the inverse images under π of the sides of the rectangle $[b_0^1, a_1^1] \times [b_0^2, a_1^2]$,

$$\partial_0 S = \pi^{-1}([b_0^1, a_1^1] \times \{b_0^2\}) \amalg \pi^{-1}([b_0^1, a_1^1] \times \{a_1^2\}),$$

and

$$\partial_1 S = \pi^{-1}(\{b_0^1\} \times [b_0^2, a_1^2]) \amalg \pi^{-1}(\{a_1^1\} \times [b_0^2, a_1^2]).$$

By condition (3) in definition 2.3.1,

$$\pi^{-1}(\{b_0^1\} \times [b_0^2, a_1^2]) \cong \pi^{-1}((b_0^1, b_0^2) \times [b_0^2, a_1^2])$$

and

$$\pi^{-1}(\{a_1^1\} \times [b_0^2, a_1^2]) \cong \pi^{-1}((a_0^1, b_0^2) \times [b_0^2, a_1^2]).$$

This makes S into a 2-bordism between the images under F of the source and target of our 2-bordism.

This assignment descends to 2-morphisms which are elements in $\pi_0((\text{Bord}_2)_{1,1})$, as any path in $(\text{Bord}_2)_{1,1}$ by definition induces a diffeomorphism $\psi_{0,1} : M_0 \rightarrow M_1$ which intertwines with the composed bordisms and thus induces an isomorphism of the images under F defined above.

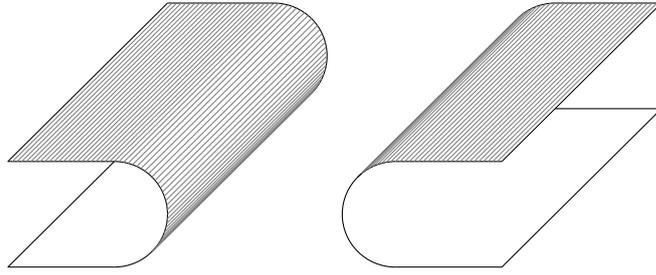
The functor is an equivalence of bicategories We check (1)-(3) of Whitehead's theorem.

For (1), the point is the image of the plane $(M = (0, 1)^2 \xrightarrow{id} (0, 1)^2, (0, 1), (0, 1))$. For k points, we can take k disjoint parallel planes in $(0, 1) \times (0, 1)^2$ which intersect $V = \mathbb{R}$ in k different points, e.g. $0, \dots, k-1$ and the intervals $I_0^1 = I_0^2 = (0, 1)$.

For (2), we use the classification of 1-dimensional manifolds with boundary. Any connected component can be cut into pieces diffeomorphic to straight lines and left and right half circles. These all lie in the image of F in a very simple way, e.g. a straight line is the image of

$$\left(M = (0, 1)^2 \xrightarrow{id} (0, 1)^2, (0, \frac{1}{3}] \leq [\frac{2}{3}, 1), (0, 1) \right),$$

and the right and left half circles are the images of the following embeddings $(0, 1)^2 \hookrightarrow \mathbb{R} \times (0, 1)^2$ with suitable choices of intervals.



By gluing these preimages in a suitable way, we get an element whose image is diffeomorphic to the connected component we started with.

For (3), to show that it is full on 2-morphisms, we use the classification theorem 3.33 of Schommer-Pries in [SP09]. He gives a set of generating 2-morphisms of 2Cob^{ext} for which one easily sees that they all are the image of an element in $(\text{Bord}_2)_{1,1}$. Moreover, the preimages can be glued. For faithfulness, a similar argument as in the proof of proposition 2.5.1 works: we use the fact that $\text{Emb}(M, \mathbb{R}^\infty \times (0, 1)^n)$ is contractible, so $\text{Emb}(M, \mathbb{R}^\infty \times (0, 1)^n)/\text{Diff}(M)$ is path connected. Using lemma 2.3.12, an isomorphism of 2-bordisms will give rise to a path in $(\text{Bord}_2)_{1,1}$.

The functor is a symmetric monoidal equivalence Similarly to in the previous subsection, the equivalence of bicategories

$$F : h_2(\text{Bord}_2) \xrightarrow{\simeq} 2\text{Cob}^{ext}$$

respects the symmetric monoidal structures. This can be seen by explicitly writing out the symmetric monoidal structure for $h_2(\text{Bord}_2)$. \square

Remark 2.5.4. In [SP09], Schommer-Pries also defined a bicategory $n\text{Cob}^{ext}$ with objects being $(n - 2)$ -dimensional manifolds, 1-morphisms being $(n - 1)$ -cobordisms, and 2-morphisms being equivalence classes of 2-bordisms, which are suitable n -dimensional $\langle 2 \rangle$ -manifolds. A similar argument should show that $h_2(L_{n-2}(\text{Bord}_n)) \simeq n\text{Cob}^{ext}$. However, one would need a suitable embedding theorem for cobordisms between cobordisms. One should be able to adapt the embedding theorem for $\langle 2 \rangle$ -manifolds from [Lau00], similarly to how we adapted the embedding theorem for manifolds with boundary.

2.6 Cobordisms with additional structure: orientations and framings

In the study of fully extended topological field theories, one is particularly interested in manifolds with extra structure, especially that of a framing. In this section we explain how to define the (∞, n) -category of structured n -bordisms, in particular for the structure of an orientation or a framing.

2.6.1 Structured manifolds

We first need to recall the definition of structured manifolds and the topology on their morphism spaces making them into a topological category. In the next subsection we will see that the simplicial set of chains on these topological spaces essentially will give rise to the spatial structure of the levels of the n -fold Segal space of structured bordisms similarly to the construction in section 2.3.2.

Throughout this subsection, let M be an n -dimensional (smooth) manifold.

Definition 2.6.1. Let X be a topological space and $E \rightarrow X$ a topological n -dimensional vector bundle which corresponds to a (homotopy class of) map(s) $e : X \rightarrow BGL(\mathbb{R}^n)$ from X to the classifying space of the topological group $GL(\mathbb{R}^n)$. More generally, we could also consider a map $e : X \rightarrow B\text{Homeo}(\mathbb{R}^n)$ to the classifying space of the topological group of homeomorphisms of \mathbb{R}^n , but for our purposes vector bundles are enough. An (X, E) -structure or, equivalently, an (X, e) -structure on an n -dimensional manifold M consists of the following data:

1. a map $f : M \rightarrow X$, and
2. an isomorphism of vector bundles

$$\text{triv} : TM \cong f^*(E).$$

Denote the set of (X, E) -structured n -dimensional manifolds by $\text{Man}_n^{(X, E)}$.

An interesting class of such structures arises from topological groups with a morphism to $O(n)$.

Definition 2.6.2. Let G be a topological group together with a continuous homomorphism $e : G \rightarrow O(n)$, which induces $e : BG \rightarrow BGL(\mathbb{R}^n)$. As usual, let $BG = EG/G$ be the classifying space of G , where EG is total space of its universal bundle, which is a weakly contractible space on which G acts freely. Then consider the vector bundle $E = (\mathbb{R}^n \times EG)/G$ on BG . A (BG, E) -structure or, equivalently, a (BG, e) -structure on an n -dimensional manifold M is called a G -structure on M . The set of G -structured n -dimensional manifolds is denoted by Man_n^G .

For us, the most important examples will be the following three examples.

Example 2.6.3. If G is the trivial group, $X = BG = *$ and E is trivial. Then a G -structure on M is a trivialization of TM , i.e. a framing.

Example 2.6.4. Let $G = O(n)$ and $e = id_{O(n)}$. Then, since the inclusion $O(n) \rightarrow \text{Diff}(\mathbb{R}^n)$ is a deformation retract, an $O(n)$ -structured manifold is just smooth manifolds.

Example 2.6.5. Let $G = SO(n)$ and $e : SO(n) \rightarrow O(n)$ is the inclusion. Then an $SO(n)$ -structured manifold is an oriented manifold.

Definition 2.6.6. Let M and N be (X, E) -structured manifolds. Then let the space of morphisms from M to N be

$$\text{Map}^{(X,E)}(M, N) = \text{Emb}(M, N) \underset{\text{Map}_{/B\text{Homeo}(\mathbb{R}^n)}(M, N)}{\times^h} \text{Map}_{/X}(M, N).$$

Taking (singular or differentiable) chains leads to a space, i.e. a simplicial set of morphisms from M to N . Thus we get a topological (or simplicial) category $\mathcal{Man}_n^{(X,E)}$ of (X, E) -structured manifolds. Disjoint union gives $\mathcal{Man}_n^{(X,E)}$ a symmetric monoidal structure.

Remark 2.6.7. For $G = O(n)$ we recover $\text{Emb}(M, N)$, and for $G = SO(n)$, the space of orientations on a manifold is discrete, so an element in $\text{Map}^{SO(n)}(M, N)$ is an orientation preserving map.

If G is the trivial group we saw above that a G -structure is a framing. In this case, the above homotopy fiber product reduces to

$$\text{Map}^{(X,E)}(M, N) = \text{Emb}(M, N) \underset{\text{Map}_{GL(d)}(\text{Fr}(TM), \text{Fr}(TN))}{\times^h} \text{Map}(M, N).$$

Thus, a framed embedding is a pair (f, h) , where $f : M \rightarrow N$ lies in $\text{Emb}(M, N)$ and h is a homotopy between the trivialization of TM induced by the framing of M and that induced by the pullback of the framing on N .

2.6.2 The (∞, n) -category of structured cobordisms

Fix a type of structure given by the pair (X, E) . In this subsection we define the n -fold (complete) Segal space of (X, E) -structured cobordisms $\text{Bord}_n^{(X,E)}$.

Compared to definition 2.3.1 we add an (X, E) -structure to the data of an element in a level set.

Definition 2.6.8. Let V be a finite dimensional vector space. For every n -tuple $k_1, \dots, k_n \geq 0$, let $(\text{PBord}_n^{(X,E),V})_{k_1, \dots, k_n}$ be the collection of tuples $(M, f, \text{triv}, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n})$, where

1. $(M, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1}^n)$ is an element in the set $(\text{PBord}_n^V)_{k_1, \dots, k_n}$, and
2. (f, triv) is an (X, E) -structure on the (abstract) manifold M .

Remark 2.6.9. Note that there is a forgetful map

$$U : (\text{PBord}_n^{(X,E),V})_{k_1, \dots, k_n} \rightarrow (\text{PBord}_n^V)_{k_1, \dots, k_n}$$

forgetting the (X, E) -structure.

Definition 2.6.10. An l -simplex of $(\text{PBord}_n^{(X,E),V})_{k_1, \dots, k_n}$ consists of the following data:

1. A family of elements

$$(M_s, f_s, \text{triv}_s) = (M_s \subseteq V \times (0, 1)^n, f_s, \text{triv}_s, (I_0^i(s) \leq \cdots \leq I_{k_i}^i(s))_{i=1, \dots, n})$$

in $(\text{PBord}_n^{(X, E), V})_{k_1, \dots, k_n}$ indexed by $s \in |\Delta^l|$, which are called the *underlying (X, E) -structured 0-simplices*;

2. For every $1 \leq i \leq k_i$,

$$(I_0^i(s) \leq \cdots \leq I_{k_i}^i(s))_{s \in |\Delta^l|}$$

is an l -simplex in Int_{k_i} with *rescaling datum* $\varphi_{s,t}^i : (0, 1) \rightarrow (0, 1)$;

3. A family of elements in $\text{Man}_n^{(X, E)}(M_s, M_t)$ with underlying *diffeomorphisms*

$$\psi_{s,t} : M_s \longrightarrow M_t,$$

indexed by $s, t \in |\Delta^l|$;

such that the triple

$$U(M_s, f_s, \text{triv}_s), \quad (\varphi_{s,t})_{s,t \in |\Delta^l|}, \quad (\psi_{s,t})_{s,t \in |\Delta^l|}$$

is an l -simplex in $(\text{PBord}_n^V)_{k_1, \dots, k_n}$.

Similarly as for PBord_n the levels can be given a spatial structure with the above l -simplices and then the collection of levels can be made into a complete n -fold Segal space $\text{Bord}_n^{(X, E)}$.

Moreover, $\text{Bord}_n^{(X, E)}$ has a symmetric monoidal structure given by (X, E) -structured versions of the Γ -object and of the tower giving Bord_n a symmetric monoidal structure.

2.6.3 Example: Objects in Bord_2^{fr} are 2-dualizable

In dimension one, a framing is the same as an orientation. Thus the first interesting case is the two-dimensional one. In this case, the existence of a framing is a rather strong condition. However, we will see that nevertheless, any object in Bord_2^{fr} is 2-dualizable. Being 2-dualizable means that it is dualizable with evaluation and coevaluation maps themselves have adjoints, see [Lur09c].

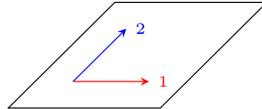
Consider an object in Bord_2^{fr} , which, since in this case $\text{Bord}_2^{fr} = \text{PBord}_2^{fr}$ by remark 2.3.25, is an element of the form

$$(M \subseteq V \times (0, 1)^2, F, (0, 1), (0, 1)),$$

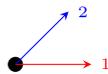
where F is a framing of M . By the submersivity condition 3 in the definition 2.3.1 of PBord_2 , M is a disjoint union of manifolds which are diffeomorphic to $(0, 1)^2$. Thus, it suffices to consider an element of the form

$$((0, 1)^2 \subseteq (0, 1)^2, F, (0, 1), (0, 1)),$$

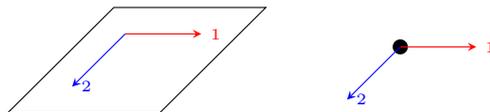
where F is a framing of $(0, 1)^2$. Depict this element by



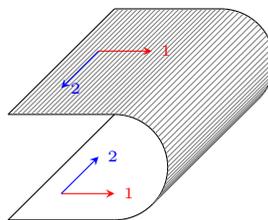
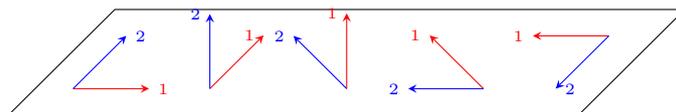
One should think of this as a point together with a 2-framing,



We claim that its dual is the same underlying unstructured manifold together with the opposite framing

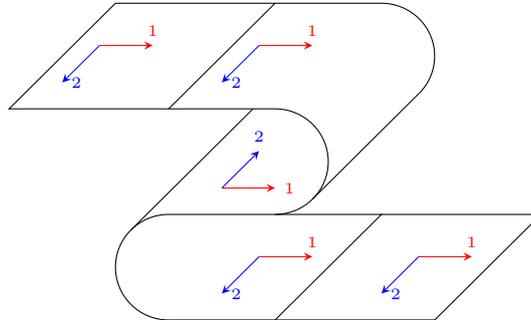


An evaluation 1-morphism ev between them is given by the element in $(\text{Bord}_2^{fr})_{1,0}$ which is a strip, i.e. $(0, 1)^2$, with the framing given by slowly rotating the framing by 180° , and is embedded into $\mathbb{R} \times (0, 1)^2$ by folding it over once as depicted further down.

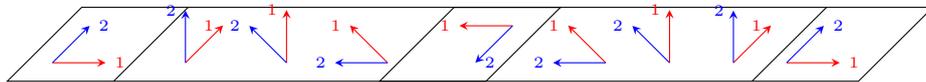


A coevaluation $coev$ is given similarly by rotating the framing along the strip in the other direction, by -180° .

The composition



is connected by a path to the flat strip with the following framing given by pulling at the ends of the strip to flatten it.

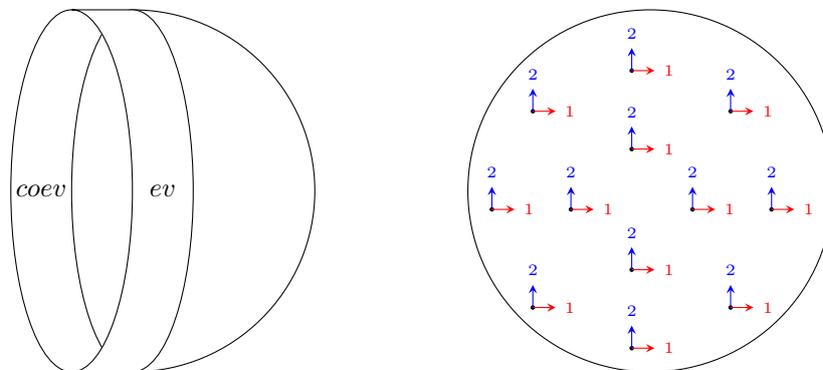


This strip is homotopic to the same strip with the trivial framing. Thus the composition is connected by a path to the identity and thus is the identity in the homotopy category. Similarly,

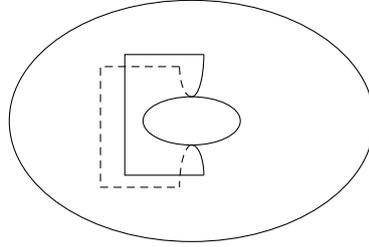
$$(ev \begin{smallmatrix} \nearrow 2 \\ \searrow 1 \end{smallmatrix} \otimes id \begin{smallmatrix} \nearrow 2 \\ \searrow 1 \end{smallmatrix}) \circ (id \begin{smallmatrix} \nearrow 2 \\ \searrow 1 \end{smallmatrix} \otimes coev \begin{smallmatrix} \nearrow 2 \\ \searrow 1 \end{smallmatrix}) \simeq id \begin{smallmatrix} \nearrow 2 \\ \searrow 1 \end{smallmatrix}.$$

In the above construction, we used $ev \begin{smallmatrix} \nearrow 2 \\ \searrow 1 \end{smallmatrix}$ and $coev \begin{smallmatrix} \nearrow 2 \\ \searrow 1 \end{smallmatrix}$ which arose from strips with framing rotating by $\pm 180^\circ$. A similar argument holds if you use for the evaluation any strip with the framing rotating by $\alpha\pi$ for any odd integer α and for the coevaluation rotation by $\beta\pi$ for any odd β . Denoting these by $ev(\alpha)$ and $coev(\beta)$, they will be adjoints to each other if $\alpha + \beta = 2$.

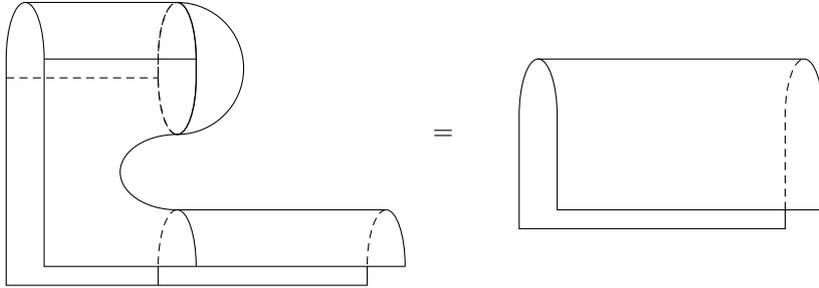
The counit of the adjunction is given by the cap with the framing coming from the trivial framing on the (flat) disk.



Similarly, the unit of the adjunction is given by a saddle with the framing coming from the one of the torus which turns by 2π along each of the fundamental loops.



Then the following 2-bordism also is framed and exhibits the adjunction.



2.7 Fully extended topological field theories

Now that we have a good definition of a symmetric monoidal (∞, n) -category of bordisms modelled as a symmetric monoidal complete n -fold Segal space, we can define fully extended topological field theories à la Lurie.

2.7.1 Definition

Definition 2.7.1. A fully extended unoriented n -dimensional topological field theory is a symmetric monoidal functor of (∞, n) -categories with source $Bord_n$.

Remark 2.7.2. Consider a fully extended unoriented n -dimensional topological field theory

$$Z : Bord_n \longrightarrow \mathcal{C},$$

where \mathcal{C} is a symmetric monoidal complete n -fold Segal space. We have seen in section 2.5 that $h_1(L_{n-1}(Bord_n)) \simeq n\text{Cob}$. The Z induces a symmetric monoidal functor

$$n\text{Cob} \simeq h_1(L_{n-1}(Bord_n)) \longrightarrow h_1(L_{n-1}(\mathcal{C}, Z(*))),$$

i.e. an ordinary n -dimensional topological field theory. The converse for $n > 1$ is not always true and poses interesting questions whether a theory can be “extended down”.

Similarly, a fully extended unoriented 2TFT with target \mathcal{C} yields an extended 2TFT

$$2\text{Cob}^{ext} \simeq h_2(Bord_2) \longrightarrow h_2(\mathcal{C}).$$

Additional structure Recall from the previous section that there are variants of Bord_n which require that the underlying manifolds of their elements to be endowed with some additional structure, e.g. an orientation or a framing. These variants lead to the following definitions.

Definition 2.7.3. Fix a type of structure given by the pair (X, E) . A *fully extended n -dimensional (X, E) -topological field theory* is a symmetric monoidal functor of (∞, n) -categories with source $\text{Bord}_n^{(X, E)}$.

In particular, the most interesting cases are the following:

Definition 2.7.4. A *fully extended n -dimensional framed topological field theory* is a symmetric monoidal functor of (∞, n) -categories with source $\text{Bord}_n^{\text{fr}}$.

Definition 2.7.5. A *fully extended n -dimensional oriented topological field theory* is a symmetric monoidal functor of (∞, n) -categories with source $\text{Bord}_n^{\text{or}}$.

Remark 2.7.6. We will sometimes be imprecise when specifying the type of fully extended TFT. From now on, if we do not specify explicitly that it is unoriented or oriented, we will usually mean that it is framed.

2.7.2 n -TFT yields k -TFT

We will see that every fully extended n -dimensional (unoriented, oriented, framed) TFT yields a fully extended k -dimensional (unoriented, oriented, framed) TFT for any $k \leq n$ by truncation from subsection 1.5.1.

Note that for $k < n$, we have a map of k -fold Segal spaces

$$\text{PBord}_k \longrightarrow \tau_k(\text{PBord}_n) = (\text{PBord}_n)_{\underbrace{\bullet, \dots, \bullet}_{k \text{ times}}, \underbrace{0, \dots, 0}_{n-k \text{ times}}}$$

induced by sending $(M \hookrightarrow V \times (0, 1)^k, (I_j^i)_{i=1}^k) \in \text{PBord}_k$ to

$$(M \times (0, 1)^{n-k} \hookrightarrow V \times (0, 1)^n, (I_j^i)_{i=1}^k, (0, 1), \dots, (0, 1)).$$

The completion map $\text{PBord}_n \rightarrow \text{Bord}_n$ induces a map on the truncations. Precomposition with the above map yields a map of (in general non-complete) n -fold Segal spaces

$$\text{PBord}_k \longrightarrow \tau_k(\text{PBord}_n) \longrightarrow \tau_k(\text{Bord}_n).$$

Recall from 1.5.1 that since $\tau_k(\text{Bord}_n)$ is complete, by the universal property of the completion we obtain a map $\text{Bord}_k \rightarrow \tau_k(\text{Bord}_n)$. This ensures that any fully extended n -dimensional (unoriented, oriented, framed) TFT with values in a complete n -fold Segal space \mathcal{C} , $\text{Bord}_n \rightarrow \mathcal{C}$ leads to a k -dimensional (unoriented, oriented, framed) TFT given by the composition

$$\text{Bord}_k \longrightarrow \tau_k(\text{Bord}_n) \longrightarrow \tau_k(\mathcal{C})$$

with values in the complete k -fold Segal space $\tau_k(\mathcal{C})$.

2.7.3 Cobordism Hypothesis à la Baez-Dolan-Lurie and outlook

In his seminal paper [Lur09c], Lurie gave a detailed sketch of proof of the *Cobordism Hypothesis*, which in its simplest form says that a fully extended framed TFT is fully determined by its value at the object given by a point which will be denoted by $*$. Conversely, any object in the target category which satisfies a suitable finiteness condition can be obtained in this way. The finiteness condition in question is called *fully dualizability*, which we will not explain here. For a full definition, we refer to [Lur09c].

Theorem 2.7.7 (Cobordism Hypothesis, [Lur09c] Theorem 1.4.9). *Let \mathcal{C} be a symmetric monoidal (∞, n) -category. The evaluation functor $Z \mapsto Z(*)$ determines a bijection between (isomorphism classes of) symmetric monoidal functors $\text{Bord}_n^{\text{fr}} \rightarrow \mathcal{C}$ and (isomorphism classes of) fully dualizable objects of \mathcal{C} .*

Thus to construct a fully extended n -dimensional framed TFT, it suffices to find a fully dualizable object in the target \mathcal{C} , and the cobordism hypothesis does the rest for us. However, fully dualizability is a condition which in general is not completely straightforward to check. Moreover, even though the proof of the cobordism hypothesis tells you that the (∞, n) -category Bord_n of cobordisms is freely generated by the point, it does not give you a simple algorithm with which one can compute all values of the fully extended n -TFT.

Our goal in this thesis is precisely this, namely, for a very special fully extended TFT, to explicitly construct it without invoking the cobordism hypothesis. In the next chapter we will construct our target, a symmetric monoidal (∞, n) -category Alg_n of E_n -algebras, and in the last chapter we will, given any object A in Alg_n , build a fully extended n -TFT by defining a strict functor of n -fold Segal spaces

$$\mathcal{FH}_n(A) : \text{Bord}_n^{\text{fr}} \longrightarrow \text{Alg}_n,$$

whose evaluation at the point is A . By the cobordism hypothesis, this in particular shows that any object in Alg_n is fully dualizable.

The Morita (∞, n) -category of E_n -algebras

In this chapter, we define the target category for our fully extended n -dimensional topological field theory, which is a symmetric monoidal Morita (∞, n) -category $\text{Alg}_n = \text{Alg}_n(\mathcal{S})$ of E_n -algebras. By an E_n -algebra, we mean an E_n -algebra object in a suitable symmetric monoidal $(\infty, 1)$ -category \mathcal{S} . In [Lur], Lurie proved that there is an equivalence of $(\infty, 1)$ -categories between E_n -algebras and locally constant factorization algebras on $(0, 1)^n \xrightarrow{\simeq} \mathbb{R}^n$, see theorem 3.2.20. We will use this equivalence to define the objects of our (∞, n) -category of E_n -algebras as a suitable space of locally constant factorization algebras on $(0, 1)^n$. As (higher) morphisms we essentially use factorization algebras which are locally constant with respect to a certain stratification to model the Morita category of E_n -algebras as a complete n -fold Segal space $\text{Alg}_n = \text{Alg}_n(\mathcal{S})$. Informally speaking, it Alg_n is the (∞, n) -category with E_n -algebras as objects, pointed (A, B) -bimodules in E_{n-1} -algebras as 1-morphisms in $\text{Hom}(A, B)$, and so on.

For the existence of factorization algebras we need the following assumption on \mathcal{S} .

Assumption 1. *Let \mathcal{S} be a symmetric monoidal $(\infty, 1)$ -category which admits all small colimits.*

3.1 The n -fold Segal space of closed covers in $(0, 1)$

In this section, we construct a (1-)fold Segal space Covers_\bullet of covers of $(0, 1)$ by closed intervals, which we will later enhance by suitable spaces of factorization algebras to give the desired complete n -fold Segal space of E_n -algebras. Before we begin with its construction, we introduce a family of collapse-and-rescale maps ϱ_a^b which will be used to define the simplicial structure.

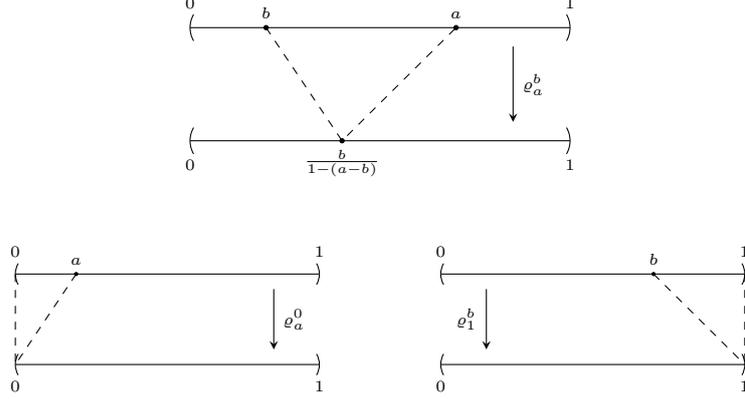
3.1.1 Collapse-and-rescale maps

We first define collapse-and-rescale maps $\varrho_a^b : [0, 1] \rightarrow [0, 1]$ which delete the interval $(b, a]$ and rescale the rest back to $[0, 1]$.

Definition 3.1.1. Let $0 \leq b, a \leq 1$ such that $(b, a) \neq (0, 1)$. If $a \leq b$, let $\varrho_a^b = \text{id}_{[0, 1]}$. If

$b < a$, let $\varrho_a^b : [0, 1] \rightarrow [0, 1]$,

$$\varrho_a^b(x) = \begin{cases} \frac{x}{1-(a-b)}, & x \leq b, \\ \frac{b}{1-(a-b)}, & b \leq x \leq a, \\ \frac{x-(a-b)}{1-(a-b)}, & a \leq x. \end{cases}$$



To simplify notation, we define the following composition of collapse-and-rescaling maps.

Definition 3.1.2. Let $0 \leq d, c, b, a \leq 1$. Then let

$$\varrho_c^d * \varrho_a^b = \varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b.$$

Remark 3.1.3. Note that if $(b, a) \subseteq (d, c)$, $\varrho_c^d * \varrho_a^b = \varrho_c^d$.

The following lemma shows that if the intervals (d, c) and (b, a) are disjoint, the composition of the respective collapse-and-rescale maps is independent of order in which we delete and rescale and so is determined by the data of the intervals which are collapsed.

Lemma 3.1.4. Let $0 \leq d, c, b, a \leq 1$ such that $(d, c) \neq (0, 1) \neq (b, a)$. Furthermore, let $(d, c) \cap (b, a) = \emptyset$. Then

$$\varrho_c^d * \varrho_a^b = \varrho_a^b * \varrho_c^d.$$

Moreover, if $b = c$ or $a = d$, the above composition is equal to $\varrho_{\max(b, a)}^{\min(d, c)}$.

Proof. Note that ϱ_a^b and ϱ_c^d are monotonically increasing and piecewise linear functions. We first consider the cases in which one of the functions in the composition is the identity.

1. If $d \geq c$, $\varrho_a^b(d) \geq \varrho_a^b(c)$ and so $\varrho_c^d = id = \varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)}$. Thus,

$$\varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b = \varrho_a^b = \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d.$$

If $b = c$, $\varrho_{\max(b, a)}^{\min(d, c)} = \varrho_{\max(b, a)}^c = \varrho_{\max(b, a)}^b = \varrho_a^b$, since if $\max(b, a) \neq a$, $a \leq b$, and $\varrho_a^b = id = \varrho_b^b$.

2. If $b \geq a$, similarly, $\varrho_a^b = id = \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)}$ and

$$\varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b = \varrho_c^d = \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d.$$

If $b = c$, $\varrho_{\max(b,a)}^{\min(d,c)} = \varrho_b^{\min(d,c)} = \varrho_c^{\min(d,c)} = \varrho_c^d$, since if $\min(d, c) \neq d$, $c \leq d$, and $\varrho_c^d = id = \varrho_c^c$.

Since ϱ_a^b and ϱ_c^d are piecewise linear functions their composition again is piecewise linear. Thus in the remaining case it suffices to compute their value at the “break points”. The computation of the composition in between the break points is essentially the same so we include it as well.

3. In the remaining case we can assume wlog that $c \leq b$ and thus $d < c \leq b < a$. This implies that

$$\begin{aligned} \varrho_a^b(d) &= \frac{d}{1-(a-b)}, & \varrho_a^b(c) &= \frac{c}{1-(a-b)}, \\ \varrho_c^d(b) &= \frac{b-(c-d)}{1-(c-b)}, & \varrho_c^d(b) &= \frac{b-(c-d)}{1-(c-b)}. \end{aligned}$$

If $x \leq d$,

$$\begin{aligned} \varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b(x) &= \frac{x}{1-(a-b)} \frac{1}{1-\frac{c-d}{1-(a-b)}} \\ &= \frac{x}{1-(a-b)-(c-d)} \\ &= \frac{x}{1-(c-d)} \frac{1}{1-\left(\frac{a-(c-d)}{1-(c-d)} - \frac{b-(c-d)}{1-(c-d)}\right)} \\ &= \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d(x). \end{aligned}$$

If $d \leq x \leq c$,

$$\varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b(x) = \frac{d}{1-(a-b)-(c-d)} = \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d(x).$$

If $c \leq x \leq b$,

$$\begin{aligned} \varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b(x) &= \frac{\frac{x}{1-(a-b)} - \frac{c-d}{1-(a-b)}}{1-\frac{c-d}{1-(a-b)}} \\ &= \frac{x-(c-d)}{1-(a-b)-(c-d)} \\ &= \frac{\frac{x-(c-d)}{1-(c-d)}}{1-\left(\frac{a-(c-d)}{1-(c-d)} - \frac{b-(c-d)}{1-(c-d)}\right)} \\ &= \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d(x). \end{aligned}$$

If $b \leq x \leq a$,

$$\varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b(x) = \frac{b - (c - d)}{1 - (a - b) - (c - d)} = \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d(x).$$

If $a \leq x$,

$$\begin{aligned} \varrho_{\varrho_a^b(c)}^{\varrho_a^b(d)} \circ \varrho_a^b(x) &= \frac{\frac{x-(a-b)}{1-(a-b)} - \frac{c-d}{1-(a-b)}}{1 - \frac{c-d}{1-(a-b)}} \\ &= \frac{x - (a - b) - (c - d)}{1 - (a - b) - (c - d)} \\ &= \frac{\frac{x-(c-d)}{1-(c-d)} - \left(\frac{a-(c-d)}{1-(c-d)} - \frac{b-(c-d)}{1-(c-d)} \right)}{1 - \left(\frac{a-(c-d)}{1-(c-d)} - \frac{b-(c-d)}{1-(c-d)} \right)} \\ &= \varrho_{\varrho_c^d(a)}^{\varrho_c^d(b)} \circ \varrho_c^d(x). \end{aligned}$$

If $b = c$,

$$\frac{d}{1 - (a - b) - (c - d)} = \frac{d}{1 - (a - d)} = \frac{b - (c - d)}{1 - (a - b) - (c - d)},$$

so the composition reduces to

$$\varrho_a^d = \varrho_{\max(b,a)}^{\min(d,c)}.$$

□

In the following, since the intervals we consider lie in $(0, 1)$ we often use the restriction of ϱ_a^b to the domain $D(\varrho_a^b)$, which is defined as follows.

Definition 3.1.5. Let $0 \leq b, a \leq 1$ such that $(b, a) \neq (0, 1)$. Let

$$D(\varrho_a^b) = \begin{cases} (0, 1), & 0 \leq b, a \leq 1, \\ (a, 1), & b = 0, \\ (0, b), & a = 1. \end{cases}$$

We might like to restrict to an even smaller domain to get a partial inverse.

Definition 3.1.6. The restriction of the collapse-and-rescale map ϱ_a^b to $D_a^b = (0, b] \cup (a, 1) \subset D(\varrho_a^b)$,

$$\varrho_a^b|_{(0,b] \cup (a,1)} : D_a^b \longrightarrow (0, 1),$$

is injective. We call D_a^b its *domain of injectivity*. In the following, let $(\varrho_a^b)^{-1}$ be the inverse of this restriction, $(\varrho_a^b|_{D_a^b})^{-1} : (0, 1) \rightarrow D_a^b$.

3.1.2 The sets Covers_k

We first define 0-simplices of the levels Covers_k as sets.

Definition 3.1.7. For an integer $k \geq 0$ let

$$\text{Covers}_k = \{I_0 \leq \dots \leq I_k\}$$

be the set consisting of ordered $(k + 1)$ -tuples of intervals $I_j \subseteq (0, 1)$ such that I_j has non-empty interior, is closed in $(0, 1)$ and $\bigcup_{j=0}^k I_j = (0, 1)$. As in the definition of Int_k in 2.1, by “ordered” we mean that the left endpoints, denoted by a_j , and the right endpoints, denoted by b_j , are ordered.

Remark 3.1.8. Note that the condition that the intervals form a cover, $\bigcup_{j=0}^k I_j = (0, 1)$, implies that $a_0 = 0$ and $b_k = 1$.

3.1.3 The spatial structure of Covers_k

The l -simplices of the space Covers_k

Similarly to the definition of Int_k , the l -simplices of Covers_k do not just consist of the data of a smooth family of covers. We add a rescaling datum which remembers how the intervals were deformed.

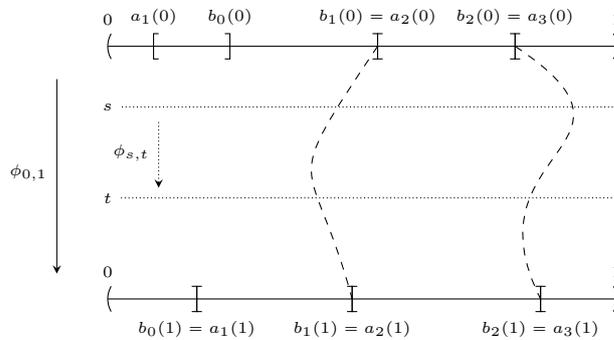
Definition 3.1.9. Let $(I_0(s) \leq \dots \leq I_k(s)) \in \text{Covers}_k$ be a smooth family of covers by closed intervals over $|\Delta^l|$, i.e. denoting the left endpoints by $a_j(s)$ and the right endpoints by $b_j(s)$, the maps $|\Delta^l| \rightarrow \mathbb{R}$, $s \mapsto a_j(s), b_j(s)$ are smooth maps. A smooth family of orientation-preserving diffeomorphisms

$$(\phi_{s,t} : (0, 1) \rightarrow (0, 1))_{s,t \in |\Delta^l|}$$

is said to *intertwine with the composed covers* if the following condition is satisfied for every morphism $f : [m] \rightarrow [l]$ in the simplex category Δ .

Let $|f| : |\Delta^m| \rightarrow |\Delta^l|$ be the induced map between standard simplices. The diffeomorphism $\phi_{s,t}$ should send the common endpoint at s to the corresponding endpoint at t for intervals which do not overlap along $|f|(|\Delta^m|)$. Explicitly, this means that for every $0 \leq j < k$ such that for every $s \in |f|(|\Delta^m|)$ the intersection $I_j(s) \cap I_{j+1}(s)$ consists of exactly one element, namely $b_j(s) = a_{j+1}(s)$, we require that for every $s \in |f|(|\Delta^m|)$,

$$b_j(s) = a_{j+1}(s) \xrightarrow{\phi_{s,t}} b_j(t) = a_{j+1}(t),$$



Remark 3.1.10. Note that as with Int_k , it is enough to check this condition for $m \leq l$.

Definition 3.1.11. An l -simplex of Covers_k consists of

1. a smooth family of *underlying 0-simplices*, i.e. for every $s \in |\Delta^l|$,

$$(I_1(s) \leq \dots \leq I_k(s)) \in \text{Covers}_k,$$

depending smoothly on s ;

2. a *rescaling datum*, which is a collection of strictly monotonically increasing homeomorphisms

$$(\phi_{s,t} : (0, 1) \rightarrow (0, 1))_{s,t \in |\Delta^l|}$$

such that

- a) $\phi_{s,s} = id$ for every $s \in |\Delta^l|$,
- b) $\phi_{s,t} = \phi_{t,s}^{-1}$ for every $s, t \in |\Delta^l|$, and
- c) $(\phi_{s,t})_{s,t \in |\Delta^l|}$ intertwines with the composed covers.

Remark 3.1.12. Note that in particular for $l = 0$ an l -simplex in this sense is an underlying 0-simplex together with $\phi_{s,s} = id : (0, 1) \rightarrow (0, 1)$, so, by abuse of language we call both a 0-simplex.

The space Covers_k

The spatial structure arises similarly to that on Int_k .

Fix $k \geq 0$ and let $f : [m] \rightarrow [l]$ be a morphism in the simplex category Δ and $|f| : |\Delta^m| \rightarrow |\Delta^l|$ the induced map between standard simplices. Then let f^* be the map sending an l -simplex in Covers_k given by

$$(I_0(s) \leq \dots \leq I_k(s))_{s \in |\Delta^l|}, \quad (\phi_{s,t} : (0, 1) \rightarrow (0, 1))_{s,t \in |\Delta^l|}$$

to

$$I_0(|f|(s)) \leq \dots \leq I_k(|f|(s))_{s \in |\Delta^m|}, \quad (\phi_{|f|(s), |f|(t)} : (0, 1) \rightarrow (0, 1))_{s \in |\Delta^m|}.$$

Analogous to lemma 2.1.7 we have

Lemma 3.1.13. *This gives a functor $\Delta^{op} \rightarrow \text{Set}$ and thus Covers_k is a space, i.e. a simplicial set.*

Notation 3.1.14. We denote the spatial face and degeneracy maps by δ_j^Δ and σ_j^Δ for $0 \leq j \leq l$.

We will need the following lemma later for the Segal condition. Its proof is similar to that of lemma 2.1.9

Lemma 3.1.15. *Each level Covers_k is contractible.*

3.1.4 The simplicial set Covers_\bullet .

In this section, we make the collection of sets Covers_\bullet (ignoring the spatial structure we just constructed) into a simplicial set by defining degeneracy and face maps, which use the family of collapse-and-rescale maps $\varrho_a^b : (0, 1) \rightarrow (0, 1)$ defined in subsection 3.1.1.

Definition 3.1.16. The j th degeneracy map is given by inserting the j th interval twice,

$$\begin{aligned} \text{Covers}_k &\xrightarrow{\sigma_j} \text{Covers}_{k+1}, \\ I_0 \leq \cdots \leq I_k &\longmapsto I_0 \leq \cdots \leq I_j \leq I_j \leq \cdots \leq I_k. \end{aligned}$$

The j th face map is given by deleting the j th interval, collapsing what now is not covered, and rescaling the rest linearly to $(0, 1)$. Explicitly,

$$\begin{aligned} \text{Covers}_k &\xrightarrow{\delta_j} \text{Covers}_{k-1}, \\ I_0 \leq \cdots \leq I_k &\longmapsto \varrho_{a_{j+1}}^{b_{j-1}}(I_0) \cap (0, 1) \leq \cdots \leq \widehat{\varrho_{a_{j+1}}^{b_{j-1}}(I_j)} \leq \cdots \leq \varrho_{a_{j+1}}^{b_{j-1}}(I_k) \cap (0, 1), \end{aligned}$$

where $\varrho_{a_{j+1}}^{b_{j-1}}$ is the collapse-and-rescale map associated to b_{j-1}, a_{j+1} from the previous section.

Proposition 3.1.17. Covers_\bullet is a simplicial set.

Proof. We need to show that the simplicial relations are satisfied. Two conditions are obviously fulfilled, namely $\sigma_l \sigma_j = \sigma_{j+1} \sigma_l$ for $l \leq j$ and

$$\delta_l \sigma_j = \begin{cases} id, & l = j, j + 1, \\ \sigma_{j-1} \delta_l, & l < j, \\ \sigma_j \delta_{l-1} & l > j + 1. \end{cases}$$

It remains to check that

$$\delta_j \delta_l = \delta_{l-1} \delta_j \quad \text{for } j < l.$$

Let $I_0 \leq \cdots \leq I_k$ be an element in Covers_k . Since the same intervals are deleted in both compositions, it is enough to show that the compositions of the respective collapse-and-rescale maps coincide on both sides. This follows from lemma 3.1.4 with

$$d = b_{j-1}, \quad c = a_{j+1}, \quad b = b_{l-1}, \quad a = a_{l+1},$$

given that $(b_{j-1}, a_{j+1}) \cap (b_{l-1}, a_{l+1}) = \emptyset$, which requires that

$$a_{j+1} = c \leq b = b_{l-1}.$$

Assume the opposite, that is, that $b_{l-1} \leq a_{j+1}$. By definition, $a_{j+1} < b_{j+1} \leq b_\alpha$ for $\alpha > j$, so this implies that $l - 1 \leq j$. Since we need to check the identity for $j < l$, this implies that $l = j + 1$. The intervals $(I_j)_j$ must form a cover of $(0, 1)$, so $b_{l-1} \geq a_l = a_{j+1}$ and therefore $a_{j+1} = b_{l-1}$. So in any case

$$a_{j+1} = c \leq b = b_{l-1}.$$

□

3.1.5 The Segal space Covers_\bullet

Face and degeneracy maps on l -simplices

We first need to extend the (simplicial) face and degeneracy maps δ_j, σ_j to l -simplices in a compatible way. They essentially arise from applying the face and degeneracy maps δ_j, σ_j to each of the 0-simplices underlying the l -simplex.

Notation 3.1.18. Let

$$\left((I_0(s) \leq \cdots \leq I_k(s))_{s \in |\Delta^l|}, \quad (\phi_{s,t})_{s,t \in |\Delta^l|} \right)$$

be an l -simplex of Covers_k . For $s \in |\Delta^l|$, denote by $\varrho_{a_{j+1}}^{b_{j-1}}(s) = \varrho_{a_{j+1}(s)}^{b_{j-1}(s)}$ the collapse-and-rescale map associated to the s th underlying 0-simplex $(I_0(s) \leq \cdots \leq I_k(s))$ of the above l -simplex, and by $D_a^b(s) = (0, b_{j-1}(s)] \cup (a_{j+1}(s), 1)$ its domain of injectivity.

Degeneracy maps on l -simplices For $0 \leq j \leq k$ the j th degeneracy map σ_j sends an l -simplex of Covers_k

$$\left((I_0(s) \leq \cdots \leq I_k(s))_{s \in |\Delta^l|}, \quad (\phi_{s,t})_{s,t \in |\Delta^l|} \right)$$

to the l -simplex of Covers_{k+1} given by

$$\left((\sigma_j(I_0(s) \leq \cdots \leq I_k(s)))_{s \in |\Delta^l|}, \quad (\phi_{s,t})_{s,t \in |\Delta^l|} \right)$$

This is well-defined, since the conditions on the $\phi_{s,t}$ stays the same.

Face maps on l -simplices For $0 \leq j \leq k$ the j th face map δ_j sends an l -simplex of Covers_k

$$\left((I_0(s) \leq \cdots \leq I_k(s))_{s \in |\Delta^l|}, \quad (\phi_{s,t})_{s,t \in |\Delta^l|} \right)$$

to the following l -simplex of Covers_{k-1} .

1. The underlying 0-simplices of the image are the images of the underlying 0-simplices under δ_j , i.e. for $s \in |\Delta^l|$,

$$\delta_j(I_0(s) \leq \cdots \leq I_k(s));$$

2. Its rescaling datum is

$$\delta_j(\phi_{s,t}) = \varrho_{a_{j+1}}^{b_{j-1}}(t) \circ \phi_{s,t}|_{D_a^b(s)} \circ \varrho_{a_{j+1}}^{b_{j-1}}(s)^{-1} : (0, 1)^n \rightarrow (0, 1)^n.$$

The complete Segal space Covers_\bullet

Proposition 3.1.19. Covers_\bullet is a complete Segal space.

Proof. That the simplicial and spatial face and degeneracy maps commute follows directly from the definition. Furthermore, we have seen in lemma 2.1.9 that every Covers_k is contractible. This ensures the Segal condition, namely that

$$\text{Covers}_k \xrightarrow{\cong} \text{Covers}_1 \times_{\text{Covers}_0}^h \cdots \times_{\text{Covers}_0}^h \text{Covers}_1,$$

and completeness. \square

Definition 3.1.20. Let

$$\text{Covers}_{\bullet, \dots, \bullet}^n = (\text{Covers}_{\bullet})^{\times n}.$$

Lemma 3.1.21. The n -fold simplicial space $\text{Covers}_{\bullet, \dots, \bullet}^n$ is a complete n -fold Segal space.

Proof. The Segal condition and completeness follow from the Segal condition and completeness for Covers_{\bullet} . Since every Covers_k is contractible by lemma 2.1.9, $(\text{Covers}_{\bullet})^{\times n}$ satisfies essential constancy, so Covers^n is an n -fold Segal space. \square

3.2 The Morita (∞, n) -category of E_n -algebras Alg_n

This section contains the main construction of the complete n -fold Segal space $\text{Alg}_n = \text{Alg}_n(\mathcal{S})$. We first recall the definition of an E_n -algebra.

3.2.1 Structured disks and E_n -algebras

As in section 2.6.1, let X be a topological space and $E \rightarrow X$ a topological n -dimensional vector bundle which corresponds to a (homotopy class of) map(s) $e : X \rightarrow \text{BGL}(\mathbb{R}^n)$ from X to the classifying space of the topological group $\text{GL}(\mathbb{R}^n)$.

Definition 3.2.1. The symmetric monoidal topological category $\text{Disk}_n^{(X, E)}$ of (X, E) -structured disks is the full topological subcategory of $\text{Man}_n^{(X, E)}$ whose objects are disjoint unions of (X, E) -structured n -dimensional Euclidean disks \mathbb{R}^n .

Example 3.2.2. Recall from section 2.6.1 that interesting examples of (X, E) -structures arise from a topological group G together with a continuous homomorphism $e : G \rightarrow O(n)$ by setting $X = BG$ and $e : BG \rightarrow \text{BGL}(\mathbb{R}^n)$. In this case, we refer to (BG, e) -structured disks as G -structured disks and use the notation $\text{Disk}_n^G = \text{Disk}_n^{(BG, e)}$.

Definition 3.2.3. Let \mathcal{S} be a symmetric monoidal $(\infty, 1)$ -category. The $(\infty, 1)$ -category $\text{Disk}_n^{(X, E)}\text{-Alg}(\mathcal{S})$ of $\text{Disk}_n^{(X, E)}$ -algebras is the $(\infty, 1)$ -category of symmetric monoidal functors $\text{Fun}^{\otimes}(\text{Disk}_n^{(X, E)}, \mathcal{S})$.

Remark 3.2.4. Recall from section 1.2 that topological categories are a model for $(\infty, 1)$ -categories. By perhaps changing to a different, suitable, model of $(\infty, 1)$ -categories, the above definition makes sense.

The most common examples are the following three special cases.

Example 3.2.5. If G is the trivial group, then $X = BG = *$, and the topological category Disk_n^G is denoted by $\text{Disk}_n^{\text{fr}}$. Using the fixed diffeomorphism $\chi : (0, 1) \cong \mathbb{R}$ it is equivalent to the topological category Cube_n whose objects are disjoint unions of $(0, 1)^n$ and whose spaces of morphisms are the spaces of embeddings $\coprod_I (0, 1)^n \rightarrow \coprod_J (0, 1)^n$ which are rectilinear on every connected component. As Cube_n -algebras are equivalent to E_n -algebras, the category $\text{Disk}_n^{\text{fr}}\text{-Alg}(\mathcal{S})$ is equivalent to the usual category of E_n -algebras in \mathcal{S} .

Remark 3.2.6. Note that morphisms in the category $\text{Disk}_n^{\text{fr}}\text{-Alg}(\mathcal{S})$ are morphisms of E_n -algebras, i.e. natural transformations of functors. In the Morita-category we will construct in this section morphisms will be bimodules of E_n -algebras.

Example 3.2.7. If $G = O(n)$, the topological category Disk_n^G is denoted by Disk_n^{un} . We call Disk_n^{un} -algebras *unoriented E_n -algebras*. Similarly to in the previous example Disk_n^{un} is equivalent to the topological category Cube_n^{un} whose objects are disjoint unions of $(0, 1)^n$ and whose spaces of morphisms are the spaces $\text{Cube}_n(\coprod_I (0, 1)^n, \coprod_J (0, 1)^n) \rtimes O(n)^{\times J}$.

Example 3.2.8. If $G = SO(n)$ and $X = BG$, the topological category $\text{Disk}_n^{(X, E)}$ is denoted by Disk_n^{or} . We call $\text{Disk}_n^{(X, E)}$ -algebras *oriented E_n -algebras*. Again similarly to above Disk_n^{or} is equivalent to the topological category Cube_n^{or} whose objects are disjoint unions of $(0, 1)^n$ and whose spaces of morphisms are the spaces $\text{Cube}_n(\coprod_I (0, 1)^n, \coprod_J (0, 1)^n) \rtimes SO(n)^{\times J}$.

3.2.2 Factorization algebras

We recall the definition of factorization algebras and basic facts. Factorization algebras were first introduced by Beilinson-Drinfeld in the algebro-geometric context for curves in [BD04]. Inspired by this, Lurie first coined the term factorizing cosheaf in [Lur]. They were further studied and developed in [CG], [GTZ10], and others. We mainly follow the conventions in [Gin]. The other main reference is [CG]. Note that all our factorization algebras will be non-lax homotopy (i.e. derived) factorization algebras in the language of [CG]. For simplicity, let us assume that \mathcal{S} is given by a relative category $(\mathcal{S}, \mathcal{W})$.

Definition 3.2.9. Let X be a topological space. A *prefactorization algebra on X with values in \mathcal{S}* is a functor

$$\mathcal{F} : \text{Open}(X) \longrightarrow \mathcal{S}$$

together with structure maps

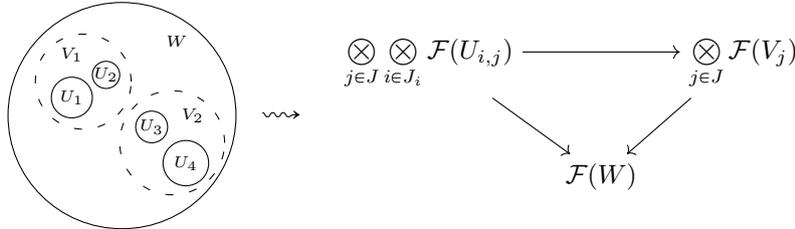
$$f_{U_1 \sqcup \dots \sqcup U_n \subseteq V} : \mathcal{F}(U_1) \otimes \dots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V)$$

for every finite disjoint union of open sets U_i (independent of the ordering of the U_i) included in another open set V such that

1. for $U \subseteq V$,

$$\mathcal{F}(U \subseteq V) = f_{U \subseteq V},$$

2. the following coherence (or associativity) condition is satisfied: for any finite collection of pairwise disjoint open subsets $(V_j)_{j \in J}$ lying in an open set W and for every $j \in J$, a finite collection of open subsets $(U_{i,j})_{i \in J_i}$ lying in V_j , the following diagram induced by the structure maps commutes:



Remark 3.2.10. The inclusion of the empty set induces map $\mathcal{F}(\emptyset) \rightarrow \mathcal{F}(U)$ for every open set U . Thus prefactorization algebras are pointed objects.

Remark 3.2.11. Prefactorization algebras on X are algebras over the colored operad with open sets in X as colors and

$$\text{PreFact}_X(U_1, \dots, U_n; V) = \begin{cases} \{*\} & \text{if } U_1 \amalg \dots \amalg U_n \subseteq V; \\ \emptyset & \text{otherwise.} \end{cases}$$

The ∞ -operad associated to this operad was denoted by $N(\text{Disk}(X))$ in [Lur].

Definition 3.2.12. Let \mathcal{U} be an open cover of U . Then the Čech complex of \mathcal{U} with values in a prefactorization algebra \mathcal{F} on U is the simplicial object in \mathcal{S} constructed as follows:

Let $P\mathcal{U}$ be the set of finite pairwise disjoint open subsets $\{U_1, \dots, U_n : U_i \in \mathcal{U}, n \geq 0\}$. For $i \geq 0$, let

$$\check{C}^i(\mathcal{U}, \mathcal{F}) = \bigoplus_{\alpha \in P\mathcal{U}^{i+1}} \bigotimes_{U_j \in \alpha_j} \mathcal{F} \left(\bigcap_{j=0}^i U_j \right).$$

To define the face maps for $0 \leq j_0 \leq i$

$$\partial_{j_0} : \check{C}^{i+1}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^i(\mathcal{U}, \mathcal{F}),$$

consider the structure maps of the inclusions of open sets

$$\bigcap_{j=0}^i U_j \hookrightarrow \bigcap_{\substack{k=0 \\ k \neq j_0}}^i U_j.$$

Then the j_0 th face map is given by the direct sum of their tensor products

$$\bigotimes_{U_j \in \alpha_j} \mathcal{F} \left(\bigcap_{j=0}^i U_j \right) \longrightarrow \bigotimes_{\substack{U_k \in \alpha_k, \\ k \neq j_0}} \mathcal{F} \left(\bigcap_{\substack{k=0 \\ k \neq j_0}}^i U_j \right).$$

Degeneracy maps are given by repeating one of the α_j 's. Summarizing we obtain a simplicial diagram in \mathcal{S} ,

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) = \left(\bigoplus_{\alpha \in P\mathcal{U}} \bigotimes_{U \in \alpha} \mathcal{F}(U) \rightrightarrows \bigoplus_{\alpha \in P\mathcal{U}^2} \bigotimes_{U_j \in \alpha_j} \mathcal{F}(U_0 \cap U_1) \rightrightarrows \dots \right).$$

Remark 3.2.13. Note that the structure maps associated to the inclusions $\bigcap_{j=0}^i U_j \subseteq U$ give maps $\mathcal{F}(\bigcap_{j=0}^i U_j) \rightarrow \mathcal{F}(U)$ and thus $\check{C}^i(\mathcal{U}, \mathcal{F}) \rightarrow \mathcal{F}(U)$ which commute with the simplicial maps by coherence. We obtain a map of simplicial objects in \mathcal{S}

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{F}(U).$$

Definition 3.2.14. An open cover \mathcal{U} of U is said to be *factorizing* if for every finite set of pairwise distinct points $\{x_1, \dots, x_n\} \subseteq U$ there is a finite subset $U_1, \dots, U_k \subseteq \mathcal{U}$ of pairwise disjoint open sets such that

$$\{x_1, \dots, x_n\} \subseteq U_1 \amalg \dots \amalg U_k.$$

A (homotopy) factorization algebra on X with values in \mathcal{S} is prefactorization algebra on X with values in \mathcal{S} for which the following *gluing condition* holds: for every open $U \subseteq X$ and every factorizing cover \mathcal{U} of U the map

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{F}(U)$$

is a weak equivalence.

Remark 3.2.15 (Unitality). If \mathcal{F} is a factorization algebra on X with values in \mathcal{S} , then $\mathcal{F}(\emptyset) \simeq \mathbb{1}$ the monoidal unit in \mathcal{S} : Consider the Čech complex for the empty cover $\mathcal{U} = \emptyset$ of $U = \emptyset$. There is exactly one $\alpha \in P\mathcal{U}$, namely $\alpha = \emptyset$ and thus every factor $\bigotimes_{U_j \in \alpha_j} \mathcal{F}\left(\bigcap_{j=0}^i U_j\right) = \bigotimes_{\emptyset} = \mathbb{1}$. Thus the gluing condition requires the map

$$(\cdots \rightrightarrows \mathbb{1} \rightrightarrows \mathbb{1}) \longrightarrow \mathcal{F}(\emptyset)$$

to be a weak equivalence.

Remark 3.2.16 (Comparison with [CG]). Let $U = U_1 \amalg \cdots \amalg U_n$. Then $\mathcal{U} = \{U_1, \dots, U_n\}$ is a factorizing open cover. Then gluing condition implies that the structure map

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \xrightarrow{\simeq} \mathcal{F}(U). \quad (3.1)$$

Thus we do not consider lax factorization algebras in the sense of [CG].

Moreover, they require a gluing condition for covers with a stronger property, namely being a *Weiss cover*, named after Michael Weiss. However, every factorizing open cover \mathcal{U} generates a Weiss open cover $\mathcal{V} = \{U_1 \amalg \cdots \amalg U_n : U_i \in \mathcal{U}\}$. Then the gluing condition for Weiss covers together with (3.1) is equivalent to the gluing property in our sense.

Definition 3.2.17. A *morphism of prefactorization algebras* $\mathcal{F} \rightarrow \mathcal{G}$ is a collection of maps

$$\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

for every open set U which commutes with the structure maps. Since \mathcal{S} was an $(\infty, 1)$ -category this gives a symmetric monoidal $(\infty, 1)$ -category of prefactorization algebras: e.g. if we start with a relative category $(\mathcal{S}, \mathcal{W})$, then define weak equivalences of prefactorization algebras to be the morphisms with values in \mathcal{W} . The symmetric monoidal structure is defined level-wise.

The category $\mathcal{F}act_X(\mathcal{S})$ of factorization algebras on X with values in \mathcal{S} is the symmetric monoidal full sub- $(\infty, 1)$ -category of the $(\infty, 1)$ -category of prefactorization algebras on X with values in \mathcal{S} .

3.2.3 Stratifications and locally constant factorization algebras

The full definition of locally constant factorization algebras on a (stratified) space can be found in [Gin]. In this paper, we will only deal with stratifications of a very special type, so we recall the definition in an easier setting suitable for the factorization algebras appearing in this thesis here.

Definition 3.2.18. Let X be an n -dimensional manifold. By a *stratification of X* we mean a filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X,$$

where X_α is an α -dimensional closed submanifold of $X_{\alpha+1}$. The connected components of $X_\alpha \setminus X_{\alpha-1}$ are called the *dimension α -strata of X* . An open disk D in X is said to have *index α* , if $D \cap X_\alpha \neq \emptyset$ and $D \subset X \setminus X_{\alpha-1}$. We say that a disk D is a *good neighborhood at X_α* if α is the index of D and $D \cap (X_\alpha \setminus X_{\alpha-1})$ is connected.

Definition 3.2.19. Let $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \dots \subset X_n = X$ be a stratification of an n -dimensional manifold X . A factorization algebra \mathcal{F} on X is called *locally constant with respect to the stratification* if for any inclusion of disks $U \hookrightarrow V$ such that both U and V are good neighborhoods at X_α for the same index $\alpha \in \{0, \dots, n\}$, the structure map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is a weak equivalence.

A factorization algebra \mathcal{F} on X is called *locally constant* if it is locally constant with respect to the stratification given by $X_\alpha = \emptyset$ for every $\alpha \neq n$, i.e.

$$\emptyset \subset X.$$

E_n -algebras as locally constant factorization algebras

We will base our construction on factorization algebras which are locally constant with respect to certain stratifications. That our objects, which will be locally constant factorization algebras on $(0, 1)^n$, indeed are E_n -algebras as defined in the previous section follows from the following theorem due to Lurie for which we need to introduce some notation.

Let X be a topological space and \mathcal{S} be a symmetric monoidal $(\infty, 1)$ -category with all small colimits. Then the category $\text{Fact}_X(\mathcal{S})$ of factorization algebras on X with values in \mathcal{S} is itself a symmetric monoidal $(\infty, 1)$ -category, see [CG]. Let Fact_X^{lc} be the full sub- $(\infty, 1)$ -category of $\text{Fact}_X(\mathcal{S})$ whose objects are locally constant factorization algebras.

Theorem 3.2.20 (Lurie, [Lur], Theorem 5.3.4.10). *There is an equivalence of $(\infty, 1)$ -categories*

$$\text{Disk}_n^{\text{fr}}\text{-Alg}(\mathcal{S}) \xrightarrow{\simeq} \text{Fact}_{\mathbb{R}^n}^{lc}.$$

Remark 3.2.21. In fact, the equivalence in the proof is given by factorization homology, i.e. the image of an E_n -algebra A is its factorization homology $\int_{\mathbb{R}^n} A$, which we will construct in the next chapter, in section 4.1.

The choice of diffeomorphism $\chi : (0, 1) \cong \mathbb{R}$ yields the following corollary, see also [Gin], Remark 23, or [Cal].

Corollary 3.2.22. *There is an equivalence of $(\infty, 1)$ -categories*

$$E_n\text{-Alg}(\mathcal{S}) \xrightarrow{\simeq} \text{Fact}_{(0,1)^n}^{lc}.$$

Bimodules as locally constant factorization algebras

Our second motivation for using factorization algebras is the following. For more details, see [Gin].

Let A, B be associative algebras in \mathcal{S} , M a pointed (A, B) -bimodule, with pointing $\mathbb{1} \xrightarrow{m} M$. Then the following assignment extends to a factorization algebra \mathcal{F}_M on $(0, 1)$: Let $0 < s < 1$. For open intervals U, V , and W in $(0, 1)$ as in the picture

$$\begin{array}{c} 0 \qquad \qquad \qquad 1 \\ \left(\text{---} (U) \text{---} \left(\begin{array}{c} s \\ \bullet \\ W \end{array} \right) \text{---} (V) \text{---} \right) \end{array}$$

we set

$$\begin{aligned} U &\longmapsto \mathcal{F}_M(U) = A, & V &\longmapsto \mathcal{F}_M(V) = B, \\ p \in W &\longmapsto \mathcal{F}_M(W) = M. \end{aligned}$$

The structure maps of the factorization algebra are given by the bimodule structure and by

$$A \otimes B \cong A \otimes \mathbb{1} \otimes B \xrightarrow{m} M.$$

This special case comes from the fact that factorization algebras naturally are pointed, as we can always include the empty set into any other open set. The inclusion $\emptyset \subseteq W$ induces a map

$$\mathbb{1} \longrightarrow M.$$

In the case where $\mathcal{S} = Ch_k$ is the $(\infty, 1)$ -category of chain complexes over a field k , the pointing is a map $k \rightarrow M$ which is determined by the image of $1 \in k$,

$$1 \longmapsto m \in M.$$

In this case the structure map of $U \amalg V \subset (0, 1)$ is given by

$$A \otimes B \longrightarrow M, \quad (a, b) \longmapsto amb.$$

The factorization algebra \mathcal{F}_M defined by a bimodule M as above is locally constant with respect to the stratification

$$\emptyset \subset \{s\} \subset (0, 1).$$

Conversely, any factorization algebra \mathcal{F} which is locally constant with respect to a stratification of the above form determines a homotopy bimodule M over homotopy algebras A, B as we show in the following lemma.

Lemma 3.2.23. *Let $0 < s < 1$ and let \mathcal{F} be a factorization algebra on $(0, 1)$ which is locally constant with respect to the stratification*

$$\emptyset \subset \{s\} \subset (0, 1).$$

Then $M = \mathcal{F}((0, 1))$ is, up to homotopy, a pointed (A, B) -bimodule for the (E_1) -algebras $A = \mathcal{F}((0, s))$ and $B = \mathcal{F}((s, 1))$ and pointing $\mathbb{1} \rightarrow M$ induced by the structure map for the inclusion $\emptyset \subset (0, 1)$.

Proof. Since $U \subset (0, s)$ and $V \subset (s, 1)$ are weak equivalences, the structure map of the factorization algebra associated to the inclusion of open sets $U \amalg V \subset (0, 1)$ as in the picture above induces the homotopy bimodule structure. \square

Corollary 3.2.24. *The data of a homotopy bimodule over E_1 -algebras is the same as the data of a factorization algebra on $(0, 1)$ which is locally constant with respect to a stratification of the form*

$$\emptyset \subset \{s\} \subset (0, 1)$$

for some $0 < s < 1$.

Locally constant factorization algebras on products

We will need the following theorem later on, which is proposition 18 and corollary 6 in [Gin].

Theorem 3.2.25. *Let X, Y be stratified manifolds with finitely many dimension α -strata for every α .*

1. *The pushforward along the projection $pr_1 : X \times Y \rightarrow X$ induces an equivalence*

$$\underline{pr}_{1*} : \mathcal{F}act_{X \times Y} \longrightarrow \mathcal{F}act_X(\mathcal{F}act_Y).$$

2. *Consider the stratification on the product $X \times Y$ given by*

$$(X \times Y)_k := \bigcup_{i+j=k} X_i \times Y_j \subset X \times Y.$$

The equivalence from 1 induces a functor

$$\underline{pr}_{1*} : \mathcal{F}act_{X \times Y}^{lc} \longrightarrow \mathcal{F}act_X^{lc}(\mathcal{F}act_Y^{lc})$$

between the subcategories of factorization algebras which are locally constant with respect to the stratifications of the respective spaces.

3.2.4 The level sets $(\text{Alg}_n)_{k_1, \dots, k_n}$

For $S \subseteq \{1, \dots, n\}$ we denote the projection from $(0, 1)^n$ onto the coordinates indexed by S by $\pi_S : (0, 1)^n \rightarrow (0, 1)^S$ and for $1 \leq i \leq n$, we abbreviate $\pi_{\{i\}}$ to π_i .

Definition 3.2.26. For every $k_1, \dots, k_n \geq 0$, let $(\text{Alg}_n)_{k_1, \dots, k_n}$ be the collection of tuples

$$(\mathcal{F}, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n}),$$

satisfying the following conditions:

1. \mathcal{F} is a factorization algebra on $(0, 1)^n$.

2. For $1 \leq i \leq n$,

$$(I_0^i \leq \dots \leq I_{k_i}^i) \in \text{Covers}_{k_i}.$$

3. \mathcal{F} is locally constant with respect to the stratification defined inductively by

$$X_n = (0, 1)^n \quad \text{and} \quad X_{n-i} = X_{n-i+1} \cap Y_i$$

for $1 \leq i \leq n$, where, denoting by $(I_j^i)^\circ = (a_j^i, b_j^i)$ the interior of the interval I_j^i ,

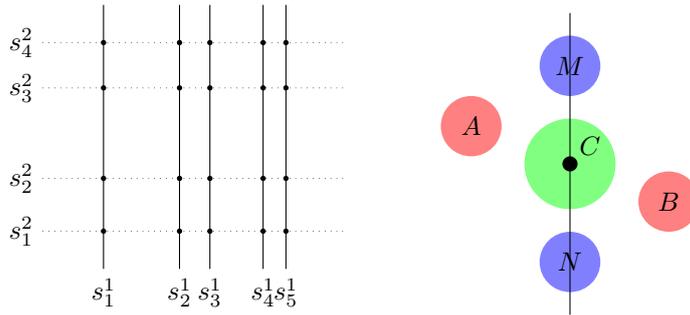
$$Y_i = \pi_i^{-1} \left((0, 1) \setminus \bigcup_{j=0}^{k_i} (a_j^i, b_j^i) \right) = (0, 1)^n \setminus \bigcup_{j=0}^{k_i} \pi_i^{-1} \left((I_j^i)^\circ \right).$$

One may think of the overlapping intervals as also giving a point of the stratification, but one which is “degenerate”, and thus gives a “degenerate” bimodule, by which we mean an E_1 -algebra viewed as a bimodule over itself.

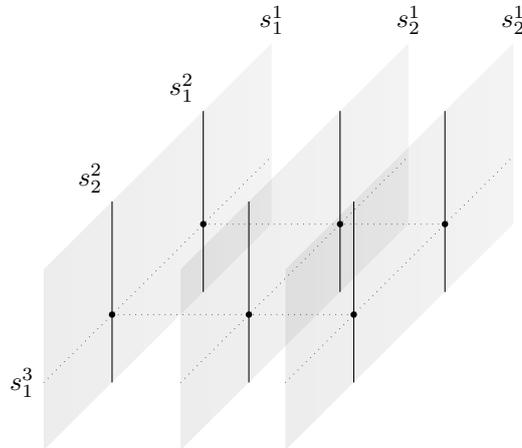


Remark 3.2.30. One should be a bit careful with the interpretation of the degenerate points of the stratification, as this data does not behave well with respect to the simplicial structure. As we explained above, this is the reason we do not use this as a definition, but keep track of the intervals instead.

Example 3.2.31. For $n = 2$, stratifications which appear in the definition of Alg_2 give pictures as in the left picture below. A 2-morphism, i.e. an element in $(\text{Alg}_2)_{1,1}$, leads to a bimodule C between bimodules M and N of E_2 -algebras A and B which are the images of open disks as in the right picture below.



For $n = 3$, stratifications which appear in the definition of Alg_3 give pictures of the following type:



3.2.5 The spaces $(\text{Alg}_n)_{k_1, \dots, k_n}$

The level sets $(\text{Alg}_n)_{k_1, \dots, k_n}$ form the underlying set of 0-simplices of a space which we construct in this section.

The space of factorization algebras

We first need suitable spaces of factorization algebras.

Recall from 3.2.17 that the category $\text{Fact}_X(\mathcal{S})$ of factorization algebras on X with values in \mathcal{S} is a symmetric monoidal $(\infty, 1)$ -category. Thus it has a space of objects, which is the space of factorization algebras we are interested in.

If we start with a symmetric monoidal relative category $(\mathcal{S}, \mathcal{W})$, then $\text{Fact}_X(\mathcal{S})$ again is a symmetric monoidal relative category. The classification diagram as explained in section 1.2.3 of $\text{Fact}_X(\mathcal{S})$ with its level-wise weak equivalences gives a symmetric monoidal complete Segal space $N(\text{Fact}_X(\mathcal{S}), \mathcal{W})$ of factorization algebras.

The objects of this Segal space form a space, i.e. a simplicial set, of factorization algebras. Explicitly, if we begin with a relative category, this space of factorization algebras is the nerve of the category of factorization algebras with weak equivalences as morphisms, i.e. a k -simplex is a sequence

$$\mathcal{F}_0 \xrightarrow{w_1} \mathcal{F}_1 \xrightarrow{w_2} \dots \xrightarrow{w_k} \mathcal{F}_k.$$

However, it will not be a Kan complex unless \mathcal{S} was an ∞ -groupoid. Thus, we use a modification of this definition which seems to be close to a Kan fibrant replacement of the above construction. We will give a more conceptual construction using families of factorization algebras in a subsequent paper.

The spatial structure of $(\text{Alg}_n)_{k_1, \dots, k_n}$

Definition 3.2.32. An l -simplex of $(\text{Alg}_n)_{k_1, \dots, k_n}$ consists of the following data:

1. A family of *underlying 0-simplices*, which is a collection of elements

$$(\mathcal{F}_s, (I_0^i(s) \leq \dots \leq I_{k_i}^i(s))_{i=1}^n) \in (\text{Alg}_n)_{k_1, \dots, k_n}$$

indexed by $s \in |\Delta^l|$;

2. For every $1 \leq i \leq n$, a *rescaling datum* $(\phi_{s,t}^i : (0, 1) \rightarrow (0, 1))_{s,t \in |\Delta^l|}$ making

$$(I_0^i(s) \leq \dots \leq I_{k_i}^i(s))_{s \in |\Delta^l|}$$

into an l -simplex in Covers_{k_i} ;

3. A collection of *weak equivalences*

$$(\phi_{s,t})_* \mathcal{F}_s \xrightarrow{w_{s,t}} \mathcal{F}_t$$

for $s, t \in |\Delta^l|$, where $\phi_{s,t} = (\phi_{s,t}^i)_{i=1}^n : (0, 1)^n \rightarrow (0, 1)^n$ is the product of the rescaling data.

Remark 3.2.33. This space can be thought of as a “rescaled” version of the space of factorization algebras explained above, namely, we compare factorization algebras after rescaling using the diffeomorphisms $\phi_{s,t}$. When the context is clear we will sometimes omit writing the pushforwards along the rescaling maps and simply write $\mathcal{F}_s \rightarrow \mathcal{F}_t$.

Similarly to the construction of PBord_n , we lift the spatial structure on $\text{Covers}_{k_1, \dots, k_n}^n$ to $(\text{Alg}_n)_{k_1, \dots, k_n}$.

Fix $k \geq 0$ and let $f : [m] \rightarrow [l]$ be a morphism in the simplex category Δ , i.e. a (weakly) order-preserving map. Then let $|f| : |\Delta^m| \rightarrow |\Delta^l|$ be the induced map between standard simplices.

Let f^Δ be the map sending an l -simplex in $(\text{Alg}_n)_{k_1, \dots, k_n}$ to the m -simplex which consists of

1. for $s \in |\Delta^m|$,

$$\mathcal{F}_{|f|(s)};$$

2. for $1 \leq i \leq n$, the m -simplex in Covers_{k_i} obtained by applying f^Δ ,

$$f^\Delta \left(I_0^i(s) \leq \cdots \leq I_{k_i}^i(s), \phi_{s,t}^i \right);$$

3. for $s, t \in |\Delta^m|$,

$$w_{|f|(s), |f|(t)} : \mathcal{F}_{|f|(s)} \longrightarrow \mathcal{F}_{|f|(t)}.$$

Proposition 3.2.34. $(\text{Alg}_n)_{k_1, \dots, k_n}$ is a space. Moreover, it is a Kan complex, i.e. fibrant in the category of simplicial sets with Quillen model structure.

Proof. The spatial structure essentially comes from the spatial structure of $\text{Covers}_{k_1, \dots, k_n}^n$ and the simplicial structure of $N(\Delta)$, so the assignments above are well-defined and behave functorially.

The proof of the Horn filling condition is similar to that in 2.3.8. A morphism $\Lambda_k^l \rightarrow \text{Alg}_n$ is the data of, for $s, t \in |\Lambda_k^l|$,

$$(\mathcal{F}_s, (I_0^i(s) \leq \cdots \leq I_{k_i}^i(s))_{i=1, \dots, n}) \in (\text{Alg}_n)_{k_1, \dots, k_n},$$

$$(\phi_{s,t}^i : (0, 1) \rightarrow (0, 1)), \text{ and } (w_{s,t} : \mathcal{F}_s \longrightarrow \mathcal{F}_t).$$

Now choose a smooth retraction $p : |\Delta^l| \rightarrow |\Lambda_k^l|$. Then for $\tilde{s}, \tilde{t} \in |\Delta^l|$, consider

$$(\mathcal{F}_{p(\tilde{s})}, (I_0^i(p(\tilde{s})) \leq \cdots \leq I_{k_i}^i(p(\tilde{s})))_{i=1, \dots, n}) \in (\text{Alg}_n)_{k_1, \dots, k_n},$$

$$(\phi_{p(\tilde{s}), p(\tilde{t})}^i : (0, 1) \rightarrow (0, 1)), \text{ and } (w_{p(\tilde{s}), p(\tilde{t})} : \mathcal{F}_{p(\tilde{s})} \longrightarrow \mathcal{F}_{p(\tilde{t})}).$$

This defines an l -simplex in PBord_n^V whose restriction to the k th horn is the given one. \square

Notation 3.2.35. We denote the *spatial face and degeneracy maps* of $(\text{Alg}_n)_{k_1, \dots, k_n}$ by δ_j^Δ and σ_j^Δ for $0 \leq j \leq l$.

3.2.6 The n -fold simplicial set Alg_n

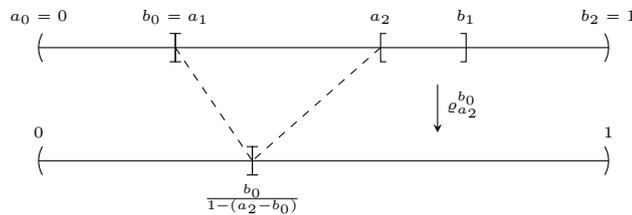
In the next two sections, we make the collection of spaces $(\text{Alg}_n)_{\bullet, \dots, \bullet}$ into an n -fold simplicial space by defining suitable face and degeneracy maps. They essentially arise from the face and degeneracy maps of the n -fold simplicial set $\text{Covers}_{\bullet, \dots, \bullet}^n$ of covers of $(0, 1)^n$ by products of closed intervals. In this section we define faces and degeneracies on 0-simplices, which makes $(\text{Alg}_n)_{\bullet, \dots, \bullet}$ into an n -fold simplicial set, ignoring the spatial structure of the levels. We will lift the n -fold simplicial set to an n -fold simplicial space using the spatial structure of the levels in the next section.

Before giving the full definition of the face and degeneracy maps of the n -fold simplicial set Alg_n , we first demonstrate them for $n = 1$.

Example 3.2.36. For $n = 1$, elements in $(\text{Alg}_1)_1$ consist of a factorization algebra \mathcal{F} on $(0, 1)$ and two intervals $(0, b]$ and $[a, 1)$ such that $a \leq b$. The source and target maps $(\text{Alg}_1)_1 \rightrightarrows (\text{Alg}_1)_0$ are given by restricting the factorization algebra which then is rescaled back to $(0, 1)$. Explicitly, the source map pushes forward the restriction of the factorization algebra \mathcal{F} to $(0, b)$ by the collapse-and-rescale map ϱ_0^b to $(0, 1)$, which is the unique affine bijection $(0, b) \rightarrow (0, 1)$. Similarly the target map pushes forward the restriction of the factorization algebra \mathcal{F} to $(a, 1)$ by the collapse-and-rescale map ϱ_a^0 to $(0, 1)$. We saw in example 3.2.29 that elements in $(\text{Alg}_1)_1$ can be viewed as pairs (A, B) of E_1 -algebras and a pointed homotopy (A, B) -bimodule M . The source and target maps $(\text{Alg}_1)_1 \rightrightarrows (\text{Alg}_1)_0$ map M to the source A , respectively the target B .

The degeneracy map $(\text{Alg}_1)_0 \rightarrow (\text{Alg}_1)_1$ sends a pair $(\mathcal{F}, (0, 1))$ consisting of a locally constant factorization algebra \mathcal{F} on $(0, 1)$ to the element $(\mathcal{F}, (0, 1) \leq (0, 1))$. In the language of algebras and bimodules, it sends an E_1 -algebra A to itself, now viewed as an (A, A) -bimodule.

Two of the face maps, $\delta_0, \delta_2 : (\text{Alg}_1)_2 \rightrightarrows (\text{Alg}_1)_1$ are defined similarly, by “forgetting” part of the data, i.e. by restricting the factorization algebra and rescaling. In the language of modules, the map δ_2 , which corresponds to the “source map”, sends an element consisting of a triple (A, B, C) of E_1 -algebras and a pair $({}_A M_{B, B} N_C)$ of bimodules to (A, B) and ${}_A M_B$. The “target map” δ_0 sends the same element to (B, C) and ${}_B M_C$. The third map δ_1 , which corresponds to composition, sends an element $(\mathcal{F}, I_0 \leq I_1 \leq I_2)$ to the pushforward along the collapse-and-rescale map $\varrho_{a_2}^{b_0} : (0, 1) \rightarrow (0, 1)$, illustrated in the following picture for the case $b_0 = a_1$.



If $b_0 \geq a_2$, then $\varrho_{a_2}^{b_0} = id$. Moreover, either $A = B$ and ${}_A M_B = {}_B B_B$, or $B = C$ and ${}_B N_C = B$ (or both). In the first case δ_1 sends $({}_B B_B, B M_C)$ to just ${}_B M_C$. In the second case δ_1 sends $({}_A M_B, B B_B)$ to just ${}_A M_B$.

If $b_0 < a_2$, the gluing axiom of factorization algebras implies that the homotopy bimodule associated to image under δ_1 of the pair of homotopy bimodules $({}_A M_B, {}_B N_C)$ is the tensor product $({}_A M_B) \otimes_B ({}_B N_C)$, i.e. composition sends an element consisting of E_1 -algebras A, B, C and bimodules ${}_A M_B$ and ${}_B N_C$ to A, C and the bimodule $({}_A M_B) \otimes_B ({}_B N_C)$.

The two degeneracy maps $\sigma_0, \sigma_1 : (\text{Alg}_1)_1 \rightrightarrows (\text{Alg}_1)_2$ send $(\mathcal{F}, I_0 \leq I_1)$ to $\sigma_0(\mathcal{F}, I_0 \leq I_1) = (\mathcal{F}, I_0 \leq I_0 \leq I_1)$, $\sigma_1(\mathcal{F}, I_0 \leq I_1) = (\mathcal{F}, I_0 \leq I_1 \leq I_1)$. In the language of modules, they send an (A, B) -bimodule ${}_A M_B$ to the pairs $({}_A A, {}_A M_B)$ respectively $({}_A M_B, {}_B B)$.

$$\begin{array}{ccc} \begin{array}{c} 0 \qquad A \qquad \bullet \qquad B \qquad 1 \\ \left(\xrightarrow{\hspace{10em}} \right) \\ \qquad \qquad \qquad M \\ \qquad \qquad \qquad \downarrow \sigma_0 \\ 0 \qquad A \qquad A \qquad \bullet \qquad B \qquad 1 \\ \left(\xrightarrow{\hspace{10em}} \right) \end{array} & & \begin{array}{c} 0 \qquad A \qquad \bullet \qquad B \qquad 1 \\ \left(\xrightarrow{\hspace{10em}} \right) \\ \qquad \qquad \qquad M \\ \qquad \qquad \qquad \downarrow \sigma_1 \\ 0 \qquad A \qquad \bullet \qquad B \qquad B \qquad 1 \\ \left(\xrightarrow{\hspace{10em}} \right) \end{array} \end{array}$$

Notation 3.2.37. Before we start defining the face and degeneracy maps, recall that we used collapse-and-rescale maps ϱ_a^b to define the simplicial structure on Covers_\bullet . More precisely, the j th face map was defined using $\varrho_{a_{j+1}}^{b_{j-1}}$. For simplicity of notation, we will denote this map by ϱ_j in the following and its domain of injectivity by D_j .

Since $1 \leq i \leq n$ will be fixed throughout the constructions, by abuse of notation, we also denote by ϱ_j the map $\varrho_{a_{j+1}^i}^{b_{j-1}^i}$ used for the j th face map in the i th direction of the n -fold simplicial structure of $\text{Covers}_{\bullet, \dots, \bullet}^n$ and its domain of injectivity by $D_j = (0, b_{j-1}^i) \cup (a_{j+1}^i, 1)$.

By even more abuse of notation we again denote by ϱ_j the map

$$\varrho_j : (0, 1)^n \rightarrow (0, 1)^n,$$

which is ϱ_j in the i th coordinate and the identity otherwise. By ϱ_j^{-1} we mean the inverse of

$$\varrho_j|_{\pi_i^{-1}(D_j)} : \pi_i^{-1}(D_j) = \prod_{\alpha \neq i} (0, 1) \times D_j \rightarrow (0, 1)^n.$$

Degeneracy maps Fix $1 \leq i \leq n$. For $0 \leq j \leq k_i$ the j th degeneracy map

$$\sigma_j^i : (\text{Alg}_n)_{k_1, \dots, k_n} \rightarrow (\text{Alg}_n)_{k_1, \dots, k_i+1, \dots, k_n}$$

applies the j th degeneracy map of Covers_\bullet to the i th tuple of intervals, i.e. it repeats the j th specified interval in the i th direction, I_j^i ,

$$\begin{aligned} (\mathcal{F}, (I_0^\alpha \leq \dots \leq I_{k_\alpha}^\alpha)_{\alpha=1}^n) &\longmapsto (\mathcal{F}, (I_0^\alpha \leq \dots \leq I_{k_\alpha}^\alpha)_{\alpha \neq i}, \sigma_j(I_1^i \leq \dots \leq I_{k_i}^i)) = \\ &(\mathcal{F}, (I_0^\alpha \leq \dots \leq I_{k_\alpha}^\alpha)_{\alpha \neq i}, I_1^i \leq \dots \leq I_j^i \leq I_j^i \leq \dots \leq I_{k_i}^i). \end{aligned}$$

Since this does not change the stratification with respect to which \mathcal{F} must be locally constant this map is well-defined.

Face maps Fix $1 \leq i \leq n$. For $0 \leq j \leq k_i$ the j th face map

$$\delta_j^i : (\text{Alg}_n)_{k_1, \dots, k_n} \rightarrow (\text{Alg}_n)_{k_1, \dots, k_i-1, \dots, k_n}$$

applies the j th face map δ_j of Covers_\bullet to the i th tuple of intervals, which forgets the j th interval I_j^i and applies the collapse-and-rescale map ϱ_j to the other intervals, and pushes the factorization algebra, restricted to $\pi_i^{-1}(D_j)$, forward along the map ϱ_j . Explicitly, $(\mathcal{F}, (I_0^\alpha \leq \dots \leq I_{k_\alpha}^\alpha)_{\alpha=1}^n)$ is sent to

$$\left((\varrho_j)_* \mathcal{F}|_{\pi_i^{-1}(D_j)}, (I_0^\alpha \leq \dots \leq I_{k_\alpha}^\alpha)_{\alpha \neq i}, \delta_j(I_1^i \leq \dots \leq I_{k_i}^i) \right).$$

This is well-defined since the restriction of the factorization algebra and the stratification with respect to which it must be locally constant are rescaled by the same rescaling map.

Remark 3.2.38. In the following, we will omit explicitly writing out the restriction of \mathcal{F} to $\pi_i^{-1}(D_j)$ for readability.

Proposition 3.2.39. *The face and degeneracy maps defined above define an n -fold simplicial set $(\text{Alg}_n)_\bullet, \dots, \bullet$.*

Proof. This follows from the fact that Covers_\bullet is a simplicial set and pushforward of factorization algebras is a functor. \square

3.2.7 The full structure of Alg_n as an n -fold simplicial space

In this section we “extend” the simplicial face and degeneracy maps δ_j^i, σ_j^i to the l -simplices of $(\text{Alg}_n)_{k_1, \dots, k_n}$ in a way that they commute with the spatial structure of $(\text{Alg}_n)_{k_1, \dots, k_n}$. This gives Alg_n the structure of an n -fold simplicial space.

Degeneracy maps on l -simplices. Fix $1 \leq i \leq n$. For $0 \leq j \leq k_i$ the j th degeneracy map σ_j^i sends an l -simplex of $(\text{Alg}_n)_{k_1, \dots, k_n}$ to the l -simplex of $(\text{Alg}_n)_{k_1, \dots, k_i+1, \dots, k_n}$ defined by applying the degeneracy map σ_j^i to each underlying 0-simplex,

$$\sigma_j^i(\mathcal{F}_s, (I_0^i(s) \leq \dots \leq I_{k_i}^i(s))) \in (\text{Alg}_n)_{k_1, \dots, k_i+1, \dots, k_n},$$

and keeping the same rescaling data $\phi_{s,t}$ and weak equivalences $(\phi_{s,t})_* \mathcal{F}_s \xrightarrow{w_{s,t}} \mathcal{F}_t$.

Face maps on l -simplices. Fix $1 \leq i \leq n$.

For $0 \leq j \leq k_i$ the j th face map δ_j^i sends an l -simplex of $(\text{Alg}_n)_{k_1, \dots, k_n}$ consisting of

$$(\mathcal{F}_s, (I_0^i(s) \leq \dots \leq I_{k_i}^i(s)))_{s \in |\Delta^l|}$$

$$(\phi_{s,t} : (0, 1)^n \longrightarrow (0, 1)^n)_{s,t \in |\Delta^l|}, \quad \text{and} \quad ((\phi_{s,t})_* \mathcal{F}_s \xrightarrow{w_{s,t}} \mathcal{F}_t)_{s,t \in |\Delta^l|}$$

to the l -simplex of $(\text{Alg}_n)_{k_1, \dots, k_i-1, \dots, k_n}$ consisting of the following data.

Denote by $\varrho_j(s)$ be the analog of the above map ϱ_j associated to the s th underlying 0-simplex $(I_0^i(s) \leq \dots \leq I_{k_i}^i(s)) \in \text{Covers}_{k_i}$.

1. The underlying 0-simplices of the image are the images of the underlying 0-simplices under δ_j^i , i.e. for $s \in |\Delta^l|$,

$$\begin{aligned} \delta_j^i(\mathcal{F}_s, (I_0^i(s) \leq \dots \leq I_{k_i}^i(s))) = \\ (\varrho_j(s)_* \mathcal{F}_s|_{\dots}, (I_0^\alpha(s) \leq \dots \leq I_{k_\alpha}^\alpha(s))_{\alpha \neq i}, \delta_j(I_0^i(s) \leq \dots \leq I_{k_i}^i(s))), \end{aligned}$$

where we omit writing down the precise restriction domain from now on. It can be checked easily that they match up where needed.

2. The underlying l -simplex in Covers_{k_i} is sent to its image under δ_j^i , i.e. its rescaling data is $\delta_j^i(\phi_{s,t})$. Recall from section 3.1.5 that this is the map

$$\delta_j^i(\phi_{s,t}) = \varrho_j(t) \circ \phi_{s,t}|_{\dots} \circ \varrho_j(s)^{-1} : (0, 1)^n \rightarrow (0, 1)^n.$$

3. Pushforward along $\varrho_j(s)$ is an endofunctor of the category of factorization algebras on $(0, 1)^n$ which preserves weak equivalences, so for every $s \in |\Delta^l|$ we have the following weak equivalences

$$\delta_j^i(\phi_{s,t})_* \left(\varrho_j(s)_* \mathcal{F}_s |_{\dots} \right) = \varrho_j(t)_* (\phi_{s,t}|_{\dots})_* \mathcal{F}_s |_{\dots} \xrightarrow{\varrho_j(t)_* w_{s,t}} \varrho_j(t)_* \mathcal{F}_t |_{\dots}.$$

Proposition 3.2.40. $(\text{Alg}_n)_{\bullet, \dots, \bullet}$ is an n -fold simplicial space.

Proof. The degeneracy and face maps σ_j^i, δ_j^i defined above satisfy the simplicial relations since we showed in lemma 3.1.21 that $\text{Covers}_{\bullet, \dots, \bullet}^n$ is an n -fold Segal space, in particular, we proved that the rescaling maps commute in the appropriate way. Moreover, they commute with each other since they modify different parts of the structure and they commute with the spatial structure maps f^Δ of the spaces $(\text{Alg}_n)_{k_1, \dots, k_n}$. \square

3.2.8 The n -fold Segal space Alg_n

Proposition 3.2.41. $(\text{Alg}_n)_{\bullet, \dots, \bullet}$ is an n -fold Segal space.

Proof. We need to prove the following conditions:

1. *The Segal condition is satisfied.* For clarity, we explain the Segal condition in the following case. The general proof works similarly. We will show that

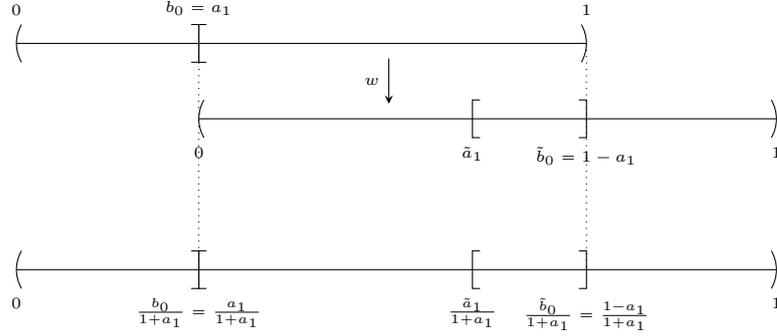
$$(\text{Alg}_n)_{k_1, \dots, 2, \dots, k_n} \xrightarrow{\sim} (\text{Alg}_n)_{k_1, \dots, 1, \dots, k_n} \times_{(\text{Alg}_n)_{k_1, \dots, 0, \dots, k_n}}^h (\text{Alg}_n)_{k_1, \dots, 1, \dots, k_n}.$$

To simplify notation, we omit the indices and the specified points in all directions except for the i th one, as this procedure only depends on this specified direction. We construct a map

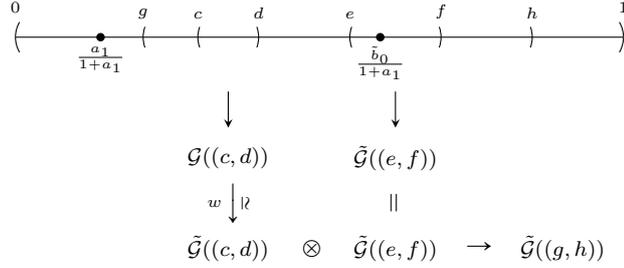
$$(\text{Alg}_n)_1 \times_{(\text{Alg}_n)_0}^h (\text{Alg}_n)_1 \xrightarrow{\text{glue}} (\text{Alg}_n)_2$$

which is a deformation retraction, i.e. $\text{glue} \circ (\delta_0 \times \delta_2) = \text{id}$, $(\delta_0 \times \delta_2) \circ \text{glue} \sim \text{id}$.

Since every level set $(\text{Alg}_n)_{k_1, \dots, k_n}$ is a Kan complex by proposition 3.2.34, i.e. fibrant, the homotopy fiber product consists of triples consisting of two points and a path between them. Thus an element in $(\text{Alg}_n)_1 \times_{(\text{Alg}_n)_0}^h (\text{Alg}_n)_1$ consists of two factorization algebras \mathcal{G} and $\tilde{\mathcal{G}}$ on $(0, 1)^n$, specified intervals $(0, b_0] \leq [a_1, 1)$, $(0, \tilde{b}_0] \leq [\tilde{a}_1, 1)$ in the i -th direction, and a path between their target and source. The path in particular gives a weak equivalence $\delta_1(\tilde{\mathcal{G}}) \xrightarrow{w} \delta_0(\mathcal{G})$. Here again, we omit the rescaling in the notation. We glue them to an element in $(\text{Alg}_n)_2$ in the following way. By first applying a piecewise linear rescaling, we can assume that $1 - a_1 = \tilde{b}_0$.



Send the above data to the factorization algebra \mathcal{F} on $(0,1)^n$ defined by \mathcal{G} on $(0, \frac{\tilde{b}_0}{1+a_1}) \times \prod_{\alpha \neq i} (0,1)$ and $\tilde{\mathcal{G}}$ on $(\frac{a_1}{1+a_1}, 1) \times \prod_{\alpha \neq i} (0,1)$ using rescaling maps which, again, we will omit for clarity of notation. It remains to “glue” them together using the weak equivalence w . On an interval (a,b) such that $\frac{a_1}{1+a_1} < a < \frac{\tilde{b}_0}{1+a_1} < b$, $\mathcal{F}((a,b)) := \mathcal{G}((a,b))$. Moreover, we define the factorization algebra structure by



Note that this way the factorization algebra is defined on a factorizing cover and can be extended by the gluing condition.

This construction extends to the spatial structure since weak equivalences are defined locally and thus can be glued along open sets. By construction, $glue \circ (\delta_0 \times \delta_2) = id$. Moreover, the weak equivalence w coming from the path gives $(\delta_0 \times \delta_2) \circ glue \sim id$.

2. For every i and every k_1, \dots, k_{i-1} , $(\text{Alg}_n)_{k_1, \dots, k_{i-1}, 0, \bullet, \dots, \bullet}$ is essentially constant.

An element in $(\text{Alg}_n)_{k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n}$ is of the form

$$(\mathcal{F}, I_0^1 \leq \dots \leq I_{k_1}^1, \dots, I_0^{i-1} \leq \dots \leq I_{k_{i-1}}^{i-1}, (0,1), \\ I_0^{i+1} \leq \dots \leq I_{k_{i+1}}^{i+1}, \dots, I_0^n \leq \dots \leq I_{k_n}^n),$$

so by definition the stratification with respect to which \mathcal{F} is locally constant reduces to

$$(0,1)^n = X_n \supseteq X_{n-1} \supseteq \dots \supseteq X_{i+1} \supseteq X_i = X_{i+1} \cap \pi_{n-i}^{-1}((0,1) \setminus (0,1)) = \emptyset.$$

Since the stratification only depends on the first $i-1$ tuples of intervals we can freely move the remaining intervals I_j^α for $\alpha > i$ and still have a well-defined element in $(\text{Alg}_n)_{k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n}$. In particular, we can move them to $I_0^\alpha = \dots = I_{k_\alpha}^\alpha = (0,1)$, which is in the image of the composition of degeneracy maps S . We can chose

the endpoints to move linearly (by setting $a_j^\alpha(t) = (1-t)a_j^\alpha$ and $b_j^\alpha(t) = (1-t)b_j^\alpha + t$), so this construction extends to a homotopy. Hence

$$S : (\text{Alg}_n)_{k_1, \dots, k_{i-1}, 0, \dots, 0} \xrightarrow{\simeq} (\text{Alg}_n)_{k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n}$$

is a weak equivalence.

□

Remark 3.2.42. One can alternatively show the Segal condition by showing that the source and target maps $s, t : (\text{Alg}_n)_{k_1, \dots, 1, \dots, k_n} \rightarrow (\text{Alg}_n)_{k_1, \dots, 0, \dots, k_n}$ are Serre fibrations and then showing that

$$(\text{Alg}_n)_{k_1, \dots, k_i, \dots, k_n} \xrightarrow{\sim} (\text{Alg}_n)_{k_1, \dots, 1, \dots, k_n} \times_{(\text{Alg}_n)_{k_1, \dots, 0, \dots, k_n}} \cdots \times_{(\text{Alg}_n)_{k_1, \dots, 0, \dots, k_n}} (\text{Alg}_n)_{k_1, \dots, 1, \dots, k_n}.$$

One can show the homotopy lifting property for cubes I^k for s, t explicitly by constructing a lift. This construction is similar to the construction of the map *glue* above. The second “strict Segal” condition follows from the fact that factorization algebras satisfy a descent condition, see e.g. 4.3.5 in [Gin].

3.2.9 Completeness of Alg_n and the Morita (∞, n) -category of E_n -algebras

Factorization algebras with values in a symmetric monoidal relative category with all coproducts \mathcal{S} are pointed in the sense that given a factorization algebra \mathcal{F} , for any open set U the inclusion of the empty set $\emptyset \hookrightarrow U$ gives a map $\mathbb{1} \rightarrow \mathcal{F}(U)$, where $\mathbb{1}$ is the unit for the monoidal product of the symmetric monoidal structure of \mathcal{S} . In this subsection we show that this pointing ensures completeness of Alg_n .

We will first explain the argument for $n = 1$ using the language of algebras and bimodules following corollary 3.2.24, and then give the general argument.

Proposition 3.2.43. *The Segal space $\text{Alg}_1(\mathcal{S})$ is complete, i.e.*

$$s_0 : (\text{Alg}_1(\mathcal{S}))_0 \longrightarrow (\text{Alg}_1(\mathcal{S}))_1^{inv}$$

is a weak equivalence.

Proof. An element in $(\text{Alg}_1)_1^{inv}$ is a pointed (A, B) -bimodule $\mathbb{1} \xrightarrow{m} M$ such that there is a pointed (B, A) -bimodule $\mathbb{1} \xrightarrow{n} N$ and weak equivalences

$$A \xrightarrow[m \otimes n]{\simeq} M \otimes_B N, \quad \text{and} \quad B \xrightarrow[n \otimes m]{\simeq} N \otimes_A M,$$

of (A, A) , respectively (B, B) -bimodules. We need to show that $A \simeq B \simeq M$. This implies that there is a path from ${}_A M_B$ to ${}_A A_A$. This construction extends to a homotopy, since a weak equivalence from ${}_A M_B$ to a different bimodule ${}_C N_D$ includes the data of a weak equivalence from A to C .

First note that

Definition 3.3.1. For every k_1, \dots, k_n , let $(\text{Alg}_n[m])_{k_1, \dots, k_n}$ be the collection of tuples

$$(\mathcal{F}_1, \dots, \mathcal{F}_m, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n}),$$

where for every $1 \leq \beta \leq m$, $(\mathcal{F}_\beta, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n}) \in (\text{Alg}_n)_{k_1, \dots, k_n}$. Similarly to Alg_n this can be made into a complete n -fold Segal space.

Proposition 3.3.2. *The assignment*

$$\begin{aligned} \Gamma &\longrightarrow \mathbf{SSpace}_n, \\ [m] &\longmapsto \text{Alg}_n[m] \end{aligned}$$

extends to a functor and endows Alg_n with a symmetric monoidal structure.

Proof. The functor sends a morphism $f : [m] \rightarrow [k]$ to

$$\begin{aligned} \text{Alg}_n[m] &\longrightarrow \text{Alg}_n[k], \\ (\mathcal{F}_1, \dots, \mathcal{F}_m, (I_j^i)_{i,j}) &\longmapsto \left(\bigotimes_{\beta \in f^{-1}(1)} \mathcal{F}_\beta, \dots, \bigotimes_{\beta \in f^{-1}(k)} \mathcal{F}_\beta, (I_j^i)_{i,j} \right). \end{aligned}$$

Here the tensor product is the tensor product of factorization algebras with values in the given symmetric monoidal category (defined level-wise). This is well-defined as every \mathcal{F}_β , and therefore also the tensor product of several \mathcal{F}_β 's are locally constant with respect to the same stratification.

To show that

$$\prod_{1 \leq \beta \leq n} \gamma_\beta : \text{Alg}_n[m] \longrightarrow (\text{Alg}[1])^m$$

is an equivalence of n -fold complete Segal spaces we need to show that for any element in the right hand side we can rescale the intervals $(I_j^i)_{i,j}$ by a smooth family of rescaling maps $\phi_{s,t}^i$ so that they coincide. This follows from the fact that rescaling $(0, 1)^n$ by some suitable rescaling data $\phi_{s,t}$ leads to weak equivalences of factorization algebras given by pushforward along $\phi_{s,t}$. This rescaling yields a path in the right hand space to an element in the image of $\prod_{1 \leq \beta \leq n} \gamma_\beta$ and the collection of these paths form a homotopy. \square

3.3.2 The monoidal structure and the tower

Our goal for this section is to endow Alg_n with a symmetric monoidal structure arising from a tower of monoidal l -hybrid $(n+l)$ -fold Segal spaces $\text{Alg}_n^{(l)}$ for $l \geq 0$.

The deloopings $\text{Alg}_n^{(l)}$

Our construction of the (∞, n) -category of E_n -algebras $\text{Alg}_n(\mathcal{S})$ relies on a symmetric monoidal $(\infty, 1)$ -category \mathcal{S} . Independent of which model for symmetric monoidal $(\infty, 1)$ -categories we choose there is a distinguished object in \mathcal{S} , the unit $\mathbb{1}$ for the symmetric monoidal structure. This object naturally is an E_n -algebra, the constant factorization algebra on \mathbb{R}^n with value $\mathbb{1}$, which determines an object $(\mathbb{1}, (0, 1), \dots, (0, 1))$ in $(\text{Alg}_n)_{0, \dots, 0}$.

The first layer of the tower

Definition 3.3.3. Let $\text{Alg}_n^{(1)}$ be the fiber of Alg_{n+1} over $\mathbb{1}_0 = \mathbb{1}$ in the first direction, i.e. $(\text{Alg}_n^{(1)})_{k_1, \dots, k_{n+1}}$ is the fiber over $\mathbb{1}^{k_1+1} \in ((\text{Alg}_{n+1})_{0, k_2, \dots, k_{n+1}})^{k_1+1}$ of the map

$$(\text{Alg}_{n+1})_{k_1, \dots, k_{n+1}} \longrightarrow ((\text{Alg}_{n+1})_{0, k_2, \dots, k_{n+1}})^{k_1+1},$$

which is the product of the $(k_1 + 1)$ different possible compositions of face maps

$$(\text{Alg}_{n+1})_{k_1, \dots, k_{n+1}} \xrightarrow{\dashrightarrow} (\text{Alg}_{n+1})_{0, k_2, \dots, k_{n+1}}.$$

Proposition 3.3.4. $\text{Alg}_n^{(1)}$ is a monoidal complete n -fold Segal space.

Proof. By construction the $(n + 1)$ -fold Segal space $\text{Alg}_n^{(1)}$ is 1-hybrid and pointed. \square

Remark 3.3.5. It may seem unnatural to take the actual fiber here instead of a homotopy fiber. This is needed as we need hybridness which requires certain spaces to be equal to a point and not just contractible. As explained in remark 3.2.42, the maps $s, t : (\text{Alg}_n)_{k_1, \dots, 1, \dots, k_n} \rightarrow (\text{Alg}_n)_{k_1, \dots, 0, \dots, k_n}$ are fibrations. Thus, in this case, the homotopy fiber and the fiber actually coincide.

The higher layers Similarly, we define the higher layers of the tower.

Assume that we have defined $\text{Alg}_n^{(0)} = \text{Alg}_n, \text{Alg}_n^{(1)}, \dots, \text{Alg}_n^{(l-1)}$ for every n such that $\text{Alg}_n^{(k)}$ is a k -hybrid $(n + k)$ -fold Segal space which is $(j - 1)$ -connected for every $0 < j \leq k$. Note that, via the degeneracy maps, $\mathbb{1}$ can be viewed as a trivial l -morphism in any $\text{Alg}_n^{(k)}$ for any $1 \leq l \leq n + k$, i.e. an element

$$\mathbb{1}_l = (\mathbb{1}, (0, 1) \leq (0, 1), \dots, (0, 1) \leq (0, 1), (0, 1), \dots, (0, 1)) \in (\text{Alg}_n^{(k)})_{\underbrace{1, \dots, 1}_{l}, 0, \dots, 0}.$$

Definition 3.3.6. Let $\text{Alg}_n^{(l)}$ be the fiber of $\text{Alg}_{n+1}^{(l-1)}$ over $\mathbb{1}_{l-1}$, i.e. $(\text{Alg}_n^{(l)})_{k_1, \dots, k_{n+l}}$ is the fiber over $\mathbb{1}_{l-1} \in (\text{Alg}_{n+l})_{1, \dots, 1, 0, k_{l+1}, \dots, k_{l+n}}$ of the product of all different possible compositions of face maps

$$(\text{Alg}_{n+1}^{(l-1)})_{k_1, \dots, k_{n+l}} \xrightarrow{\dashrightarrow} ((\text{Alg}_{n+1}^{(l-1)})_{1, \dots, 1, 0, k_{l+1}, \dots, k_{l+n}}).$$

Proposition 3.3.7. $\text{Alg}_n^{(l)}$ is a k -monoidal complete n -fold Segal space.

Proof. Again by construction the $(n + l)$ -fold Segal space $\text{Alg}_n^{(l)}$ is l -hybrid and $(j - 1)$ -connected for every $0 < j \leq l$. \square

The tower and the symmetric monoidal structure

The monoidal complete n -fold Segal space $\text{Alg}_n^{(1)}$ turns out to be a delooping of Alg_n . The following proposition shows that the collection of the l -monoidal complete n -fold Segal spaces $(\text{Alg}_n^{(l)})_l$ forms the tower which gives Alg_n a symmetric monoidal structure.

Proposition 3.3.8. *For $n, l \geq 0$, there are weak equivalences*

$$L(\text{Alg}_n^{(l)}, \mathbb{1}) \begin{array}{c} \xrightarrow{u} \\ \xleftarrow{\ell} \end{array} \text{Alg}_n^{(l-1)},$$

defined as follows.

1. The map u sends an element $(\mathcal{F}) = (\mathcal{F}, (0, b_0^1] \leq [a_1^1, 1), (I_0^i \leq \dots \leq I_{k_i}^i)_{i=2}^{n+l}) \in L(\text{Alg}_n^{(l)})$ to

$$(u(\mathcal{F})) = (u(\mathcal{F}), (I_0^i \leq \dots \leq I_{k_i}^i)_{i=2}^{n+l}),$$

where $u(\mathcal{F}) = (\pi_{\{2, \dots, n+1\}})_* \mathcal{F}$ is the pushforward of \mathcal{F} along the projection $\pi_{\{2, \dots, n+1\}} : (0, 1)^{\{1, \dots, n+1\}} \rightarrow (0, 1)^{\{2, \dots, n+1\}}$

2. The map ℓ sends an element $(\mathcal{G}) = (\mathcal{G}, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=2}^{n+l}) \in \text{Alg}_n^{(l-1)}$ to

$$(\ell(\mathcal{G})) = (\ell(\mathcal{G}), (0, \frac{1}{2}] \leq [\frac{1}{2}, 1), (I_0^i \leq \dots \leq I_{k_i}^i)_{i=2}^{n+l}),$$

where $\ell(\mathcal{G}) = \iota_*(\mathcal{G})$ is the pushforward of \mathcal{G} along the inclusion

$$\iota : (0, 1)^{n+l-1} \rightarrow (0, 1)^{n+l}, (x_2, \dots, x_{n+l}) \mapsto (\frac{1}{2}, x_2, \dots, x_{n+l}).$$

The map ℓ is called the looping and u the delooping map.

We will need a refinement of theorem 3.2.25 suitable for our situation.

Definition 3.3.9. Let M be a topological space and $N \subseteq M$ be a closed subspace. Then a factorization algebra on M is said to be *supported on N* , if

$$\mathcal{F}|_{M \setminus N} = \mathbb{1}.$$

Recall from theorem 3.2.25 that there is a functor $\underline{pr}_{1*} : \text{Fact}_{X \times Y} \longrightarrow \text{Fact}_X(\text{Fact}_Y)$ given by the pushforward along the projection $pr_1 : X \times Y \rightarrow X$.

Lemma 3.3.10. *Let $X = (0, 1)$ with stratification $X \supset \{s\} \supset \emptyset$ for $s \in X$ and let Y be a stratified manifold with stratification $Y = Y_n \supset Y_{n-1} \supset Y_0 \supset Y_{-1} = \emptyset$. Consider the stratification on $X \times Y$ given by*

$$X \times Y \supset \{s\} \times Y \supset \{s\} \times Y_{n-1} \supset \dots \supset \{s\} \times Y_1 \supset \{s\} \times Y_0 \supset \emptyset,$$

which is coarser than the one from theorem 3.2.25. Then factorization algebras on X lying in $\text{Fact}_X^{lc}(\text{Fact}_Y^{lc})$ which are supported on $\{s\}$ arise from factorization algebras on $X \times Y$ which are locally constant with respect to this coarser stratification and are supported on $\{s\} \times Y$ via the functor \underline{pr}_1 . Moreover, this is a one-to-one correspondence.

Proof. Note that factorization algebras which are locally constant with respect to this coarser stratification (the space of which is denoted by $\text{Fact}_{X \times Y}^{lc, coarse}$) also are locally constant with respect to the finer stratification from 3.2.25. We need to show that the

composition of the inclusion $\mathcal{F}act_{X \times Y}^{lc, coarse} \hookrightarrow \mathcal{F}act_{X \times Y}^{lc}$ with \underline{pr}_{1*} yields an equivalence between elements supported on $\{s\} \times Y \subset X \times Y$ and elements supported on $\{s\} \subset X$.

First, let $\mathcal{F} \in \mathcal{F}act_{X \times Y}^{lc, coarse}$ be supported on $\{s\} \times Y$. We need to check that its image is supported on $\{s\}$. By definition, $\underline{pr}_{1*}\mathcal{F}$ is the factorization algebra on X such that

$$X \supseteq U \longmapsto \mathcal{F}_U, \quad \mathcal{F}_U : V \longmapsto \mathcal{F}_U(V) = \mathcal{F}(U \times V),$$

and \mathcal{F}_U is a factorization algebra on Y . Let $U \subset X \setminus \{s\}$, i.e. $s \notin U$. Then for every $V \subset Y$, $U \times V \subset (X \times Y) \setminus (\{s\} \times Y)$, and since \mathcal{F} is supported on $\{s\} \times Y$,

$$\mathcal{F}_U(V) = \mathcal{F}(U \times V) = \mathbb{1}.$$

Conversely, consider an element in $\mathcal{F}act_X^{lc}(\mathcal{F}act_Y^{lc})$ which is supported in $\{s\}$. From theorem 3.2.25, 1 we know that it arises from a factorization algebra $\mathcal{F} \in \mathcal{F}act_{X \times Y}$. We need to check that \mathcal{F} is supported on $\{s\} \times Y$ and that it is locally constant with respect to the coarse stratification. It is enough to check the conditions on a factorizing basis, so it is enough to check them on products of open sets.

Let $U \times V \subset (X \times Y) \setminus (\{s\} \times Y)$, then $U \subset X \setminus \{s\}$, and

$$\mathcal{F}(U \times V) = \mathcal{F}_U(V) = \mathbb{1}.$$

Now let $U \times V \subset U' \times V'$ be an inclusion of disks such that both $U \times V$ and $U' \times V'$ are good neighborhoods at α for the same index $\alpha \in \{0, \dots, n+1\}$.

If $\alpha = n+1$, then $U \times V \subset U' \times V' \subset (X \times Y) \setminus (\{s\} \times Y)$ and by the above,

$$\mathcal{F}(U \times V) = \mathbb{1} = \mathcal{F}(U' \times V').$$

If $0 \leq \alpha \leq n$, then $(U \times V) \cap (\{s\} \times Y) \neq \emptyset$ and $(U' \times V') \cap \{s\} \times Y \neq \emptyset$, so we get

1. The inclusion $U \subset U'$ is a weak equivalence and the intersections $U \cap \{s\}$ and $U' \cap \{s\}$ are not empty, so they are disks of the same index for X ,
2. Both V and V' are disks of index α for the same α .

Then

$$\mathcal{F}(U \times V) = \mathcal{F}_U(V) \stackrel{1}{\simeq} \mathcal{F}_{U'}(V) \stackrel{2}{\simeq} \mathcal{F}_{U'}(V') = \mathcal{F}(U' \times V').$$

□

The main step in the proof of proposition 3.3.8 is the following observation.

Lemma 3.3.11. *Let $X = (0, 1)$ and $s \in (0, 1)$. Then the data of a factorization algebra \mathcal{F} on X with values in \mathcal{S} which is locally constant with respect to the stratification $X \supset \{s\} \supset \emptyset$ and is supported on s is equivalent to the data of its global sections $\mathcal{F}(X) \in \mathcal{S}$. It can be recovered from its global sections by pushforward along the map $* \rightarrow (0, 1), * \mapsto s$.*

Proof. We showed in corollary 3.2.24 that factorization algebras on Y which are locally constant with respect to a stratification of the form $Y \supset \{s\} \supset \emptyset$ are equivalent to (homotopy) bimodules. The fact that \mathcal{F} is supported on s implies that \mathcal{F} is equivalent to a $(\mathbb{1}, \mathbb{1})$ -bimodule which is the data of an object in \mathcal{S} . \square

Proof of Proposition 3.3.8. Let

$$u : L(\text{Alg}_n^{(l)}) \longrightarrow \text{Alg}_n^{(l-1)},$$

$$(\mathcal{F}) = (\mathcal{F}, (0, b_0^1] \leq [a_1^1, 1), (I_0^i \leq \dots \leq I_{k_i}^i)_{i=2}^{n+l}) \longmapsto (u(\mathcal{F})) = (u(\mathcal{F}), (I_0^i \leq \dots \leq I_{k_i}^i)_{i=2}^{n+l}),$$

where $u(\mathcal{F}) = (\pi_{\{2, \dots, n+1\}})_* \mathcal{F}$ is the pushforward of \mathcal{F} along the projection $\pi_{\{2, \dots, n+1\}} : (0, 1)^{\{1, \dots, n+1\}} \rightarrow (0, 1)^{\{2, \dots, n+1\}}$ forgetting the first coordinate. So it “forgets” the data associated to the first coordinate, which includes the first specified intervals.

Note that setting $X = (0, 1)$ as in the lemmas above and $Y = (0, 1)^{\{2, \dots, n+1\}}$,

$$u(\mathcal{F}) = (\pi_{\{2, \dots, n+1\}})_* \mathcal{F} = \underline{pr}_{1*}(\mathcal{F})(X).$$

By lemma 3.3.10, $\underline{pr}_{1*}(\mathcal{F})$ is locally constant on X , supported on $\{b_0^1\}$, and its global sections $u(\mathcal{F})$ is locally constant with respect to the stratification on $Y = (0, 1)^{\{2, \dots, n+1\}}$. Hence u is well-defined.

Conversely, let

$$\ell : \text{Alg}_n^{(l-1)} \longrightarrow L(\text{Alg}_n^{(l)})$$

$$(\mathcal{G}) = (\mathcal{G}, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=2}^{n+l}) \longmapsto (\ell(\mathcal{G})) = (\ell(\mathcal{G}), (0, \frac{1}{2}] \leq [\frac{1}{2}, 1), (I_0^i \leq \dots \leq I_{k_i}^i)_{i=2}^{n+l}),$$

where $\ell(\mathcal{G}) = \iota_*(\mathcal{G})$ is the pushforward of \mathcal{G} along the inclusion

$$\iota : (0, 1)^{n+l-1} \rightarrow (0, 1)^{n+l}, (x_2, \dots, x_{n+l}) \mapsto (\frac{1}{2}, x_2, \dots, x_{n+l}).$$

By applying first lemma 3.3.11 and then lemma 3.3.10, the map ℓ is well-defined. Moreover, by definition, $u \circ \ell = id$. It remains to show that $\ell \circ u \simeq id$.

Given an element $(\mathcal{F}) \in (L(\text{Alg}_n^{(l)}))_{1, k_2, \dots, k_{n+l}}$, note that the associated stratification on $X = (0, 1)$ is given by $(0, 1) \setminus ((0, b_0^1) \cup (a_1^1, 1))$ either is empty or is equal to a point $s = b_0^1 = a_1^1$. This data is lost when applying u . By the lemmas above, the factorization algebra is recovered under $\ell \circ u$ except for the data of s , which in the definition of ℓ we chose to be $s = \frac{1}{2}$. However, a homotopy from $\ell \circ u$ to the identity is given by the following construction. Let $\xi \in [0, 1]$. Send an element (\mathcal{F}) to its pushforward along f_ξ , which is the (restriction to $(0, 1)$ of the) unique piecewise affine map $\mathbb{R} \rightarrow \mathbb{R}$ such that

$$0 \mapsto 0, \quad s \mapsto s\xi + (1-s)\frac{1}{2}, \quad 1 \mapsto 1.$$

\square

Corollary 3.3.12. *The l -monoidal complete n -fold Segal spaces $\text{Alg}_n^{(l)}$ endow Alg_n with a symmetric monoidal structure.*

3.4 The homotopy category of Alg_1 and the Morita category

The idea behind our construction of Alg_1 was to model an $(\infty, 1)$ -category of algebras and pointed bimodules between them. Indeed, the homotopy category of Alg_1 turns out to be what we expect.

Definition 3.4.1. Let Mor_1 be the category whose objects are algebras and whose morphisms from an algebra A to an algebra B are equivalence classes of (A, B) -bimodules ${}_A M_B$, where ${}_A M_B$ is equivalent to ${}_{A'} M'_{B'}$ iff $A \simeq A'$, $B \simeq B'$, $M \simeq M'$.

Remark 3.4.2. Keep in mind that we are considering algebra and bimodule objects in some symmetric monoidal relative category \mathcal{S} , e.g. $\mathcal{S} = \text{Ch}_k$. If we choose $\mathcal{S} = \text{Vect}_k$ with isomorphisms as weak equivalences, we get the classical category of algebras and bimodules. If we want to specify which relative category the algebra and module objects take values in, we write $\text{Mor}_1(\mathcal{S})$. The symmetric monoidal structure comes from the one on \mathcal{S} , which sends (A, A') to their tensor product $A \otimes A'$ in \mathcal{S} and $({}_A M_B, {}_C N_D)$ to ${}_{A \otimes C} M \otimes N_{B \otimes D}$.

Proposition 3.4.3. *There is an equivalence of symmetric monoidal categories*

$$h_1(\text{Alg}_1) \simeq \text{Mor}_1.$$

Proof. We have seen in examples 3.2.29 and 3.2.36 using 3.2.20 that objects of Alg_1 , and thus also of $h_1(\text{Alg}_1)$ are equivalent to (homotopy) algebras. A (1-)morphism in Alg_1 from A to B is a factorization algebra \mathcal{F} on \mathbb{R} which gives the data of an (A, B) -bimodule ${}_A M_B$. The extra information it encodes is a choice of intervals $(0, b] \leq [a, 1)$ which corresponds to choosing where on $(0, 1)$ the module is located. The space of this extra information is the space of $s \in (0, 1)$ and thus contractible. Moreover, paths from ${}_A M_B$ to ${}_{A'} M'_{B'}$, by definition give weak equivalences $A \simeq A'$, $B \simeq B'$, $M \simeq M'$. Thus, a connected component of the space of (1-)morphisms in Alg_1 from A to B is an equivalence class of (A, B) -bimodules M . Summarizing, there is an equivalence of categories

$$F : h_1(\text{Alg}_1) \longrightarrow \text{Mor}_1,$$

which sends an object $(\mathcal{F}, (0, 1)) \in (\text{Alg}_1)_0$ to $\mathcal{F}((0, 1))$ and a 1-morphism represented by $(\mathcal{F}, (0, b] \leq [a, 1))$ to the $(\mathcal{F}((0, b)), \mathcal{F}([a, 1)))$ -bimodule $\mathcal{F}((0, 1))$.

We saw in example 1.6.9 that the symmetric monoidal structure on Alg_1 induces one on the ordinary category $h_1(\text{Alg}_1)$ coming from the diagram

$$\text{Alg}_1[1] \times \text{Alg}_1[1] \xleftarrow[\gamma_1 \times \gamma_2]{\simeq} \text{Alg}_1[2] \xrightarrow{\gamma} \text{Alg}_1[1],$$

where the first arrow is an equivalence of complete Segal spaces. By the definition of the map γ , it is clear that the equivalence of categories F respects the monoidal structure. \square

3.4.1 The $(\infty, n + 1)$ -category of E_n -algebras

As mentioned above, starting with a symmetric monoidal $(\infty, 1)$ -category \mathcal{S} with all products, factorization algebras on any space X with values in \mathcal{S} again form a symmetric monoidal $(\infty, 1)$ -category. Thus, Alg_n can be extended to an n -fold complete Segal object in $(\infty, 1)$ -categories $\widetilde{\text{Alg}}_n$, and from this one can extract an $(\infty, n + 1)$ -category, which

is moreover symmetric monoidal. It can also be explicitly constructed as a complete $(n+1)$ -fold Segal spaces using, as $(n+1)$ -morphisms morphisms of stratified factorization algebras, which are weak equivalences outside of the lowest dimensional stratum. In the full $(\infty, n+1)$ -category every object is dualizable, but there are fewer fully dualizable objects. This construction will be explained in more detail in [JFS].

In the case of $n = 1$ we saw in the previous section that the homotopy category of Alg_1 is just the Morita category Mor_1 of algebras and equivalence classes of bimodules. This equivalence can be extended to an equivalence of the homotopy bicategory of the $(\infty, 2)$ -category Alg_1 with the full bicategory of algebras, bimodules, and intertwiners, which one might want to call the full “Morita bicategory $\widetilde{\text{Mor}}_1$ of E_1 -algebras”.

3.4.2 An unpointed version

Note that in our construction we use factorization algebras and weak equivalences to model objects, 1-morphisms, and 2-morphisms. As we discussed in section 3.2.9, factorization algebras are pointed, with pointing coming from the monoidal unit $\mathbb{1}$ of the underlying category \mathcal{S} . This pointing leads to pointed bimodules and intertwiners.

For applications one might be interested in an unpointed version to obtain a category with unpointed bimodules as morphisms, which leads to the usual Morita category. For such a construction an unpointed version of factorization algebras which are locally constant with respect to the same stratifications is needed. Such “unpointed factorization algebras” can be defined using an operad similar to the one used in the definition of factorization algebras, see remark 3.2.11, but allowing only certain inclusions of empty sets. However, there is no reason for such an unpointed version of an (∞, n) -category of E_n -algebras to be complete.

Factorization homology as a fully extended topological field theory

Recall that the main task of this thesis is the following. Given any E_n -algebra A , i.e. any object in Alg_n , we would like to define a map of symmetric monoidal n -fold Segal spaces

$$\mathcal{FH}_n(A) : \text{Bord}_n^{fr} \longrightarrow \text{Alg}_n$$

essentially given by taking factorization homology of A . As complete n -fold Segal spaces are models for (∞, n) -categories, this defines a fully extended topological field theory with values in $\mathcal{C} = \text{Alg}_n$.

This chapter deals with the construction of this functor. For better overview we split the construction in two steps. First we construct a map, which is just a map of n -fold simplicial sets, to an auxiliary complete n -fold Segal space of factorization algebras which is essentially given by factorization homology. Then we construct a map which can be understood as “collapsing” and then “rescaling” a factorization algebra. Their composition yields the desired map of n -fold Segal spaces. The construction can be summarized in the following diagram. We indicate in which section the individual maps are constructed.

$$\begin{array}{ccccc}
 \text{PBord}_n^{fr} & \xrightarrow{4.3} & \text{Fact}_n & \xrightarrow{4.4} & \text{FAlg}_n \\
 \uparrow & & \searrow^{4.5} & & \uparrow^{4.4.2} \\
 \text{Bord}_n & \xrightarrow{\mathcal{FH}_n(A)} & & & \text{Alg}_n
 \end{array}$$

This map extends to the symmetric monoidal structures and yields the desired fully extended topological field theory.

4.1 Factorization Homology

Inspired by an algebro-geometric version by Beilinson and Drinfeld in [BD04] and a similar construction by Salvatore in [Sal01], Lurie introduced factorization homology in [Lur] calling it topological chiral homology. It has been studied, amongst others, in [Fra12,

Fra13, AFT12, GTZ10, GTZ12, Hor14b]. We briefly recall the definition and the most important properties we will use in this chapter.

Assumption 2. *From now on, we will require that the symmetric monoidal $(\infty, 1)$ -category \mathcal{S} satisfies the conditions such that factorization homology exists, i.e. following [AFT14], it should be \otimes -sifted cocomplete. In particular, if the symmetric monoidal $(\infty, 1)$ -category \mathcal{S} , which we required to have all small colimits in the previous section, additionally is tensored over spaces, factorization homology will exist.*

Again, as in sections 2.6.1 and 3.2.1, let X be a topological space and $E \rightarrow X$ a topological n -dimensional vector bundle which corresponds to a (homotopy class of) map(s) $e : X \rightarrow BGL(\mathbb{R}^n)$ from X to the classifying space of the topological group $GL(\mathbb{R}^n)$.

Recall from definition 3.2.3 that a $\mathcal{D}isk_n^{(X,E)}$ -algebra in \mathcal{S} is a symmetric monoidal (co-variant) functor

$$A : \mathcal{D}isk_n^{(X,E)} \longrightarrow \mathcal{S}.$$

Now consider an (X, E) -structured n -dimensional manifold M . Since $\mathcal{D}isk_n^{(X,E)} \subseteq \mathcal{M}an_n^{(X,E)}$, it yields a contravariant functor

$$\begin{aligned} \underline{M} : (\mathcal{D}isk_n^{(X,E)})^{op} &\longrightarrow \mathcal{S}pace, \\ \coprod_I \mathbb{R}^n &\longmapsto \text{Emb}^{(X,E)}(\coprod_I \mathbb{R}^n, M). \end{aligned}$$

Definition 4.1.1. Let the *factorization homology of M with coefficients in A* be the homotopy coend of the functor $\underline{M} \times A \longrightarrow \mathcal{S}pace \times \mathcal{S} \xrightarrow{\otimes} \mathcal{S}$ and denote it by

$$\int_M A = \underline{M} \otimes_{\mathcal{D}isk_n^{(X,E)}} A.$$

Remark 4.1.2. Recall that we required \mathcal{S} to contain all small colimits. This ensures the existence of the coend.

Remark 4.1.3. Equivalently, the homotopy coend can be computed by the following homotopy colimit,

$$\int_M A = \text{hocolim}_{\sqcup_{i=1}^n U_i \rightarrow M} \bigotimes_{i=1}^n A(U_i),$$

i.e. the homotopy colimit is taken over

$$(\mathcal{D}isk_n^{(X,E)})_{/M} = \mathcal{D}isk_n^{(X,E)} \otimes_{\mathcal{M}an_n^{(X,E)}} (\mathcal{M}an_n^{(X,E)})_{/M}.$$

Example 4.1.4. If $M = \mathbb{R}^n$, then $\int_{\mathbb{R}^n} A \simeq A$.

In [GTZ10] it was proven that if we consider this construction locally on M for $X = BG$, where G is the trivial group, we obtain a locally constant factorization algebra on M .

Theorem 4.1.5 ([GTZ10], Proposition 13). *Given an E_n -algebra A , i.e. a $\mathcal{D}isk_n^{fr}$ -algebra, the rule*

$$U \mapsto \int_U A$$

for open subsets $U \subseteq M$ with the induced framing extends to a locally constant factorization algebra on M .

Remark 4.1.6. By abuse of notation, we will denote this factorization algebra by $\int_M A$, i.e. for an open subset $U \subseteq M$,

$$\left(\int_M A\right)(U) = \int_U A = \underline{U} \otimes_{\text{Disk}_n^{\text{fr}}} A \in \mathcal{S}.$$

4.2 The auxiliary (∞, n) -category Fact_n

The main idea for our functor is that, given an E_n -algebra A , we define a map, also called $\int_{(-)} A$, which should be given by first taking factorization homology to obtain a factorization algebra on the manifold M and then pushing it forward to obtain a factorization algebra on $(0, 1)^n$, i.e.

$$\begin{array}{ccc} M & \xleftarrow{\iota} & V \times (0, 1)^n \\ & \searrow \pi & \downarrow \\ & & (0, 1)^n \end{array} \quad \xrightarrow{\int_{(-)} A} \quad \pi_* \left(\int_M A \right).$$

We define an auxiliary complete n -fold Segal space Fact_n by translating the properties 1.-3. in the definition of PBord_n to conditions on the factorization algebra. We will show that this is the correct translation in section 4.3.

Similarly as to in the definition of Alg_n , for $S \subseteq \{1, \dots, n\}$, denote by $\pi_S : \mathbb{R}^n \rightarrow \mathbb{R}^S$ the projection onto the coordinates indexed by S .

Definition 4.2.1. Let elements in $(\text{Fact}_n)_{k_1, \dots, k_n}$ be pairs

$$(\mathcal{F}, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n}),$$

satisfying the following conditions:

1. \mathcal{F} is a factorization algebra on $(0, 1)^n$.
2. For $1 \leq i \leq n$,

$$(I_0^i \leq \dots \leq I_{k_i}^i) \in \text{Int}_{k_i}.$$

3. For $1 \leq i \leq n$, the factorization algebra \mathcal{F} is an E_{n-i+1} -algebra in factorization algebras on $(0, 1)^{\{1, \dots, i-1\}}$ in a neighborhood of $\pi_i^{-1}(I_0^i \cup \dots \cup I_{k_i}^i) \subset (0, 1)^n$.

Remark 4.2.2. In condition 3 we first use theorem 3.2.25 to view \mathcal{F} as a factorization algebra on $(0, 1)^{\{i, \dots, n\}}$ in the $(\infty, 1)$ -category of factorization algebras on $(0, 1)^{\{1, \dots, i-1\}}$ and then require that this factorization algebra on $(0, 1)^{\{i, \dots, n\}}$ is locally constant. This is translated to saying that it is an E_{n-i+1} -algebra by using theorem 3.2.20.

4.2.1 The spaces $(\text{Fact}_n)_{k_1, \dots, k_n}$

The spatial structure of $(\text{Fact}_n)_{k_1, \dots, k_n}$ is a mixture of that on Bord_n , essentially coming from the one on the spaces Int_{k_i} , and that on Alg_n .

Definition 4.2.3. An l -simplex in $(\text{Fact}_n)_{k_1, \dots, k_n}$ is given by the data of

1. *underlying 0-simplices*, i.e. for every $s \in |\Delta^l|$,

$$(\mathcal{F}_s, (I_0^i(s) \leq \cdots \leq I_{k_i}^i(s))) \in (\text{Fact}_n)_{k_1, \dots, k_n};$$

2. for every $1 \leq i \leq k_i$,

$$(I_0^i(s) \leq \cdots \leq I_{k_i}^i(s))_{s \in |\Delta^l|}$$

is an l -simplex in Int_{k_i} with rescaling datum $\varphi_{s,t}^i : (0, 1) \rightarrow (0, 1)$;

3. for every $s, t \in |\Delta^l|$, weak equivalences

$$(\varphi_{s,t})_* \mathcal{F}_s \xrightarrow{w_{s,t}} \mathcal{F}_t,$$

where $\varphi_{s,t} = (\varphi_{s,t}^i)_{i=1}^n : (0, 1)^n \rightarrow (0, 1)^n$ is the product of the rescaling data.

The spatial face and degeneracy maps $\delta_l^\Delta, \sigma_l^\Delta$ arise from the face and degeneracy maps of (Δ^l) similarly to those of PBord_n , and we obtain a space $(\text{Fact}_n)_{k_1, \dots, k_n}$.

4.2.2 The n -fold Segal space Fact_n .

We now define face and degeneracy maps on the 0-simplices of the levels of $\text{Fact}_{\bullet, \dots, \bullet}$, essentially coming from those of the n -fold Segal space $(\text{Int}_{\bullet, \dots, \bullet})^n = \text{Int}_{\bullet} \times \cdots \times \text{Int}_{\bullet}$. They are similar to those of Alg_n , but use the rescaling maps ρ_j coming from Int_{\bullet} instead of the collapse-and-rescale maps ϱ_a^b coming from Covers_{\bullet} . Recall that for $j = 0$ or $j = k$, in the usual notation they are the linear rescaling maps

$$\rho_0 : D_0 = (a_1, 1) \rightarrow (0, 1), \quad x \mapsto \frac{x-a_1}{1-a_1}, \quad \rho_k : D_k = (0, b_{k-1}) \rightarrow (0, 1), \quad x \mapsto \frac{x}{b_{k-1}}.$$

Since $1 \leq i \leq n$ will be fixed throughout the following constructions, by abuse of notation we define

$$\rho_j : \pi_i^{-1}(D_j) = \prod_{\alpha \neq i} (0, 1) \times D_j \rightarrow (0, 1)^n,$$

which is ρ_j in the i th coordinate and the identity otherwise.

Degeneracy maps on 0-simplices Fix $1 \leq i \leq n$. For $0 \leq j \leq k_i$ the j th *degeneracy map*

$$s_j^i : (\text{Fact}_n)_{k_1, \dots, k_n} \rightarrow (\text{Fact}_n)_{k_1, \dots, k_i+1, \dots, k_n}$$

applies the j th degeneracy map of Int_{\bullet} to the i th tuple of intervals, i.e. it repeats the j th interval in the i th direction,

$$(\mathcal{F}, (I_0^\alpha \leq \cdots \leq I_{k_\alpha}^\alpha)_{\alpha=1}^n) \longmapsto (\mathcal{F}, (I_0^\alpha \leq \cdots \leq I_{k_\alpha}^\alpha)_{\alpha \neq i}, s_j(I_0^i \leq \cdots \leq I_{k_i}^i)) = (\mathcal{F}, (I_0^\alpha \leq \cdots \leq I_{k_\alpha}^\alpha)_{\alpha \neq i}, I_1^i \leq \cdots \leq I_j^i \leq I_j^i \leq \cdots \leq I_{k_i}^i).$$

Face maps on 0-simplices Fix $1 \leq i \leq n$. For $0 \leq j \leq k_i$ the j th *face map*

$$d_j^i : (\text{Fact}_n)_{k_1, \dots, k_n} \rightarrow (\text{Fact}_n)_{k_1, \dots, k_i-1, \dots, k_n}$$

applies the j th face map of Int_{\bullet} to the i th tuple of intervals, which forgets the j th interval, and, if necessary, rescales them and pushes the factorization algebra forward along the

rescaling map ρ_j . Explicitly, for $j \neq 0, k_i$, the 0-simplex $(\mathcal{F}) = (\mathcal{F}, (I_0^\alpha \leq \dots \leq I_{k_\alpha}^\alpha)_{\alpha=1}^n)$ is sent to

$$\begin{aligned} (\mathcal{F}, (I_0^\alpha \leq \dots \leq I_{k_i}^\alpha)_{\alpha \neq i}, d_j(I_0^i \leq \dots \leq I_{k_i}^i)) &= \\ &= (\mathcal{F}, (I_0^\alpha \leq \dots \leq I_{k_i}^\alpha)_{\alpha \neq i}, I_0^i \leq \dots \leq I_{j-1}^i \leq I_{j+1}^i \leq \dots \leq I_{k_i}^i). \end{aligned}$$

For $j = 0$ or $j = k_i$, the 0-simplex (\mathcal{F}) is sent to

$$((\rho_j)_* \mathcal{F}|_{\pi_i^{-1}(D_j)}, (I_0^\alpha \leq \dots \leq I_{k_i}^\alpha)_{\alpha \neq i}, d_j(I_0^i \leq \dots \leq I_{k_i}^i)).$$

The full structure as an n -fold Segal space Face and degeneracy maps on l -simplices are defined analogous to for Alg_n , by which we obtain an n -fold simplicial space Fact_n .

Proposition 4.2.4. $(\text{Fact}_n)_{\bullet, \dots, \bullet}$ is an n -fold Segal space.

Proof. The proof of the Segal condition works similarly as for Alg_n and essentially follows from the fact that paths of objects arise from weak equivalences and rescaling, which we can use to glue.

It remains to check that for every i and every k_1, \dots, k_{i-1} , the $(n-i)$ -fold Segal space $(\text{Fact}_n)_{k_1, \dots, k_{i-1}, 0, \bullet, \dots, \bullet}$ is essentially constant.

We claim that the composition of degeneracy maps

$$(\text{Fact}_n)_{k_1, \dots, k_{i-1}, 0, \dots, 0} \hookrightarrow (\text{Fact}_n)_{k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n}$$

is a deformation retract.

For $s \in [0, 1]$, consider the path γ_s in $(\text{Fact}_n)_{k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n}$ sending an element represented by

$$(\mathcal{F}) := (\mathcal{F}, (I_0^\beta \leq \dots \leq I_{k_\beta}^\beta)_{1 \leq \beta < i}, (0, 1), (I_0^\alpha \leq \dots \leq I_{k_\alpha}^\alpha)_{i < \alpha \leq n})$$

to

$$(\mathcal{F})_s := (\mathcal{F}, (I_0^\beta \leq \dots \leq I_{k_\beta}^\beta)_{1 \leq \beta < i}, (0, 1), (I_0^\alpha(s) \leq \dots \leq I_{k_\alpha}^\alpha(s))_{i < \alpha \leq n}),$$

where for $\alpha > i$, $a_j^\alpha(s) = (1-s)a_j^\alpha$ and $b_j^\alpha(s) = (1-s)b_j^\alpha + s$. Note that for $s = 0$, $I_0^\alpha(0) = I_0^\alpha$, $I_j^\alpha(0) = I_j^\alpha$ and for $s = 1$, $I_j^\alpha(1) = (0, 1)$.

The collection of paths γ_s form a deformation retraction provided that each path is well-defined, i.e. indeed maps to $(\text{Fact}_n)_{k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n}$. It suffices to check condition (3) in definition 4.2.1 for $(\mathcal{F})_s$. Since $(\mathcal{F}) \in (\text{Fact}_n)_{k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n}$, this reduces to checking

For every $i < \alpha \leq n$, \mathcal{F} is an $E_{n-\alpha+1}$ -algebra in factorization algebras on $(0, 1)^{\{1, \dots, \alpha-1\}}$ in a neighborhood of $\pi_\alpha^{-1}(I_0^\alpha(s) \cup \dots \cup I_{k_\alpha}^\alpha(s)) \subseteq (0, 1)^n$.

Condition (3) on (\mathcal{F}) for i implies that in particular, \mathcal{F} is an E_{n-i+1} -algebra in factorization algebras on $(0, 1)^{\{1, \dots, i-1\}}$ in (a neighborhood of) $\pi_i^{-1}((0, 1)) = (0, 1)^n$.

This in turn implies that for every $\alpha > i$, \mathcal{F} is an $E_{n-\alpha+1}$ -algebra in factorization algebras on $(0, 1)^{\{1, \dots, \alpha-1\}}$ in a neighborhood of $\pi_\alpha^{-1}((0, 1)) = (0, 1)^n \supseteq \pi_\alpha^{-1}(I_0^\alpha(s) \cup \dots \cup I_{k_\alpha}^\alpha(s))$. \square

4.2.4 The symmetric monoidal structure on Fact_n

Fact_n has a symmetric monoidal structure defined by a Γ -object which arises similarly to the structure of Alg_n as a Γ -object.

Definition 4.2.6. For every k_1, \dots, k_n and $m \geq 0$, let $(\text{Fact}_n[m])_{k_1, \dots, k_n}$ be the collection of tuples

$$(\mathcal{F}_1, \dots, \mathcal{F}_m, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n}),$$

where for every $1 \leq \beta \leq m$, $(\mathcal{F}_\beta, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n}) \in (\text{Fact}_n)_{k_1, \dots, k_n}$. Similarly to Fact_n this can be made into a complete n -fold Segal space.

Proposition 4.2.7. *The assignment*

$$\begin{aligned} \Gamma &\longrightarrow \mathbf{SSpace}_n, \\ [m] &\longmapsto \text{Fact}_n[m] \end{aligned}$$

extends to a functor and endows Fact_n with a symmetric monoidal structure.

Proof. Just as for Alg_n , a morphism $f : [m] \rightarrow [k]$ is sent to the functor

$$\begin{aligned} \text{Fact}_n[m] &\longrightarrow \text{Fact}_n[k], \\ (\mathcal{F}_1, \dots, \mathcal{F}_m, I's) &\longmapsto \left(\bigotimes_{\beta \in f^{-1}(1)} \mathcal{F}_\beta, \dots, \bigotimes_{\beta \in f^{-1}(k)} \mathcal{F}_\beta, I's \right). \end{aligned}$$

□

Remark 4.2.8. It is not quite as straightforward to write down the symmetric monoidal structure as a tower and we will not need it later on.

4.3 The map of n -fold simplicial sets $\int_{(-)} A$

In this section, given a fixed E_n -algebra A , we define a map of n -fold simplicial sets

$$\int_{(-)} A : \text{PBord}_n^{fr} \longrightarrow \text{Fact}_n$$

from the framed bordism category to the auxiliary category of factorization algebras. This map essentially translates the properties of the bordisms to factorization algebras on $(0, 1)$. It thus in a certain sense encodes the geometry of the embedded manifold. It will not, however, be a map of n -fold Segal spaces as it does not extend to the simplicial structure of the “levels”, as we explain below in problem 4.3.4.

Recall from definition 2.3.1 that for an element (M) in PBord_n^{fr} we used the following notation, where $S \subseteq \{1, \dots, n\}$:

$$\begin{array}{ccc} M & \longleftrightarrow & V \times (0, 1)^n \\ & \searrow \pi & \downarrow \\ & p_S & (0, 1)^n \\ & \searrow & \downarrow \pi_S \\ & & (0, 1)^S \end{array}$$

The following proposition shows that the third condition on factorization algebras in Fact_n is the exact translation via the map $\pi_*(\int_{(-)} A)$ of the third condition on elements in PBord_n^{fr} .

Proposition 4.3.1. *Let A be an E_n -algebra and let M be an n -dimensional framed manifold. For $S \subseteq \{1, \dots, n\}$, let $p_S : M \rightarrow (0, 1)^S$ be submersive at $x \in p_S^{-1}((t^\alpha)_{\alpha \in S})$. Then $\mathcal{F} := \pi_*(\int_M A)$ is an $E_{|S|}$ -algebra in factorization algebras on $(0, 1)^{\mathbb{Z}^S}$ in a neighborhood of $\pi_S^{-1}((t^\alpha)_{\alpha \in S})$.*

Proof. \mathcal{F} is a factorization algebra on $(0, 1)^n$, so by theorem 3.2.25 it is a factorization algebra on $(0, 1)^S$ with values in factorization algebras on $(0, 1)^{\mathbb{Z}^S}$. We denote it by $\tilde{\mathcal{F}} : U \mapsto \mathcal{F}_U$ for $U \subset (0, 1)^S$, where

$$\mathcal{F}_U : W \mapsto \mathcal{F}(U \times W) \text{ for } W \subset (0, 1)^{\mathbb{Z}^S}.$$

We need to show that $\tilde{\mathcal{F}}$ is locally constant in a neighborhood of $(t^\alpha)_{\alpha \in S}$. Take $V \subset U \subset (0, 1)^S$ two sufficiently small open sets containing $(t^\alpha)_{i \in S}$ such that $U \simeq V$. The structure map $\mathcal{F}_V \rightarrow \mathcal{F}_U$ is a weak equivalence if for every open set $W \subset (0, 1)^{\mathbb{Z}^S}$, the map $\mathcal{F}_V(W) \rightarrow \mathcal{F}_U(W)$ is a weak equivalence. Consider

$$\begin{aligned} \mathcal{F}_U(W) &= \mathcal{F}(U \times W) = \pi_* \left(\int_M A \right) (U \times W) = \left(\int_M A \right) (\pi^{-1}(U \times W)), \\ \mathcal{F}_V(W) &= \mathcal{F}(V \times W) = \pi_* \left(\int_M A \right) (V \times W) = \left(\int_M A \right) (\pi^{-1}(V \times W)). \end{aligned}$$

Since $\int_M A$ is locally constant, it is enough to show that the inclusion $\pi^{-1}(V \times W) \subset \pi^{-1}(U \times W)$ is a weak equivalence. Since

$$\pi^{-1}(V \times W) = p_S^{-1}(V) \cap p_{\mathbb{Z}^S}^{-1}(W) \quad \text{and} \quad \pi^{-1}(U \times W) = p_S^{-1}(U) \cap p_{\mathbb{Z}^S}^{-1}(W),$$

it is enough to show that $p_S^{-1}(V) \hookrightarrow p_S^{-1}(U)$ is a weak equivalence. This holds because we assumed that $V \simeq U$ and that p_S is a submersion at $p_S^{-1}((t^i)_{i \in S})$, so locally a projection map. \square

Definition 4.3.2. Let A be an E_n -algebra. Let

$$\int_{(-)} A : \text{PBord}_n^{fr} \longrightarrow \text{Fact}_n$$

send

$$(M \hookrightarrow V \times (0, 1)^n, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n}) \in (\text{PBord}_n)_{k_1, \dots, k_n}$$

to

$$\left(\pi_* \left(\int_M A \right), (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n} \right) \in (\text{Fact}_n)_{k_1, \dots, k_n},$$

where, as in the previous sections, $\pi : M \hookrightarrow V \times (0, 1)^n \twoheadrightarrow (0, 1)^n$.

Proposition 4.3.3. $\int_{(-)} A$ is a well-defined map of n -fold simplicial sets.

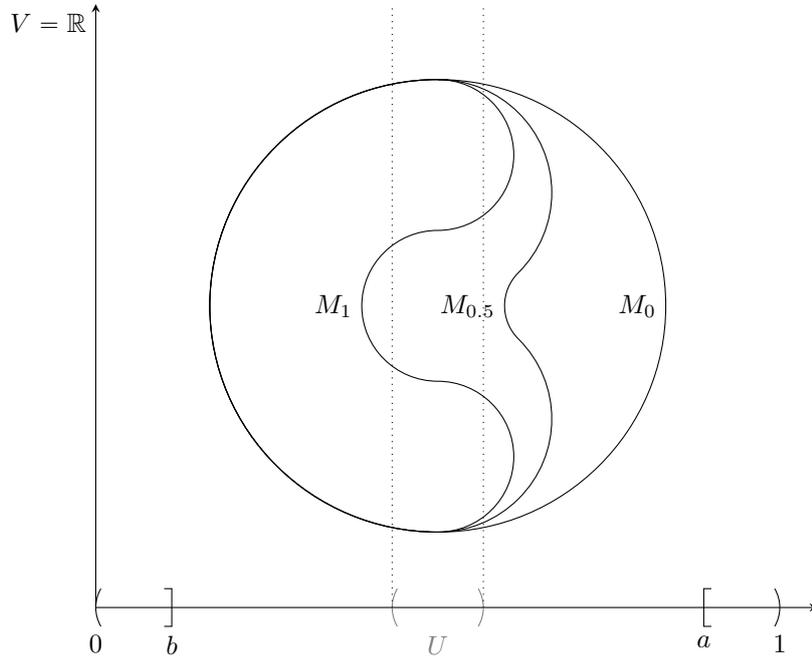
Proof. By the above proposition, $(\int_{(-)} A)((M))$ is an element in $(\text{Fact}_n)_{k_1, \dots, k_n}$. Moreover, $\int_{(-)} A$ commutes with the face and degeneracy maps d_j^i, s_j^i and δ_j^i, σ_j^i of the n -fold simplicial sets $(\text{PBord}_n^{fr})_{\bullet, \dots, \bullet}$ and $(\text{Fact}_n)_{\bullet, \dots, \bullet}$ by construction. \square

Problem 4.3.4. $\int_{(-)} A$ does not extend to a map between l -simplices of the levels, i.e. $(\int_{(-)} A)_{k_1, \dots, k_n}$ is not a map of simplicial sets

$$\left(\int_{(-)} A\right)_{k_1, \dots, k_n} : (\text{PBord}_n)_{k_1, \dots, k_n} \longrightarrow (\text{Fact}_n)_{k_1, \dots, k_n},$$

as can be seen in the following example.

Consider the following 1-simplex in $(\text{Bord}_1)_1$, which is given by a smooth deformation of the standard embedding of the circle, $[0, 1] \times S^1 \hookrightarrow [0, 1] \times \mathbb{R} \times (0, 1)$, and the pair of intervals $(0, b] \leq [a, 1)$.



The factorization algebra $\mathcal{F}_1 = (\pi_1)_*(\int_{S^1} A)$ associated to $s = 1$ is not weakly equivalent to that associated to $s = 0$ (even after any rescaling of $(0, 1)$), as its value on the open set U as given in the picture is

$$\mathcal{F}_1(U) = A^{\otimes 2} \otimes (A^{op})^{\otimes 2},$$

but $\mathcal{F}_0 = (\pi_0)_*(\int_{S^1} A)$ on intervals takes on values $\mathbb{1}, A, A^{op}$, or $A \otimes A^{op}$.

4.4 Collapsing the factorization algebra and FAlg_n

In this section, we explain how to “collapse” a factorization algebra in Fact_n . We define a map of n -fold simplicial sets

$$\nabla : \text{Fact}_n \longrightarrow \text{FAlg}_n$$

to an n -fold Segal space $\text{FAlg}_n \supseteq \text{Alg}_n$ of factorization algebras on $(0, 1)^n$, which have certain locally constancy properties, but do not lead to bimodules.

We first define a collapse-and-rescale map $\mathfrak{v} : \text{Int}_\bullet \rightarrow \text{Covers}_\bullet$ given by applying a collapse-and-rescale map $\varrho_{\underline{a}}^{\underline{b}} : (0, 1) \rightarrow (0, 1)$ to a tuple of intervals with endpoints $\underline{a}, \underline{b}$. This map is lifted to a map $\mathfrak{V} : \text{Fact}_n \rightarrow \text{FAlg}_n$ by pushing forward the factorization algebra along the product of the collapse-and-rescale maps.

4.4.1 The collapse-and-rescale map $\mathfrak{v} : \text{Int}_\bullet \rightarrow \text{Covers}_\bullet$

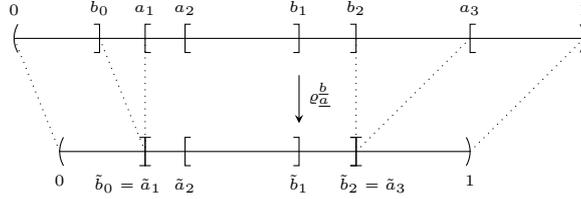
... on the levels

Informally speaking, we first collapse the complement of all intervals and then rescale the rest to $(0, 1)$. We saw in lemma 3.1.4 that the collapse-and-rescale maps ϱ_a^b commute in a suitable way. This ensures that we can define the collapse-and-rescale map $\varrho_{\underline{a}}^{\underline{b}}$ can be defined as a successive application of ϱ_a^b 's.

Definition 4.4.1. Let I_0, \dots, I_k be closed intervals in $(0, 1)$ with non-empty interior and endpoints $\underline{a} = (a_0, \dots, a_k)$, $\underline{b} = (b_0, \dots, b_k)$. Then, let

$$\varrho_{\underline{a}}^{\underline{b}} = \varrho_{a_1}^{b_0} * \varrho_{a_2}^{b_1} \cdots * \varrho_{a_k}^{b_{k-1}}.$$

Note that since by definition $(a_\alpha, b_\alpha) \neq \emptyset$, $(b_{\alpha-1}, a_\alpha) \cap (b_\alpha, a_{\alpha+1}) = \emptyset$. So we can apply lemma 3.1.4 and the map $\varrho_{\underline{a}}^{\underline{b}}$ is independent of the order of maps $\varrho_{a_{\alpha+1}}^{b_\alpha}$. In the following, we will apply this to $(I_0 \leq \dots \leq I_k) \in \text{Int}_k$.



Notation 4.4.2. For I_0, \dots, I_k as above let $\{j_1, \dots, j_l\} \subseteq \{0, \dots, k-1\}$ be the indices for which $b_{j_\beta} < a_{j_\beta+1}$, i.e. $\varrho_{a_{j_\beta+1}}^{b_{j_\beta}} \neq id$. Then, similarly as we saw for ϱ_a^b ,

$$\varrho_{\underline{a}}^{\underline{b}}|_{D_{\underline{a}}^{\underline{b}}},$$

for $D_{\underline{a}}^{\underline{b}} = (0, b_{j_1}] \cup (a_{j_1+1}, b_{j_2}] \cup \dots \cup (a_{j_l+1}, 1)$ is bijective. We denote its inverse by

$$(\varrho_{\underline{a}}^{\underline{b}})^{-1} = (\varrho_{\underline{a}}^{\underline{b}}|_{D_{\underline{a}}^{\underline{b}}})^{-1} : (0, 1) \longrightarrow D_{\underline{a}}^{\underline{b}}.$$

... as a map of complete Segal spaces

Proposition 4.4.3. *The map*

$$\begin{aligned} \text{Int}_k &\xrightarrow{\mathfrak{v}_k} \text{Covers}_k, \\ (I_0 \leq \dots \leq I_k) &\longmapsto (\varrho_{\underline{a}}^{\underline{b}}(I_0) \leq \dots \leq \varrho_{\underline{a}}^{\underline{b}}(I_k)), \end{aligned}$$

extends to a map of complete Segal spaces.

Proof. We first need to show that the map \mathfrak{V}_k extends to a map of spaces $\mathfrak{V}_k : \text{Int}_k \rightarrow \text{Covers}_k$, i.e. we need to define it on l -simplices and show that it commutes with the spatial face and degeneracy maps s_l^Δ, d_l^Δ of Int_k and $\sigma_l^\Delta, \delta_l^\Delta$ of Covers_k . Finally we need to show that all \mathfrak{V}_k together form a map of simplicial spaces, i.e. they commute with the simplicial face and degeneracy maps s_j, d_j of Int_\bullet and σ_j, δ_j of Covers_\bullet .

... on l -simplices Consider an l -simplex in Int_k consisting of underlying 0-simplices $(I_1(s) \leq \dots \leq I_k(s))_{s \in |\Delta^l|}$ and a rescaling datum $(\varphi_{s,t} : (0,1) \rightarrow (0,1))_{s,t \in |\Delta^l|}$. It is sent to the l -simplex in Covers_k defined as follows:

1. for $s \in |\Delta^l|$, the s th underlying 0-simplex of the image is

$$\left(\varrho_{\underline{a}}^b(I_0(s)) \leq \dots \leq \varrho_{\underline{a}}^b(I_k(s)) \right) \in \text{Covers}_k;$$

2. for $s, t \in |\Delta^l|$, the rescaling datum is

$$\phi_{s,t} = \varrho_{\underline{a}(t)}^{b(t)} \circ \varphi_{s,t} \big|_{D_{\underline{a}(s)}^b} \circ (\varrho_{\underline{a}(s)}^{b(s)})^{-1} : (0,1) \rightarrow (0,1).$$

... commutes with the spatial degeneracy and face maps The map \mathfrak{V}_k commutes with spatial degeneracy and face maps since these come from the degeneracy and face maps of the simplicial set $(\Delta^l)_l$.

... commutes with the simplicial degeneracy and face maps This essentially follows from the behaviour of the collapse-and-rescale maps $\varrho_{\underline{a}}^b$, which is summarized in the following lemma. \square

Lemma 4.4.4. *The following diagram commutes:*

$$\begin{array}{ccc} \text{Int}_{k+1} & \xrightarrow{\mathfrak{V}_{k+1}} & \text{Covers}_{k+1} \\ \uparrow d_j & & \uparrow \delta_j \\ \text{Int}_k & \xrightarrow{\mathfrak{V}_k} & \text{Covers}_k \\ \downarrow s_j & & \downarrow \sigma_j \\ \text{Int}_{k-1} & \xrightarrow{\mathfrak{V}_{k-1}} & \text{Covers}_{k-1} \end{array}$$

Proof. The collapse-and-rescaling maps on the top and in the middle coincide, since $\varrho_{\underline{a}_j}^{b_j} = \text{id}$ and therefore

$$\varrho_{d_j(\underline{a})}^{d_j(\underline{b})} = \dots * \varrho_{\underline{a}_j}^{b_{j-1}} * \varrho_{\underline{a}_j}^{b_j} * \varrho_{\underline{a}_{j+1}}^{b_{j+1}} = \varrho_{\underline{a}}^b.$$

Thus the top diagram commutes.

For the lower diagram, we need to compare the composition of the (collapse-and-)rescaling maps.

$$\begin{array}{ccc}
\text{Int}_k & \xrightarrow{\varrho_{\underline{a}}^b} & \text{Covers}_k \\
\text{id or } \rho_j \downarrow & & \downarrow \varrho_{a_{j+1}}^{b_{j-1}} \\
\text{Int}_{k-1} & \xrightarrow{\varrho_{\underline{a}}^b} & \text{Covers}_{k-1}
\end{array}$$

The upper right composition $\sigma_j \circ \mathbf{x}_k$ has as rescaling map $\varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{\underline{a}}^b$. Using lemma 3.1.4 and remark 3.1.3 we obtain

$$\begin{aligned}
\varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{\underline{a}}^b &= \varrho_{a_{j+1}}^{b_{j-1}} * (\varrho_{a_1}^{b_0} * \varrho_{a_2}^{b_1} \cdots * \varrho_{a_k}^{b_{k-1}}) \\
&\stackrel{3.1.4}{=} \varrho_{a_{j+1}}^{b_{j-1}} * (\varrho_{a_j}^{b_{j-1}} * \varrho_{a_{j+1}}^{b_j} * \underset{\alpha \neq j-1, j}{*} \varrho_{a_{\alpha+1}}^{b_\alpha}) \\
&= (\varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{a_j}^{b_{j-1}}) * \varrho_{a_{j+1}}^{b_j} * \underset{\alpha \neq j-1, j}{*} \varrho_{a_{\alpha+1}}^{b_\alpha} \\
&\stackrel{3.1.3}{=} (\varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{a_{j+1}}^{b_j}) * \underset{\alpha \neq j-1, j}{*} \varrho_{a_{\alpha+1}}^{b_\alpha} \\
&\stackrel{3.1.3}{=} \varrho_{a_{j+1}}^{b_{j-1}} * \underset{\alpha \neq j-1, j}{*} \varrho_{a_{\alpha+1}}^{b_\alpha} \\
&\stackrel{3.1.4}{=} \varrho_{a_1}^{b_0} * \cdots * \varrho_{a_{j-1}}^{b_{j-2}} * \varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{a_{j+2}}^{b_j} * \cdots * \varrho_{a_k}^{b_{k-1}} \\
&= \varrho_{\widehat{\underline{a}}^j}^{\widehat{\underline{b}}^j},
\end{aligned}$$

where $\widehat{\underline{a}}^j = (a_0, \dots, a_{j-1}, a_{j+1}, \dots, a_k)$ and $\widehat{\underline{b}}^j = (b_0, \dots, b_{j-1}, b_{j+1}, \dots, b_k)$.

For $j \neq 0, k$, we have that $s_j(\underline{a}) = \widehat{\underline{a}}^j$ and $s_j(\underline{b}) = \widehat{\underline{b}}^j$ and thus the lower left composition $\mathbf{x}_{k-1} \circ s_j$ has as rescaling map

$$\varrho_{s_j(\underline{a})}^{s_j(\underline{b})} = \varrho_{\widehat{\underline{a}}^j}^{\widehat{\underline{b}}^j}.$$

For $j = 0$ or $j = k$ we have that $s_j(\underline{a}) = \rho_j(\widehat{\underline{a}}^j) = \varrho_{a_{j+1}}^{b_{j-1}}(\widehat{\underline{a}}^j)$ and $s_j(\underline{b}) = \rho_j(\widehat{\underline{b}}^j) = \varrho_{a_{j+1}}^{b_{j-1}}(\widehat{\underline{b}}^j)$, and thus the lower left composition $\mathbf{x}_{k-1} \circ s_j$ has as rescaling map

$$\begin{aligned}
\varrho_{s_j(\underline{a})}^{s_j(\underline{b})} &= \varrho_{\widehat{\underline{a}}^j}^{\widehat{\underline{b}}^j} * \varrho_{a_{j+1}}^{b_{j-1}} \\
&= (\varrho_{a_1}^{b_0} * \cdots * \varrho_{a_{j-1}}^{b_{j-2}} * \varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{a_{j+2}}^{b_j} * \cdots * \varrho_{a_k}^{b_{k-1}}) * \varrho_{a_{j+1}}^{b_{j-1}} \\
&= \varrho_{a_1}^{b_0} * \cdots * \varrho_{a_{j-1}}^{b_{j-2}} * \varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{a_{j+2}}^{b_j} * \cdots * \varrho_{a_k}^{b_{k-1}} \\
&= \varrho_{\widehat{\underline{a}}^j}^{\widehat{\underline{b}}^j},
\end{aligned}$$

since similarly to above, by lemma 3.1.4 we can first reorder the terms in the parentheses, use $\varrho_{a_{j+1}}^{b_{j-1}} * \varrho_{a_{j+1}}^{b_{j-1}} = \varrho_{a_{j+1}}^{b_{j-1}}$ by remark 3.1.3, and then reorder again. \square

4.4.2 The “faux” Alg_n , the n -fold Segal space FAlg_n

Recall that in definition 3.2.26, given $(I_0^i \leq \cdots \leq I_{k_i}^i)_{i=1, \dots, n} \in (\text{Covers}^n)_{k_1, \dots, k_n}$, we inductively defined a stratification of $(0, 1)^n$ by

$$X_n = (0, 1)^n, \quad X_{n-i} = X_{n-i+1} \cap Y_i,$$

for $1 \leq i \leq n$, where

$$Y_i = \pi_i^{-1}(S^i) \quad \text{for } S^i = (0, 1) \setminus \bigcup_{j=0}^{k_i} (a_j^i, b_j^i) = (0, 1) \setminus \bigcup_{j=0}^{k_i} (I_j^i)^\circ,$$

and $(I_j^i)^\circ = (a_j^i, b_j^i)$ is the interior of the interval I_j^i . Note that the set $X \setminus Y_i = \bigcup_{j=0}^{k_i} (a_j^i, b_j^i) \times (0, 1)^{\underline{n} \setminus i}$ is a disjoint union of products of the form

$$(0, s_1^i) \times (0, 1)^{\underline{n} \setminus i}, \quad (s_j^i, s_{j+1}^i) \times (0, 1)^{\underline{n} \setminus i}, \quad \text{or } (s_{l_i}^i, 1) \times (0, 1)^{\underline{n} \setminus i},$$

where $\underline{n} = \{1, \dots, n\}$.

We now define a ‘‘faux’’ n -fold Segal space FAlg_n of E_n -algebras, whose objects are E_n -algebras, but the morphisms do not behave like modules.

Definition 4.4.5. For every $k_1, \dots, k_n \geq 0$, let $(\text{FAlg}_n)_{k_1, \dots, k_n}$ be the collection of tuples

$$(\mathcal{F}, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1, \dots, n}),$$

satisfying the following conditions:

1. \mathcal{F} is a factorization algebra on $(0, 1)^n$.
2. For $1 \leq i \leq n$,

$$(I_0^i \leq \dots \leq I_{k_i}^i) \in \text{Covers}_{k_i},$$
3. For $1 \leq i \leq n$, on every connected component of $X \setminus Y_i$, the factorization algebra \mathcal{F} is an E_{n-i+1} -algebra in factorization algebras on $(0, 1)^{\{1, \dots, i-1\}}$.

We make the collection $(\text{FAlg}_n)_{\bullet, \dots, \bullet}$ into an n -fold Segal space similarly to $(\text{Alg}_n)_{\bullet, \dots, \bullet}$.

Remark 4.4.6. Similarly to definition 4.2.1 we use theorem 3.2.25 to formulate the condition on the factorization algebra \mathcal{F} .

Example 4.4.7. For $n = 1$, $(\text{FAlg}_1)_k$ consists of elements of the form

$$(\mathcal{F}, I_0 \leq \dots \leq I_k),$$

where \mathcal{F} is a factorization algebra on $(0, 1)$ and is locally constant everywhere except at the points $S = \{s_1, \dots, s_l\} = (0, 1) \setminus (I_0 \cup \dots \cup I_k)$. In particular, $(\text{FAlg}_1)_0 = (\text{Alg}_1)_0$ and consists of locally constant factorization algebras on $(0, 1)$, i.e. E_1 -algebras. However, for $k > 1$, $(\text{Alg}_1)_k$ is the proper subset of $(\text{FAlg}_1)_k$ of elements which furthermore satisfy the condition that if U, V are intervals containing the same point s_j , $\mathcal{F}(U) \simeq \mathcal{F}(V)$.

Proposition 4.4.8. *There is an inclusion of n -fold Segal spaces*

$$\text{Alg}_n \subset \text{FAlg}_n.$$

Proof. Recall from definition 3.2.26 that

$$X_{n-\alpha} = S^1 \times \dots \times S^\alpha \times (0, 1)^{\{\alpha+1, \dots, n\}}.$$

Thus, the stratification induces a stratification on $X \setminus Y_i$ of the form

$$\begin{aligned} (X \setminus Y_i) \cap X_{n-\alpha} &= S^1 \times \cdots \times S^\alpha \times (0, 1)^{\{\alpha+1, \dots, i-1\}} \times ((0, 1) \setminus S^i) \times (0, 1)^{\{i+1, \dots, n\}} \\ &= \tilde{X}_{n-\alpha} \times (0, 1)^{\{i+1, \dots, n\}}, \end{aligned}$$

where

$$\tilde{X}_{n-\alpha} = S^1 \times \cdots \times S^\alpha \times (0, 1)^{\{\alpha+1, \dots, i-1\}} \times ((0, 1) \setminus S^i).$$

for $0 \leq \alpha < i$.

Let $(\mathcal{F}, (I_0^i \leq \cdots \leq I_{k_i}^i)_{1 \leq i \leq n}) \in (\text{Alg}_n)_{k_1, \dots, k_n}$. The restriction $\mathcal{F}|_{X \setminus Y_i}$ is locally constant with respect to the stratification $(X \setminus Y_i) \cap X_{n-\alpha}$. Thus, as a factorization algebra on $(0, 1)^{\{i+1, \dots, n\}}$ with values in factorization algebras on $(0, 1)^{\{1, \dots, i\}}$ it is locally constant. \square

4.4.3 The collapsing map $\mathbb{V} : \text{Fact}_n \rightarrow \text{FAlg}_n$

We can now lift the collapsing map $\mathfrak{v} : \text{Int} \rightarrow \text{Covers}$ to a collapsing map $\mathbb{V} : \text{Fact}_n \rightarrow \text{FAlg}_n$.

Notation 4.4.9. Let $(I_0^i \leq \cdots \leq I_{k_i}^i)_{i=1, \dots, n} \in \text{Int}_{k_1, \dots, k_n}^n$. For $1 \leq i \leq n$ denote the collapse-and-rescale map associated to $(I_0^i \leq \cdots \leq I_{k_i}^i) \in \text{Int}_{k_i}$ by $\varrho_{\underline{a}^i}^{b^i}$, and denote their product by

$$\bar{\varrho}_{\underline{a}} = (\varrho_{\underline{a}^1}^{b^1}, \dots, \varrho_{\underline{a}^n}^{b^n}) : (0, 1)^n \longrightarrow (0, 1)^n.$$

Note that

$$\bar{\varrho}_{\underline{a}} = \varrho_{\underline{a}^1}^{b^1} \circ \cdots \circ \varrho_{\underline{a}^n}^{b^n},$$

where as before we again denote by $\varrho_{\underline{a}^i}^{b^i}$ the map $(0, 1)^n \longrightarrow (0, 1)^n$ which is $\varrho_{\underline{a}^i}^{b^i}$ in the i th coordinate and the identity otherwise, and the order in the above composition does not matter.

Proposition 4.4.10.

$$\begin{aligned} (\text{Fact}_n)_{k_1, \dots, k_n} &\xrightarrow{\mathbb{V}} (\text{FAlg}_n)_{k_1, \dots, k_n} \\ (\mathcal{F}, (I_0^i \leq \cdots \leq I_{k_i}^i)_{1 \leq i \leq n}) &\longmapsto \left((\bar{\varrho}_{\underline{a}})_* (\mathcal{F}), (\mathfrak{v}(I_0^i \leq \cdots \leq I_{k_i}^i))_{1 \leq i \leq n} \right) \end{aligned}$$

is a map of n -fold Segal spaces.

Proof. As we have seen in lemma 4.4.4 that the (collapse-and-)rescaling maps behave well with respect to face and degeneracy maps of the simplicial space, it is enough to show that \mathbb{V} indeed maps to FAlg_n .

We need to check the third condition in definition 4.4.5, i.e. that for $1 \leq i \leq n$, on

$$X \setminus Y_i = \pi_i^{-1} \left(\bigcup_{j=0}^{k_i} \varrho_{\underline{a}^i}^{b^i} (I_j^i)^\circ \right)$$

$(\bar{\varrho}_{\underline{a}})_* \mathcal{F}$ is an E_{n-i+1} -algebra in factorization algebras on $(0, 1)^{\{1, \dots, i-1\}}$.

For this it is enough to show that for every $0 \leq j \leq k_i$, we have

$$\left((\varrho_{\underline{a}}^{\bar{b}})_* \mathcal{F} \right) \Big|_{\pi_i^{-1}(\varrho_{\underline{a}^i}^{b^i}(I_j^i)^\circ)} = (\varrho_{\underline{a}}^{\bar{b}})_* \left(\mathcal{F} \Big|_{\pi_i^{-1}(I_j^i)^\circ} \right)$$

is an E_{n-i+1} -algebra in factorization algebras on $(0, 1)^{\{1, \dots, i-1\}}$.

Since $(\mathcal{F}, (I_0^i \leq \dots \leq I_{k_i}^i)_{1 \leq i \leq n}) \in (\text{Fact}_n)_{k_1, \dots, k_n}$, $\mathcal{F} \Big|_{\pi_i^{-1}(I_j^i)^\circ}$ is an E_{n-i+1} -algebra in factorization algebras on $(0, 1)^{\{1, \dots, i-1\}}$, so the following lemma finishes the proof. \square

Lemma 4.4.11. *Let \mathcal{G} be a locally constant factorization algebra on $(0, 1)$ and let ϱ_a^b be a collapse-and-rescaling map. Then $(\varrho_a^b)_* \mathcal{G}$ is locally constant on $(0, 1)$.*

Proof. This follows from the fact that preimages of intervals under ϱ_a^b again are intervals. \square

4.5 The functor of (∞, n) -categories \mathcal{FH}_n

We now show that, given an E_n -algebra A , the composition $\mathbb{V} \circ \int_{(-)} A$ lands in Alg_n and thus yields a map

$$\mathcal{FH}_n = \mathcal{FH}_n(A) : \text{PBord}_n^{fr} \longrightarrow \text{Alg}_n.$$

Proposition 4.5.1. *Let $(M) = (M \hookrightarrow V \times (0, 1)^n, (I_0^i \leq \dots \leq I_{k_i}^i)_{1 \leq i \leq n}) \in (\text{PBord}_n)_{k_1, \dots, k_n}$. Then*

$$\left(\mathbb{V} \circ \int_{(-)} A \right) ((M)) \in (\text{Alg}_n)_{k_1, \dots, k_n},$$

Proof. Let $\pi : M \hookrightarrow V \times (0, 1)^n \rightarrow (0, 1)^n$ and as usual denote the endpoints of the interval I_j^i by a_j^i, b_j^i . We need to show that the underlying factorization algebra of $\mathcal{FH}_n((M))$, which is

$$\mathcal{F}_{(M)} = (\varrho_{\underline{a}}^{\bar{b}})_* \pi_* \int_M A,$$

is locally constant with respect to the stratification associated to the intervals $(\varrho_{\underline{a}^i}^{b^i}(I_0^i) \leq \dots \leq \varrho_{\underline{a}^i}^{b^i}(I_{k_i}^i))_{i=1}^n$.

Let $V \subseteq U$ be good neighborhoods at $X_{n-\alpha} = S^1 \times \dots \times S^\alpha \times (0, 1)^{\{\alpha+1, \dots, n\}}$ from definition 3.2.26 respectively remark 3.2.27. We can assume that they are boxes, i.e. products of intervals

$$U = U^1 \times \dots \times U^n, \quad V = V^1 \times \dots \times V^n$$

and meet exactly one connected component

$$(s_j^i)_{\beta=1}^\alpha \times (0, 1)^{\{\alpha+1, \dots, n\}}$$

of $X_{n-\alpha}$. We need to show that the structure map $\mathcal{F}_{(M)}(V) \rightarrow \mathcal{F}_{(M)}(U)$ is a weak equivalence. By definition,

$$\mathcal{F}_{(M)}(V) = \left(\int_M A \right) (\pi^{-1}(\tilde{V})) \quad \text{and} \quad \mathcal{F}_{(M)}(U) = \left(\int_M A \right) (\pi^{-1}(\tilde{U})),$$

where $\tilde{V} = (\varrho_{\bar{a}}^{\bar{b}})^{-1}(V)$ and $\tilde{U} = (\varrho_{\bar{a}}^{\bar{b}})^{-1}(U)$. Since $\int_M A$ is locally constant, it is enough to show that the inclusion $\pi^{-1}(\tilde{V}) \hookrightarrow \pi^{-1}(\tilde{U})$ is a weak equivalence.

The open sets \tilde{V} and \tilde{U} are boxes of open intervals,

$$\tilde{V} = (e^1, f^1) \times \cdots \times (e^n, f^n) \quad \text{and} \quad \tilde{U} = (c^1, d^1) \times \cdots \times (c^n, d^n)$$

where $(c^i, d^i) = (\varrho_{\bar{a}^i}^{\bar{b}^i})^{-1}(U^i)$ and $(e^i, f^i) = (\varrho_{\bar{a}^i}^{\bar{b}^i})^{-1}(V^i)$. The endpoints c^i and e^i , respectively d^i and f^i , either lie in the same closed specified interval $I_{j^i}^i$ or in ones connected by a chain of overlapping intervals. An argument similar to that in corollary 2.3.4 or, if an endpoint is 0 or 1, example 2.3.2 gives a diffeomorphism

$$\pi^{-1}(\tilde{U}) \longrightarrow \pi^{-1}(\tilde{V}).$$

□

Definition 4.5.2. Let

$$\mathcal{FH}_n = \mathcal{FH}_n(A) = \mathbb{V} \circ \int_{(-)} A : \text{PBord}_n^{fr} \longrightarrow \text{Alg}_n,$$

$$\mathcal{FH}_n((M)) = (\mathcal{F}_M, (\mathbb{V}(I_0^i \leq \dots \leq I_{k_i}^i)_{i=1}^n)),$$

where $\mathcal{F}_{(M)} = (\varrho_{\bar{a}}^{\bar{b}})_* \pi_* \int_M A$. By the universal property of the completion it extends to a map of complete n -fold Segal spaces

$$\mathcal{FH}_n = \mathcal{FH}_n(A) : \text{Bord}_n^{fr} \longrightarrow \text{Alg}_n.$$

Example 4.5.3. A point viewed as an object $*$ in the bordism category is represented by the trivial bordism

$$(M = (0, 1)^n \hookrightarrow (0, 1)^n, ((0, 1))_{i=1}^n).$$

We have seen in example 4.1.4 that $\int_{(0,1)^n} A \simeq \int_{\mathbb{R}^n} A \simeq A$ as a factorization algebra on $M = (0, 1)^n$. Then $\pi : M = (0, 1)^n \rightarrow (0, 1)^n$ is the identity map and $\varrho_{\bar{a}}^{\bar{b}} = id$. So,

$$\mathcal{FH}_n(A)(*) = (A, ((0, 1))_{i=1}^n) \in (\text{Alg}_n)_{0, \dots, 0}.$$

4.6 The fully extended topological field theory \mathcal{FH}_n

To obtain a fully extended topological field theory the functor $\mathcal{FH}_n(A)$ needs to be symmetric monoidal. In this section we extend it to a symmetric monoidal functor, both by defining a natural transformation of Γ -objects and by defining compatible functors between the layers of the towers.

4.6.1 Symmetric monoidality via Γ -objects

We extend the map $\mathcal{FH}_n(A)$ to a natural transformation between functors $\Gamma \rightarrow \mathbf{SSpace}_n$, $[m] \mapsto \text{PBord}_n^{fr}[m], \text{Alg}_n[m]$.

Proposition 4.6.1. *For every object $[m] \in \Gamma$, let $\mathcal{FH}_n[m] = \mathcal{FH}_n(A)[m]$ be the map of n -fold Segal spaces*

$$\begin{aligned} \text{PBord}_n^{fr}[m] &\longrightarrow \text{Alg}_n[m], \\ (M_1, \dots, M_m, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1}^n) &\longmapsto (\mathcal{F}_{(M_1)}, \dots, \mathcal{F}_{(M_m)}, \mathfrak{V}(I_0^i \leq \dots \leq I_{k_i}^i)_{i=1}^n). \end{aligned}$$

This assignment endows the functor $\mathcal{FH}_n(A)$ of (∞, n) -categories with a symmetric monoidal structure.

Proof. The map $\mathcal{FH}_n[m]$ is well-defined since the the image of the left-hand element under the inclusion $\text{PBord}_n[m] \subseteq (\text{PBord}_n[1])^m$ is the collection of

$$(M_\beta, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1}^n) \in (\text{PBord}_n^{fr, V})_{k_1, \dots, k_n}$$

which under $\mathcal{FH}_n(A)$ are sent to elements in $(\text{Alg}_n)_{k_1, \dots, k_n}$ with underlying factorization algebras $\mathcal{F}_{(M_\beta)}$ and the same underlying element in $(\text{Covers}^n)_{k_1, \dots, k_n}$

$$(\mathfrak{V}(I_0^i \leq \dots \leq I_{k_i}^i)_{i=1}^n)^n.$$

Thus the collection of the images lies in the image of the inclusion $\text{Alg}_n[m] \subseteq (\text{Alg}[1])^m$. The map $\mathcal{FH}_n[m]$ is a map of n -fold Segal spaces by the same argument as for \mathcal{FH}_n .

To see that this assignment defines a natural transformation, let $f : [m] \rightarrow [k]$, and $1 \leq \alpha \leq k$. Let $\pi = \pi[1] \amalg \dots \amalg \pi[m] : M_1 \amalg \dots \amalg M_m \rightarrow (0, 1)^n$. By the following lemma we have

$$\pi_* \int_{\amalg_{\beta \in f^{-1}(\alpha)} M_\beta} A = \bigotimes_{\beta \in f^{-1}(\alpha)} \pi[\beta]_* \int_{M_\beta} A,$$

and thus the following diagram commutes.

$$\begin{array}{ccc} \text{PBord}_n^{fr}[m] & \xrightarrow{\mathcal{FH}_n[m]} & \text{Alg}_n[m] \\ \downarrow f & & \downarrow f \\ \text{PBord}_n^{fr}[k] & \xrightarrow{\mathcal{FH}_n[k]} & \text{Alg}_n[k] \end{array}$$

□

Lemma 4.6.2. *Let $f : X \rightarrow Z$, $g : Y \rightarrow Z$ be continuous maps of topological spaces and let \mathcal{F} be a factorization algebra on $X \amalg Y$. Then*

$$(f \amalg g)_* \mathcal{F} = f_* \mathcal{F}|_X \otimes g_* \mathcal{F}|_Y.$$

Proof. Let $U \subset Z$ be open. Then $(f \amalg g)^{-1}(U) = f^{-1}(U) \amalg g^{-1}(U)$ and by the gluing property of \mathcal{F} , we have

$$\mathcal{F}(f^{-1}(U) \amalg g^{-1}(U)) = \mathcal{F}(f^{-1}(U)) \otimes \mathcal{F}(g^{-1}(U)).$$

□

4.6.2 Symmetric monoidality via the tower

In this section we extend the map to the layers of the tower in a compatible way.

On the l th layer the extension $\mathcal{FH}_n^{(l)}$ is the composition of maps $\int_{(-)} A$ and $\mathfrak{V}^{(l)}$ analogous to those for $l = 0$. For simplicity, instead of defining the layers for the auxiliary spaces Fact_n and FAlg_n we define $\mathcal{FH}_n^{(l)}$ directly.

Proposition 4.6.3. *For every $l \geq 0$, the assignment*

$$\begin{aligned} \text{PBord}_n^{fr,l} &\longrightarrow \text{Alg}_n^{(l)}, \\ (\pi^{(l)} : M \rightarrow (0,1)^{n+l}, (I_0^i \leq \dots \leq I_{k_i}^i)_{i=1}^{n+l}) &\longmapsto (\mathcal{F}_{M^{(l)}}, \mathfrak{V}(I_0^i \leq \dots \leq I_{k_i}^i)_{i=1}^{n+l}), \end{aligned}$$

where $\mathcal{F}_{M^{(l)}} = (\bar{g}_{\bar{a}})_*(\pi^{(l)})_* \int_M A$, is a map of n -fold Segal spaces $\mathcal{FH}_n^{(l)} = \mathcal{FH}_n^{(l)}(A)$. It commutes with the looping and delooping maps u, ℓ from propositions 2.4.8 and 3.3.8.

Proof. We need to check:

1. $\mathcal{FH}_n^{(l)}$ is well-defined, i.e. its image indeed lies in $\text{Alg}_n^{(l)}$.

Similarly to propositions 4.4.10 and 4.5.1 one can show that $\mathcal{F}_{M^{(l)}}$ is locally constant with respect to the stratification associated to $\mathfrak{V}(I_0^i \leq \dots \leq I_{k_i}^i)_{i=1}^{n+l}$, and thus $\mathcal{FH}_n^{(l)}$ maps to Alg_{n+l} . Moreover, as noted in remark 2.4.6, $(\text{PBord}_n^{fr,l})_{1, \dots, 1, 0, \bullet, \dots, \bullet}$, with $(l-1)$ 1's, is the point viewed as a constant $(n-l)$ -fold Segal space. This implies that $\mathcal{FH}_n^{(l)}$ indeed maps to $\text{Alg}_n^{(l)} \subset \text{Alg}_{n+l}$.

2. $\mathcal{FH}_n^{(l)}$ commutes with the looping and delooping maps u, ℓ from propositions 2.4.8 and 3.3.8, i.e. the following diagram commutes:

$$\begin{array}{ccc} \text{PBord}_n^{fr,l} & \xrightarrow{\mathcal{FH}_n^{(l)}} & \text{Alg}_n^{(l)} \\ \begin{array}{c} \uparrow u \\ \downarrow \ell \end{array} & & \begin{array}{c} \uparrow u \\ \downarrow \ell \end{array} \\ L(\text{PBord}_n^{fr,l+1}) & \xrightarrow{\mathcal{FH}_n^{(l+1)}} & L(\text{Alg}_n^{(l+1)}) \end{array}$$

It is straightforward to see from the constructions of u that the diagram for u commutes. The commutativity for ℓ follows from the properties of the collapse-and-rescale maps.

□

By the universal property of the $(l$ -hybrid) completion, we obtain maps

$$\mathcal{FH}_n^{(l)} : \text{Bord}_n^{fr,(l)} \longrightarrow \text{Alg}_n^{(l)}.$$

which endow the functor $\mathcal{FH}_n(A) : \text{Bord}_n^{fr} \rightarrow \text{Alg}_n$ of (∞, n) -categories with a symmetric monoidal structure.

Corollary 4.6.4. *The maps $\mathcal{FH}_n^{(l)}$ make the functor*

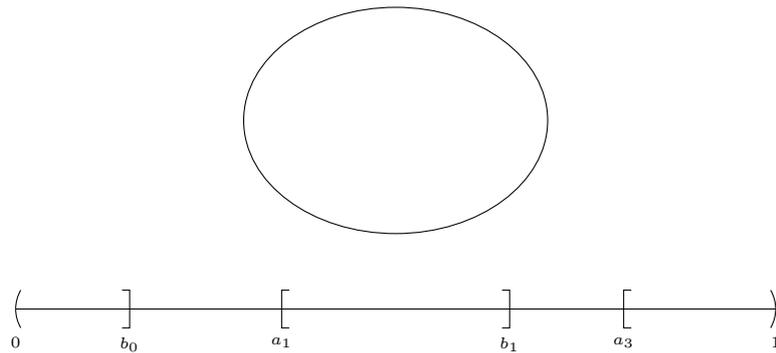
$$\mathcal{FH}_n(A) : \text{Bord}_n^{fr} \longrightarrow \text{Alg}_n$$

into a fully extended topological field theory.

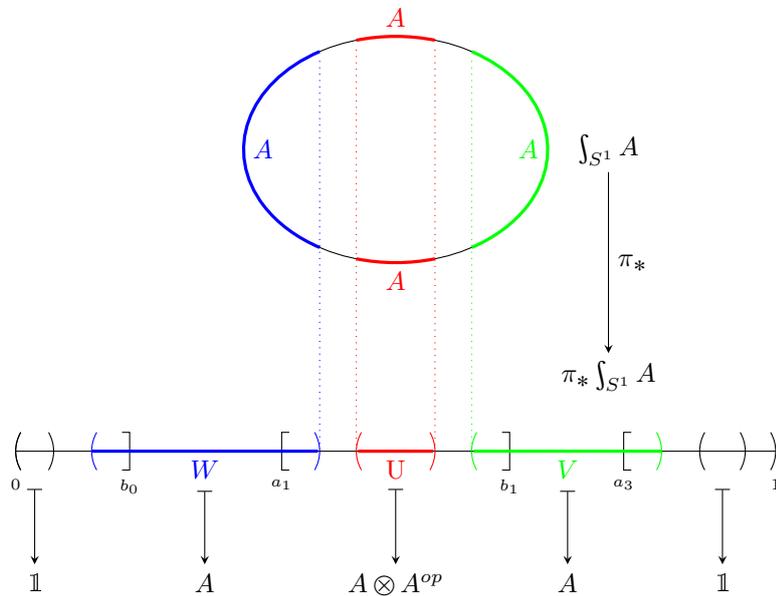
4.7 The simplest example: $n = 1$

Let A be an algebra. Then $\mathcal{FH}_1(A)$ is the field theory we expect: It sends S^1 considered as a 1-morphism from \emptyset to \emptyset to $\mathrm{HC}(A) = A \otimes_{A \otimes A^{op}} A$, the Hochschild chains of A . Note that we work in the derived setting, so tensor products are derived tensor products.

Consider the following element in $(\mathrm{Bord}_1)_2$, a pair of composable morphisms:

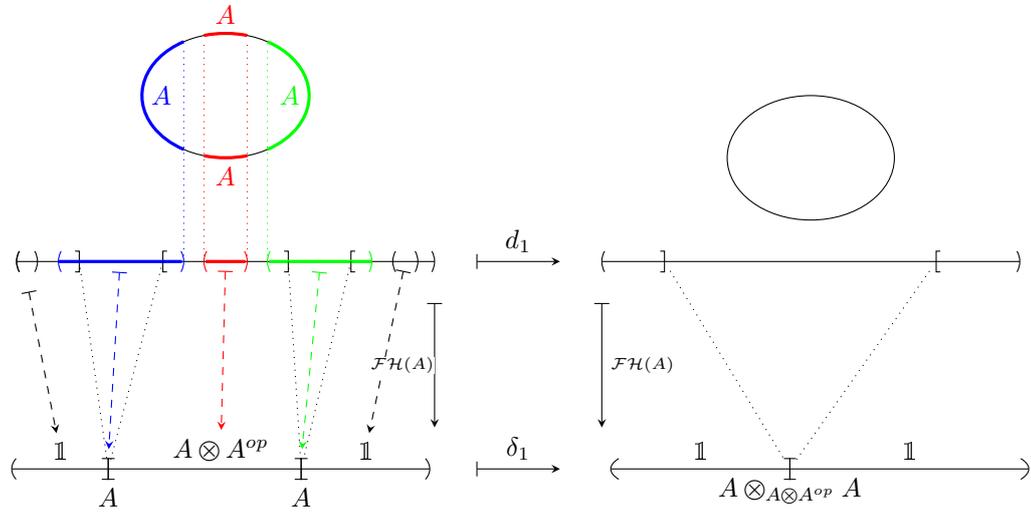


Taking factorization homology and pushing it to the base leads to the following factorization algebra on $(0, 1)$:



Thus pushing forward again along the collapse-and-rescale map $\rho_a^{\bar{b}}$ gives the following

diagram



Bibliography

- [Abr96] Lowell Abrams. Two-dimensional topological quantum field theories and Frobenius algebras. *J. Knot Theory Ramifications*, 5, 1996.
- [AFT12] David Ayala, John Francis, and Hiro L. Tanaka. Structured singular manifolds and factorization homology. *ArXiv e-prints*, June 2012.
- [AFT14] David Ayala, John Francis, and Hiro L. Tanaka. Factorization homology of stratified spaces. *ArXiv e-prints*, September 2014.
- [Ati88] Michael Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, (68):175–186 (1989), 1988.
- [BD95] John C. Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *J. Math. Phys.*, 36(11):6073–6105, 1995.
- [BD04] Alexander Beilinson and Vladimir Drinfeld. *Chiral algebras*, volume 51 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.
- [Ber10] Julia E. Bergner. A survey of $(\infty, 1)$ -categories. In *Towards higher categories*, volume 152 of *IMA Vol. Math. Appl.*, pages 69–83. Springer, New York, 2010.
- [BK11] C. Barwick and D. M. Kan. Partial model categories and their simplicial nerves. *ArXiv e-prints*, February 2011.
- [BR13] Julia E. Bergner and Charles Rezk. Comparison of models for (∞, n) -categories, I. *Geom. Topol.*, 17(4):2163–2202, 2013.
- [BS11] C. Barwick and C. Schommer-Pries. On the Unicity of the Homotopy Theory of Higher Categories. *ArXiv e-prints*, November 2011.
- [Cal] Damien Calaque. Around Hochschild (co-)homology. Habilitation Thesis.
- [Cam14] J. A. Campbell. Derived Koszul Duality and Topological Hochschild Homology. *ArXiv e-prints*, January 2014.
- [CG] Kevin Costello and Owen Gwilliam. Factorization algebras in perturbative quantum field theory.

- [Fra12] John Francis. Factorization homology of topological manifolds. *ArXiv e-prints*, June 2012.
- [Fra13] John Francis. The tangent complex and Hochschild cohomology of \mathcal{E}_n -rings. *Compos. Math.*, 149(3):430–480, 2013.
- [Gal11] Søren Galatius. Stable homology of automorphism groups of free groups. *Ann. of Math. (2)*, 173(2):705–768, 2011.
- [Gin] Grégory Ginot. Notes on factorization algebras, factorization homology and applications. In *Mathematical aspects of quantum field theories*. Springer. To appear.
- [GTZ10] Grégory Ginot, Thomas Tradler, and Mahmoud Zeinalian. Derived Higher Hochschild Homology, Topological Chiral Homology and Factorization algebras. *To appear in Communications in Mathematical Physics*, November 2010.
- [GTZ12] G. Ginot, T. Tradler, and M. Zeinalian. Higher Hochschild cohomology, Brane topology and centralizers of E_n -algebra maps. *ArXiv e-prints*, May 2012.
- [GWW] David Gay, Katrin Wehrheim, and Chris Woodward. Connected cerf theory. Available at <http://math.mit.edu/~katrin/papers/cerf.pdf>.
- [Hor14a] G. Horel. A model structure on internal categories in simplicial sets. *ArXiv e-prints*, March 2014.
- [Hor14b] G. Horel. Operads, modules and topological field theories. *ArXiv e-prints*, May 2014.
- [HS98] Andre Hirschowitz and Carlos Simpson. Descente pour les n-champs (Descent for n-stacks). *ArXiv Mathematics e-prints*, July 1998.
- [JFS] Theo Johnson-Freyd and Claudia Scheimbauer. The oplax square of an (∞, n) -category. In preparation.
- [Joy] A. Joyal. Notes on quasicategories. available from author.
- [Lau00] Gerd Laures. On cobordism of manifolds with corners. *Trans. Amer. Math. Soc.*, 352(12):5667–5688 (electronic), 2000.
- [Lur] Jacob Lurie. Higher algebra.
- [Lur09a] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.
- [Lur09b] Jacob Lurie. (Infinity,2)-Categories and the Goodwillie Calculus I. *ArXiv e-prints*, May 2009.
- [Lur09c] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.
- [Mil63] J. Milnor. *Morse theory*. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

- [Rez01] Charles Rezk. A model for the homotopy theory of homotopy theory. *Trans. Amer. Math. Soc.*, 353(3):973–1007 (electronic), 2001.
- [Rez10] Charles Rezk. A Cartesian presentation of weak n -categories. *Geom. Topol.*, 14(1):521–571, 2010.
- [Sal01] Paolo Salvatore. Configuration spaces with summable labels. In *Cohomological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*, pages 375–395. Birkhäuser, Basel, 2001.
- [Seg04] Graeme Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 421–577. Cambridge Univ. Press, Cambridge, 2004.
- [Sim98] C. Simpson. On the Breen-Baez-Dolan stabilization hypothesis for Tamsamani’s weak n -categories. *ArXiv Mathematics e-prints*, October 1998.
- [SP09] Christopher John Schommer-Pries. *The classification of two-dimensional extended topological field theories*. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)—University of California, Berkeley.
- [SW94a] N. Seiberg and E. Witten. Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory. *Nuclear Phys. B*, 426(1):19–52, 1994.
- [SW94b] N. Seiberg and E. Witten. Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD. *Nuclear Phys. B*, 431(3):484–550, 1994.
- [Thu74] William Thurston. Foliations and groups of diffeomorphisms. *Bulletin of the American Mathematical Society*, 80(2):304–307, 03 1974.
- [Toe] Bertrand Toen. Dualité de tannaka supérieure i: structures monoidales. preprint available at <http://www.math.univ-toulouse.fr/~toen/tan.pdf>.
- [Toë05] Bertrand Toën. Vers une axiomatisation de la théorie des catégories supérieures. *K-Theory*, 34(3):233–263, 2005.
- [TV05] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. I. Topos theory. *Adv. Math.*, 193(2):257–372, 2005.
- [TV09] Bertrand Toen and Gabriele Vezzosi. Caractères de Chern, traces équivariantes et géométrie algébrique dérivée. *ArXiv e-prints*, March 2009.
- [Wit82] Edward Witten. Supersymmetry and Morse theory. *J. Differential Geom.*, 17(4):661–692 (1983), 1982.
- [Wit88] Edward Witten. Topological quantum field theory. *Comm. Math. Phys.*, 117(3):353–386, 1988.
- [Wit89] Edward Witten. Quantum field theory and the Jones polynomial. In *Braid group, knot theory and statistical mechanics*, volume 9 of *Adv. Ser. Math. Phys.*, pages 239–329. World Sci. Publ., Teaneck, NJ, 1989.
- [Wit94] Edward Witten. Monopoles and four-manifolds. *Math. Res. Lett.*, 1(6):769–796, 1994.

- [Zha13] Yan Zhao. Extended topological field theories and the cobordism hypothesis. Master's thesis, Université Paris XI (Paris-Sud) and Università degli studi di Padova, 2013.

Curriculum Vitae

Claudia Isabella Scheimbauer

Born on June 15, 1986 in Vienna, Austria; citizen of Austria.

EDUCATION

exp. Aug. 26, 2014	Ph.D. in Mathematics, ETH Zurich
since July 2011	Advisor Damien Calaque, Coreferees Giovanni Felder and Bertrand Toën
Oct. 2009	DI (equivalent to M.Sc.) in Technical Mathematics, Vienna University of Technology
June 2004	Matura, Bundesrealgymnasium Krottenbachstrasse, Vienna

RESEARCH INTERESTS

My research area is *mathematical physics*: higher category theory, fully extended topological field theories, factorization algebras, derived algebraic geometry