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Pointless factorization algebras, pointless higher Morita categories and enriched skein categories

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Abstract

We define a variant of factorization algebras called *pointless* factorization algebras and prove that the tool-box of gluing techniques available for ordinary factorization algebras carries over to the pointless version. We then construct the pointless higher Morita categories $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$. Explicitly, we use Scheimbauer's construction of the pointed higher Morita categories $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ which uses constructible factorization algebras, and adapt it to the pointless setting. Then we translate the n-dualizability results of Gwilliam–Scheimbauer from $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ to $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$, exhibiting that the latter is also fully n-dualizable, and assuming a lifting lemma we prove a criterion for 3-dualizability in $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$.

In the last part of the thesis we prove that enriched skein categories compute factorization homology in $\mathcal{V}$-Cat.

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Contents

1. Introduction

1.1. General motivation. In a classical field theory on a spacetime manifold M one assigns to each open region $U \subset M$ a "space" of physical field configurations. Classical observables are then typically given by functions on this space of field configurations. When moving from a classical theory to a quantum field theory a crucial question is how one can capture the essence of what observables should be. One possible answer is the notion of *factorization algebras* in the sense of Costello–Gwilliam [**CG17, CG21**]. By design, the definition of a factorization algebra is meant to be a minimal axiom system for the observables of a quantum field theory.

Following Costello–Gwilliam, a factorization algebra $\mathcal{F}$ on a space X with values in Vect, the category of vector spaces, corresponds to giving a vector space $\mathcal{F}(U)$ for each open set $U \subseteq X$. Heuristically, this can be thought of as giving a space of measurements that can be performed at the spacetime U. For each inclusion $U_1 \uplus \ldots \uplus U_n \hookrightarrow V$, where $\uplus$ denotes the disjoint union, we assign a linear map

$$\mathcal{F}(U_1 \uplus \ldots \uplus U_n) \longrightarrow \mathcal{F}(V)$$

respecting composition. This can be seen as induced by performing measurements in disjoint regions of the spacetime. Moreover, these maps induce correlation functions ([**CG17**, §2, Definition 5.0.2]). We also require $\mathcal{F}$ to be *multiplicative* in the sense that we require the following maps

$$\mathbb{1} \xrightarrow{\simeq} \mathcal{F}(\emptyset) \qquad \text{and} \qquad \mathcal{F}(U) \otimes \mathcal{F}(V) \xrightarrow{\simeq} \mathcal{F}(U \uplus V)$$

to be isomorphisms. Lastly, $\mathcal{F}$ is required to satisfy descent for so-called *Weiss covers*.

The above is a very crude explanation. In general we want to replace Vect with a sufficiently nice symmetric monoidal ∞-category $\mathcal{C}$. Thus, we also need to take into account all of the corresponding coherences coming from being in a higher categorical setting. To capture this one instead assembles the above as a functor of ∞-operads. Explicitly, opens of X and inclusions thereof are assembled into an ∞-operad denoted $\text{open(X)}^{\otimes}$. A factorization algebra is then an algebra for this ∞-operad in $\mathcal{C}$ which satisfies multiplicativity and descent for Weiss covers.

A motivation for the above definition of a factorization algebra is the algebro-geometric version of Beilinson–Drinfeld [**BD04**], called *chiral algebras*. Their version is related to vertex operator algebras, which is an important notion in the setting of conformal (quantum) field theories. Moreover, topological versions of the Beilinson-Drinfeld chiral algebras and corresponding chiral homology were recognized by multiple people. This is developed by Lurie under the name of *topological chiral homology* ([**Lur17**]), by Morrison–Walker under the name of *blob homology* ([**MW12**]) or by Ayala–Francis under the name of *factorization homology* ([**AF15**]). We use the name factorization homology here. In short, this is a homology theory for topological manifolds which satisfy a version of the Eilenberg–Steenrod axioms for ordinary homology ([**AF15, AFT17a**]).

These notions are related to quantum field theories where the correlation functions do not depend on the metric of the spacetime, i.e. they are topologically invariant. When this is the case one talks about having a *topological (quantum) field theory*, or a TFT for short. At the level of factorization algebras one can capture topological invariance by imposing an additional condition called *local constancy*. That is, given an inclusion $U \hookrightarrow V$ we impose that the map $\mathcal{F}(U) \to \mathcal{F}(V)$ is an equivalence whenever U and V are isotopy equivalent.

The data of a topological (quantum) field theory can be packaged in a functorial way. The original definition by Atiyah [**Ati88**] and Segal [**Seg88**] says that a TFT is a symmetric monoidal functor (of ordinary categories)

$$\mathcal{Z}\colon \textsc{Cob}_d \longrightarrow \textsc{Vect}\ , \tag{1.1}$$

where, informally, $\textsc{Cob}_d$ is the category whose objects are closed $(d-1)$-dimensional manifolds and morphisms are (diffeomorphism classes) of d-dimensional cobordisms between these.[1] The symmetric monoidal structure on $\textsc{Cob}_d$ is given by disjoint union. One can additionally decorate the manifolds with tangential structures like orientations or framings in a compatible way. In that case we get variants of topological field theories like *oriented* or *framed* TFTs.

On one side of (1.1) we have the category of bordisms which can be thought of as topological input. Similarly, we can think of the target category as encoding algebraic data. From this point of view it is not all that surprising that topological field theories exhibit an interesting relationship between algebra and topology. For example, setting $n = 2$ and working with the oriented version of $\textsc{Cob}_d$ in (1.1) it is well-known that the category of oriented 2-dimensional TFTs is equivalent to the category of commutative Frobenius algebras.

A higher categorical version of a TFT is a *fully extended* topological field theory [**BD95, Lur09b**]. For this one replaces the ordinary category of cobordisms $\textsc{Cob}_d$ in (1.1) by a symmetric monoidal (∞, d)-category of bordisms denoted $\textsc{Bord}_d$. Informally, objects are (compact) 0-manifolds, 1-morphisms are "bordisms between objects", 2-morphisms are "bordisms between bordisms", and so on until one reaches d-morphisms which are d-dimensional bordisms. For the higher morphisms the flavour changes and we instead have that $(d+1)$-morphisms are diffeomorphisms of d-dimensional bordisms, and so on. In particular, this means that for $k > d$, all of the k-morphisms are invertible. The symmetric monoidal structure is again given by disjoint union. With this, a fully extended d-dimensional topological field theory is a symmetric monoidal functor

$$\mathcal{Z}\colon \textsc{Bord}_d \longrightarrow \mathcal{C}\ ,$$

where the target category is also upgraded to be a symmetric monoidal (∞, d)-category. As before, one can coherently equip the bordisms with tangential structures like orientations and framings to obtain *oriented* or *framed* fully extended TFTs.

The name "fully extended" comes from the fact that this (fully) extends the data of the bordisms all the way down to 0-dimensional manifolds, i.e. points. This provides tremendous computational power. In the setting of an ordinary d-dimensional TFT one can perform computations by decomposing a d-dimensional cobordism into easier d-dimensional cobordisms glued along $(d-1)$-dimensional cobordisms. Meanwhile, in the fully extended setting one can decompose a d-dimensional bordism into d-dimensional bordisms which glue along $(d-1)$-dimensional bordisms, which themselves are allowed to glue along $(d-2)$-dimensional bordisms and so on all the way down to points. The above decomposition-procedure is also the heuristics behind the "Cobordism hypothesis", conjectured by Baez and Dolan in [**BD95**]. Very informally, it states that a d-dimensional fully extended TFT is fully determined by its value at a point. More precisely, evaluation at a point induces an equivalence of ∞-groupoids between isomorphism classes of d-dimensional fully extended TFTs valued in $\mathcal{C}$ and isomorphism classes of so-called *d-dualizable* objects of $\mathcal{C}$.

[1] Explicitly, a cobordism from M to N is a d-dimensional manifold Σ with an identification of the boundary $\partial\Sigma$ with $M \coprod N$.

We again have an interplay between topological and algebraic data: while constructing a fully extended topological field theory requires exceptionally good understanding of the (∞, d)-category $\mathrm{B{\scriptsize ORD}}_d$, the condition of an object being d-dualizable is phrased entirely within the target category $\mathcal{C}$. Hence, the latter is a purely algebraic (or category theoretic) condition. For this reason it is in many cases considerably easier to investigate dualizability in the target category $\mathcal{C}$ than it is to construct a fully extended topological field theory.

One target category which has been widely studied in the literature is the *higher Morita category*. Classically, the Morita 2-category is the 2-category whose objects are associative unital algebras, 1-morphisms are bimodules and 2-morphisms are bimodule homomorphisms. When generalizing this to get a higher categorical version there are, at the level of objects, two mathematical tools available in the literature. That is, either E_n-algebras, i.e. algebras over the little disk operad, or locally constant factorization algebras. The former is used in Haugseng's version of the higher Morita category $\mathrm{A{\scriptsize LG}}_n^{\mathrm{H}}(\mathcal{C})$ from [**Hau17**], while the latter is used in Scheimbauer's version $\mathrm{A{\scriptsize LG}}_n^{\mathrm{ptd}}(\mathcal{C})$ from [**Sch14, GS18**]. In this thesis we construct a third version $\mathrm{A{\scriptsize LG}}_n^{\mathrm{pl}}(\mathcal{C})$ by using a new variant of factorization algebras. [2]

In many cases dualizability can be thought of as some kind of finiteness-condition. For example, a 1-dualizable object in the category $\mathrm{V{\scriptsize ECT}}$ is a finite-dimensional vector space. The Morita 2-category introduced above corresponds to both $\mathrm{A{\scriptsize LG}}_1^{\mathrm{H}}(\mathrm{V{\scriptsize ECT}})$ and $\mathrm{A{\scriptsize LG}}_1^{\mathrm{pl}}(\mathrm{V{\scriptsize ECT}})$. Every object is dualizable with the opposite algebra providing the dual, while the 2-dualizable objects are the separable, finitely-generated algebras, c.f. [**DP80**]. This is in stark contrast to if we would have chosen to work with $\mathrm{A{\scriptsize LG}}_1^{\mathrm{ptd}}(\mathrm{V{\scriptsize ECT}})$, where only the trivial object is 2-dualizable. We explain the reason for this, and the motivation for constructing a third version of the higher Morita categories in detail in Section 1.2.

When working with fully extended framed TFTs with values in the pointed higher Morita category $\mathrm{A{\scriptsize LG}}_n^{\mathrm{ptd}}(\mathcal{C})$ the cobordism hypothesis is a very precise statement. Lurie outlined in [**Lur09b**] that given an E_n-algebra $\mathcal{A}$, factorization homology computes a fully extended n-dimensional TFT. This was explicitly constructed in [**Sch14**] and exactly corresponds to a fully extended n-dimensional TFT with values in $\mathrm{A{\scriptsize LG}}_n^{\mathrm{ptd}}(\mathcal{C})$. In some more detail, let $\mathcal{M}\mathrm{fld}_d^{\otimes}$ denote the symmetric monoidal ∞-category of d-dimensional manifolds and embeddings, and let $\mathcal{D}\mathrm{isks}_d^{\otimes}$ be the subcategory whose objects are d-dimensional disks. An E_n-algebra $\mathcal{A}$ is a symmetric monoidal functor as indicated below

$$\begin{tikzcd} \mathcal{D}\mathrm{isks}_d^{\otimes} \arrow[r, "\mathcal{A}"] \arrow[d, hook, "\iota"] & \mathcal{C}^{\otimes} \\ \mathcal{M}\mathrm{fld}_d^{\otimes} \arrow[ur, dashed, "\int_{\_} \mathcal{A}"'] & \end{tikzcd} . \tag{1.2}$$

Factorization homology (with coefficients in $\mathcal{A}$), denoted $\int_{\_} \mathcal{A}$ is defined as the left Kan extension of $\mathcal{A}$ along ι the inclusion of disks into all manifolds [**AF15**]. One can think of the assignment $\int_M \mathcal{A}$ as "integrating" over the value of all disks included into the manifold M.

Using factorization homology to compute topological field theories has also been exploited in more concrete examples. For example, in [**BZBJ18a, BZBJ18b**] Ben-Zvi, Brochier and

[2] We write "ptd" as an abbreviation for "pointed" to denote Scheimbauer's higher Morita categories. Meanwhile the third version is decorated with "pl" which is an abbreviation for "pointless". See Section 1.2 for more details.

Jordan compute factorization homology on surfaces (i.e. $d = 2$) with values in $\textsc{Cat}$, interpreted as some suitable (2,1)-category of categories. For their computations they use certain categories of representations of quantum groups and their results provide a concrete example of the general idea that there is an interesting interplay between quantization and factorization homology. Another concrete example is provided by *skein categories*, which Cooke shows to compute factorization homology on oriented surfaces (with values in the category of $\Bbbk$-linear categories) in [**Coo23**]. A skein category is a categorical analogue of a skein algebra first defined by Walker and Johnson-Freyd in [**Wal**] and [**JF19**], respectively.

1.2. The point of pointless factorization algebras. We start by giving a brief explanation of the two constructions of the higher Morita categories present in the literature, as well as explain their advantages and drawbacks as it relates to dualizability. Along the way we will see why there is need for a third version, and we build a list of requirements for such a third version. Then we present the solution developed in this thesis; namely the pointless higher Morita categories built from constructible pointless factorization algebras.

In [**Lur09b**, Definition 4.1.11] Lurie sketched an inductive definition of the higher Morita categories using E_n-algebras. Scheimbauer made a variant of this sketch precise in [**Sch14, GS18**] by constructing an $(\infty, n+1)$-category denoted $\textsc{Alg}_n^{\mathrm{ptd}}(\mathcal{C})$ using *constructible* factorization algebras. That is, factorization algebras $\mathcal{F}$ defined on a stratified space such that $\mathcal{F}$ is "locally constant with respect to the stratification". Explicitly, this means that $\mathcal{F}$ sends any inclusion $U \hookrightarrow V$ of stratified isotopy equivalent disks to an equivalence. We give an informal dictionary between the precise construction of Scheimbauer and what it morally speaking corresponds to for E_n-algebras.

n-category	E_n**-algebra version**	**factorization version**
0-morphism	E_n-algebra	locally constant on $(0,1)^n$
1-morphism	bimodule of E_n-algebras	constructible for $\{a^1\} \times (0,1)^{n-1}$
2-morphism	bimodule of bimodules of E_{n-1}-algebras	constructible for $\{a^1, a^2\} \times (0,1)^{n-2} \subset \{a^1\} \times (0,1)^{n-1}$
$\vdots$	$\vdots$	$\vdots$
n-morphism	**pointed** bimodule of ... bimodules	constructible for full flags
$(n+1)$-morphism	**point-preserving** bimodule map	map of factorization algebras

At the level of objects the dictionary can be made precise, i.e. there is a known equivalence between E_n-algebras and locally constant factorization algebras on $(0,1)^n$.[3] We highlight that here the n-morphisms are *pointed* bimodules (of ... bimodules). At the level of constructible factorization algebras this comes from the fact that for every open $U \subseteq X$ there is an inclusion $\emptyset \hookrightarrow U$ which in turn induces a structure map

$$\mathcal{F}(\emptyset) \simeq \mathbb{1}_{\mathcal{C}} \longrightarrow \mathcal{F}(U) \tag{1.3}$$

[3] There is an equivalence of E_n-algebras and locally constant multiplicative prefactorization algebras on $\mathrm{disk}(\mathbb{R}^n)$ by [**Lur17**, Theorem 5.4.5.9] and [**Lur17**, Example 5.4.5.3]. By [**KSW24**, Corollary 5.24] the latter is equivalent to locally constant factorization algebras on $\mathbb{R}^n \cong (0,1)^n$.

for any constructible factorization algebra $\mathcal{F}$. Similarly, the $(n+1)$-morphisms are maps that preserve the structure of the n-morphisms, i.e. *point-preserving* bimodule maps.

As already mentioned, higher Morita categories are tractable targets for framed fully extended topological field theories, and in this context it is interesting to understand dualizability in the higher Morita categories. Gwilliam and Scheimbauer proves that every object of $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ is fully n-dualizable in [**GS18**, Theorem 4.1]. Their constructions of duals and adjoints leverages the geometric nature of constructible factorization algebras. However, they also prove that only the trivial object of $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ is $(n+1)$-dualizable ([**GS18**, Theorem 5.1]). The obstacle for interesting $(n+1)$-dualizability in $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ is exactly the pointed nature of the n- and $(n+1)$-morphisms.

The next natural question is to compare this to the version of the higher Morita categories $\mathrm{ALG}_n^{\mathrm{H}}(\mathcal{C})$ constructed by Haugseng in [**Hau17**]. Briefly, Haugseng uses generalized non-symmetric ∞-operads to model E_n-algebras, and this version does not have the pointed feature of $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$. That is, n-morphisms of $\mathrm{ALG}_n^{\mathrm{H}}(\mathcal{C})$ are simply bimodules (of .. bimodules) and the $(n+1)$-morphisms are bimodule maps. In light of pointings being the obstruction for non-trivial $(n+1)$-dualizable objects in $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ it is natural to expect interesting $(n+1)$-dualizability in $\mathrm{ALG}_n^{\mathrm{H}}(\mathcal{C})$. However, this would assume that we know how to prove n-dualizability in $\mathrm{ALG}_n^{\mathrm{H}}(\mathcal{C})$, which we do *not*. The reason for this is that the generalized non-symmetric ∞-operads used to construct $\mathrm{ALG}_n^{\mathrm{H}}(\mathcal{C})$ are very rigid, and there is at the moment no good way of translating the geometric constructions of [**GS18**] to this setting.

In summary, there is need for a third version of the higher Morita categories to be able to prove interesting $(n+1)$-dualizability results. We approach this by looking for a variant of constructible factorization algebras satisfying the following list of requirements.

The list of requirements 1.1.

(1) On $(0,1)^n$ with strata of dimension ≥ 1 we get back the ordinary notion of constructible factorization algebras.
(2) On $(0,1)^n$ stratified by a full flag (i.e. containing 0-dimensional strata) we get bimodule structures (instead of *pointed* bimodule structures).
(3) They must satisfy the same key properties as constructible factorization algebras used to construct the factorization higher Morita categories. Explicitly,
    (a) they need to glue,
    (b) and push forward along certain nice maps.
(4) The n-dualizability results of [**GS18**] must transfer to the new version.

In this thesis we define and develop the theory of constructible *pointless* factorization algebras as a solution to the above problem. We highlight that the the list of requirements 1.1 is developed exactly to get the following new informal dictionary.

n-category	Lurie's version	pointless factorization version
0-morphism	E_n-algebra	locally constant on $(0,1)^n$
1-morphism	bimodule of E_n-algebras	constructible for $\{a^1\} \times (0,1)^{n-1}$
2-morphism	bimodule of bimodules of E_{n-1}-algebras	constructible for $\{a^1, a^2\} \times (0,1)^{n-2} \subset \{a^1\} \times (0,1)^{n-1}$
$\vdots$	$\vdots$	$\vdots$
n-morphism	bimodule of ... bimodules	constructible and **pointless** for full flags
$(n+1)$-morphism	bimodule map	map of **pointless** factorization algebras

We now give a brief explanation of the definition of pointless factorization algebras, where pointless refers to the fact that they do *not* have the same pointed nature as ordinary (constructible) factorization algebras. Recall that ordinary factorization algebras are defined as algebras for the ∞-operad $\mathrm{open(X)}^{\otimes}$, which satisfy multiplicativity and the Weiss condition. Here, $\mathrm{open(X)}^{\otimes}$ is the ∞-operad produced from the ordinary poset $\mathrm{open}(X)$ of opens of X and inclusions thereof. The pointed structure of ordinary factorization algebras arises from the fact that $\emptyset$ is an initial object of $\mathrm{open}(X)$, and hence also of $\mathrm{open(X)}^{\otimes}$.

For the pointless version of constructible factorization algebras we work with a wide subcategory $\mathrm{open}(X)_{\mathrm{mrk}} \subseteq \mathrm{open}(X)$ of *marked* opens and *marked* inclusions. Explicitly, we declare the 0-dimensional strata of X to be marked points, and define marked inclusions to be inclusions which are *bijective* on marked points. For example, let $X = (0,1)$ with a 0-dimensional strata (and hence also marked point) at $\frac{1}{2}$. If $\frac{1}{2} \in U$, for some open $U \subseteq X$, the ordinary inclusion of the emptyset into U is *not* a marked inclusion because $\emptyset$ contains no marked points while U contains one. In other words, in this situation we do not get the structure maps from (1.3) which is what induces the unwanted pointed structure. This is exactly tailored to satisfy 1 and 2 of the list of requirements 1.1. The multiplicativity-condition stays the same in this setting, while the Weiss condition needs to be made compatible with the marked inclusions in a natural way.

Let us elaborate a bit more on the list of requirements 1.1. As explained in 3 gluing and pushing forward is necessary to be able to use constructible pointless factorization algebras to construct the pointless higher Morita categories $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$. In particular, both properties are crucial to define composition as well as to construct the symmetric monoidal structure on $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$. Lastly, 4 follows from proving that constructible ordinary factorization algebras restrict to constructible pointless factorization algebras. This in turn induces a symmetric monoidal functor from $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ to $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ along which we can transfer the dualizability-results of [**GS18**].

1.3. Part 1: Pointless factorization algebras. In this part we develop the theory of pointless factorization algebras. In particular, we generalize the toolbox of techniques for ordinary factorization algebras developed in [**KSW24**] to the pointless setting. A main motivation for writing the article [**KSW24**] joint with Claudia Scheimbauer and Tashi Walde was to have a proof that constructible factorization algebras glue along an open cover which would be generalizable to the pointless setting. This is for example used to equip the ∞-category of constructible factorization algebras with a symmetric monoidal structure, and is also imperative for the construction of the pointless higher Morita categories in Part 2.

We stress that none of the results in this part are a formal consequence of those in [**KSW24**] for ordinary factorization algebras. This is because [**KSW24**] proves everything for open(X) and full subposets thereof, which open$(X)_{\mathrm{mrk}}$ by design is *not.* Regardless, many of the proofs directly translate, and for this we introduce the following notation.

Notation 1.2. To highlight which results are straightforward adaptations of results from [**KSW24**] we will add an asterisk to the reference. For example, 7.7* indicates that the proposition is a straightforward adaptation, while no asterisk indicates that the result relies on non-trivial modifications.

In the literature on factorization algebras it has long been claimed that they satisfy gluing along open covers. We explain what we mean by this in a simple example in the setting of factorization algebras. Fix a space X and let U and V be an open cover of X. Let $\mathcal{F}_U$ and $\mathcal{F}_V$ be two factorization algebras defined on U and V, respectively, such that they agree on the intersection $U \cap V$. That is, such that $(\mathcal{F}_U)_{|U\cap V} \simeq (\mathcal{F}_V)_{|U\cap V}$. By saying that factorization algebras "glue" we mean that, from the above data, one can construct a factorization algebra $\mathcal{F}$ on X which extends $\mathcal{F}_U$ and $\mathcal{F}_V$. For locally constant factorization algebras this was proven in [**Mat17**], and for the constructible factorization algebras this is [**KSW24**, Theorem 6.1]. In this thesis we prove that this result also modifies to give a proof that constructible *pointless* factorization algebras glue. Explicitly, we prove:

Theorem A (Theorem 5.3). Let $\mathcal{U}$ be an open cover of a smooth conical maximally marked manifold X. Then the map induced by restricting constructible pointless factorization algebras

$$\mathrm{Fact}^{\mathrm{pl,cstr}}_X(\mathcal{C}) \longrightarrow \lim_{\{U_1,\dots,U_n\}\subseteq\mathcal{U}} \mathrm{Fact}^{\mathrm{pl,cstr}}_{U_1\cap\cdots\cap U_n}(\mathcal{C})$$

is an equivalence of ∞-categories. In other words, $\mathrm{Fact}^{\mathrm{pl,cstr}}_{(-)}(\mathcal{C})$ is a sheaf of ∞-categories.

We outline how this is proven and highlight the parts which required non-trivial modifications compared to the results of [**KSW24**] in Section 1.3.1 below. However, let us first explain some consequences of the above result. Just like for constructible factorization algebras we can use Theorem A to equip the ∞-category of constructible pointless factorization algebras $\mathrm{Fact}^{\mathrm{pl,cstr}}_X$ with a symmetric monoidal structure.

Theorem B (Proposition 5.11). For every smooth conical maximally marked manifold X with enough good marked disks, the functor

$$\mathrm{Fact}^{\mathrm{pl,cstr},\otimes}_X \colon \mathrm{Fin}_* \longrightarrow \mathrm{Cat}_\infty, \qquad I_+ \mapsto \mathrm{Fact}^{\mathrm{pl,cstr}}_{X^{\amalg I}} \tag{1.4}$$

exhibits a monoid-object in $(\mathrm{Cat}_\infty, \times)$. That is, a symmetric monoidal structure on the ∞-category $\mathrm{Fact}^{\mathrm{pl,cstr}}_X$ of constructible pointless factorization algebras.

We also prove that constructible factorization algebras restrict to constructible pointless factorization algebras. At the level of ∞-operads we have an inclusion

$$\iota \colon \mathrm{open}(X)^{\otimes}_{\mathrm{mrk}} \longrightarrow \mathrm{open(X)}^{\otimes} \tag{1.5}$$

which induces a restriction functor ι^* between the corresponding ∞-categories of algebras. This restricts to the corresponding ∞-subcategories of factorization algebras, respectively pointless factorization algebras, and simply corresponds to forgetting some of the structure maps. If we further restrict to the ∞-subcategories of constructible (pointless) factorization algebras, the restriction functor ι^* respects the symmetric monoidal structure on $\mathrm{F}\text{ACT}_X^{\mathrm{cstr}}$ from [**KSW24**, Corollary 6.8], respectively that on $\mathrm{F}\text{ACT}_X^{\mathrm{pl,cstr}}$ from Theorem B.

Theorem C (Proposition 5.12). Let $\mathcal{C}$ be a symmetric monoidal $\otimes$-presentable ∞-category. Assume that X is a smooth conical manifold which is maximally marked, and let ι be the map (1.5). Restriction along ι induces a symmetric monoidal functor

$$\iota^* \colon \mathrm{F}\text{ACT}_X^{\mathrm{cstr}}(\mathcal{C}) \longrightarrow \mathrm{F}\text{ACT}_X^{\mathrm{pl,cstr}}(\mathcal{C}) ,$$

where on the left hand side we forget about the marking of X and only remember the underlying smooth conical manifold.

Moreover, we establish a pointless analogue of [**KSW24**, Theorem 6.12] which was prompted by a question of David Ayala. Roughly, the question translates to whether one can assemble the data of a constructible pointless factorization algebra on a marked cone $\mathbf{C}Z$ from a constructible factorization algebra $\mathcal{A}$ on $Z \times (0,\infty)$ (thought of as an associative algebra $\int_Z \mathcal{A}$ in $\mathrm{F}\text{ACT}_{(0,\infty)}^{\mathrm{pl,cstr}}$) and a constructible pointless factorization algebra on $[0,\infty)$ (thought of as a module for the associative algebra $\int_Z \mathcal{A}$).[4] See Section 5.3 for more details.

Theorem D (Theorem 5.16). Let $\mathcal{C}^{\otimes}$ be a $\otimes$-presentable symmetric monoidal ∞-category. Let Z be a compact smooth conical manifold with enough good disks, and let $\mathbf{C}Z$ be the corresponding cone marked at the cone point. We have a pullback square of ∞-categories

$$\begin{tikzcd} \mathrm{F}\text{ACT}_{\mathbf{C}Z}^{\mathrm{pl,cstr}} \arrow[r] \arrow[d, "p_\star"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{F}\text{ACT}_{Z\times(0,\infty)}^{\mathrm{pl,cstr}} \arrow[d, "p_\star"] & \mathcal{A} \arrow[d, mapsto] \\ \mathrm{F}\text{ACT}_{[0,\infty)}^{\mathrm{pl,cstr}} \arrow[r] & \mathrm{F}\text{ACT}_{(0,\infty)}^{\mathrm{pl,cstr}} & \int_Z \mathcal{A} \end{tikzcd} . \tag{1.6}$$

Here, the vertical maps are given by pushforward along the quotient map $p \colon \mathbf{C}Z \to [0,\infty)$, while the horizontal functors are given by restriction to open subspaces.

As explained in [**KSW24**, Construction 6.16, Corollary 6.17 & Remark 6.18] one can globalize the above result to inductively assemble a constructible pointless factorization algebra on a smooth conical maximally marked from (morally speaking) algebras and modules in locally constant factorization algebras on the individual strata.

Lastly, we highlight the constructibility-tools which required substantial effort to translate to the setting of constructible pointless factorization algebras. First of all, we have the pointless analogue of [**KSW24**, Corollary 5.22] which states that checking the marked Weiss cosheaf condition is equivalent to checking if a constructible copresheaf is a left Kan extension:

Theorem E (Corollary 7.34). Let X be a smooth conical maximally marked manifold, $\mathfrak{B} \subseteq \mathrm{disk}(X)_{\mathrm{mrk}}$ a decomposable multiplicative disk-basis of X and $\mathfrak{U} \supseteq \mathfrak{B}$ a presieve. Then for every constructible copresheaf on $\mathfrak{B}$ the left Kan extension to $\mathfrak{U}$ is a marked Weiss cosheaf.

The key ingredient of the proof of Theorem E is a localization-result at the level of opens and marked inclusions. In the setting of ordinary opens and inclusions it is shown in [**KSW24**] that localizing a full subposet $\mathfrak{B}$ of $\mathrm{open}(X)$ at all stratified isotopy equivalences $\mathcal{J}_{\mathfrak{B}}$ is equivalent to the slice ∞-category $\mathcal{D}\mathrm{isks}_{/X}$, where $\mathcal{D}\mathrm{isks} \subseteq \mathcal{M}\mathrm{fld}$ is the full ∞-subcategory spanned

[4] In the ordinary setting one instead gets a *pointed* module here, which is why the pointless version is in fact the more natural version to consider.

by those smooth conical manifolds which are disjoint unions of conical disks. Explicitly, [**KSW24**, Theorem 5.10] says that for $\mathfrak{B} \subset \mathrm{disk}(X)$ a decomposable multiplicative disk-basis of X the induced map

$$\mathfrak{B}[\mathcal{J}_{\mathfrak{B}}^{-1}] \xrightarrow{\simeq} \mathcal{D}\mathrm{isks}_{/X}$$

is an equivalence. This is a generalization of [**AFT17a**, Proposition 2.22] which only considers $\mathfrak{B} = \mathrm{disk}(X)$ and unstratified spaces, and also contains a gap. Moreover, at the same time as [**KSW24**] was written Arakawa provided a full proof for $\mathfrak{B} = \mathrm{disk}(X)$ in the unstratified setting in [**Ara24**, Theorem 2.24].

For the pointless version of factorization algebras we work with marked opens and inclusions which are *bijective* on marked points. In this setting we first need to define marked variants of the ∞-categories of manifolds and disks, i.e. we define $\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}} \subseteq \mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}$ in Section 6. We also establish modifications of some key results from [**AFT17a**] which are used in proving [**KSW24**, Theorem 5.10]. With this we prove the following result:

Theorem F (Theorem 7.25). *Let X be a smooth conical maximally marked manifold and let $\mathfrak{B} \subseteq \mathrm{disk}(X)_{\mathrm{mrk}}$ be a decomposable multiplicative disk-basis of X. Let $V \in \mathrm{open}\,(X)_{\mathrm{mrk}}$ be any object. Then the induced map*

$$\left(\mathfrak{B} \cap V_{\downarrow_{\mathrm{m}}}\right)[(\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}})^{-1}] \xrightarrow{\simeq} \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/V} \tag{1.7}$$

is an equivalence of ∞-categories.

We explain the insight into why this version is sufficient to prove Theorem E and in turn Theorem A in Remark 7.26. In Remark 7.32 we spell out which results of [**KSW24**] we do *not* (yet) have for constructible pointless factorization algebras as a consequence of Theorem F being a different statement than that of [**KSW24**, Theorem 5.10].

1.3.1. Outline of the proof of Theorem A. Let $\mathcal{U}$ be an open cover of a conically smooth maximally marked manifold X, which moreover has enough good disks. Let β be an ordinal and $V_\bullet \colon \beta \twoheadrightarrow \mathcal{U}$ be a surjection, so we can write $\mathcal{U} = \{U_i\}_{i<\beta}$. For any finite subset $I \subset \beta$ we set $U_I := \bigcap_{i\in I} U_i$. Let $\mathfrak{B} := \mathrm{open}\,(\mathcal{U})_{\mathrm{mrk}}$ be as in Definition 5.2. Explicitly, this is the full subposet of $\mathrm{open}(X)_{\mathrm{mrk}}$ whose objects are all marked opens $U \in \mathrm{open}\,(U_i)_{\mathrm{mrk}}$ for some $U_i \in \mathcal{U}$. Additionally, let $\mathfrak{B}' := \mathrm{open}_{\neq\emptyset}(\mathcal{U})_{\mathrm{mrk}}$, where $\mathrm{open}_{\neq\emptyset}(\mathcal{U})_{\mathrm{mrk}} = \mathrm{open}\,(\mathcal{U})_{\mathrm{mrk}} \setminus \{\emptyset\}$ as seen from Definition 7.1. In the special case of a single marked open U_I we have that $\mathrm{open}_{\neq\emptyset}\,(U_I)_{\mathrm{mrk}}$ is the full subposet of the non-empty marked opens of $\mathrm{open}\,(U_I)_{\mathrm{mrk}}$ (Definition 5.20).

Consider the commuting diagram given in Equation (1.8). The three different columns correspond to the three main steps in the proof of Theorem A, and we now explain each in some more detail.

Leftmost column: The leftmost column corresponds to results from Section 7.1.1 where most notably it is proven that the ∞-operad $\mathfrak{B}' = \mathrm{open}_{\neq\emptyset}(\mathcal{U})^{\otimes}_{\mathrm{mrk}}$ is the colimit of the ∞-operads $\mathrm{open}_{\neq\emptyset}(U_I)^{\otimes}_{\mathrm{mrk}}$. This gives rise to an equivalence of the corresponding ∞-categories of algebras (Corollary 7.5*) which moreover restricts to the full subcategories of multiplicative algebras (see (7.3)* and the surrounding text). Combining this result with Proposition 7.13 one also gets an equivalence between the corresponding ∞-categories of multiplicative pointless prefactorization algebras (see (7.4)* and the surrounding text). Finally, this equivalence restricts to an equivalence at the level of pointless factorization algebras and their constructible counterpart (Proposition 7.7* and Proposition 7.7*).

Rightmost column: The rightmost column concerns extending pointless factorization algebras from a factorizing basis as developed in Section 7.1.2. Since $\mathfrak{B}_{\cup}$ is a factorizing basis of X, we have an equivalence at the level of Weiss algebras (Proposition 7.10*) which restricts to an equivalence between the corresponding categories of pointless factorization algebras (Proposition 7.8*). Furthermore, this equivalence also restricts to constructible pointless factorization algebras (Proposition 5.4*) by using that constructibility is a local condition (Theorem 7.20*).

Middle column: The middle column concerns going from pointless prefactorization algebras defined on $\mathfrak{B}$ to its disjoint union completion $\mathfrak{B}_{\sqcup}$, as recorded in Section 7.1.3. The main insight is that there is an equivalence at the level of multiplicative algebras (Proposition 7.13*). However, just like in the ordinary setting of [**KSW24**] it is not clear if this equivalence restricts to an equivalence on pointless factorization algebras. This is indicated by a dashed arrow in the above diagram, and we refer the reader to [**KSW24**, Question 4.1 & 4.25 and Remark 4.26] for some more details on why this is far from straightforward to prove.

As a solution we further restrict to the corresponding categories of constructible pointless factorization algebras, in which case we get an equivalence (Proposition 5.5). This proof relies heavily on results from Section 7.2 and Section 7.3, which are highlighted as Theorem F and Theorem E here in the introduction. The upshot is that the marked Weiss condition is automatic in this specific setting by leveraging constructibility.

$$
\begin{tikzcd}[column sep=small]
 & \lim\limits_{\mathfrak{B}\supset I\supsetneq\emptyset} \mathrm{Fact}^{\mathrm{pl,cstr}}_{U_I} \arrow[d, hook] & & \mathrm{Fact}^{\mathrm{pl,cstr}}_{\mathfrak{B}} \arrow[ll, "\simeq", "\text{Prop. 7.7}^*"'] \arrow[d, hook] & & & \mathrm{Fact}^{\mathrm{pl,cstr}}_{\mathfrak{B}_\cup} \arrow[lll, "\simeq", "\text{Prop. 5.5}"'] \arrow[d, hook] & & & \mathrm{Fact}^{\mathrm{cstr}}_{X} \arrow[lll, "\simeq", "\text{Prop. 5.4}^*"'] \arrow[d, hook] & \\
 & \lim\limits_{\mathfrak{B}\supset I\supsetneq\emptyset} \mathrm{Fact}^{\mathrm{pless}}_{U_I} \arrow[dl, hook] & & \mathrm{Fact}^{\mathrm{pless}}_{\mathfrak{B}} \arrow[ll, "\simeq", "\text{Prop. 7.7}^*"'] \arrow[dl, hook] \arrow[ddr, hook] & & & \mathrm{Fact}^{\mathrm{pless}}_{\mathfrak{B}_\cup} \arrow[lll, dashed, "{[\textbf{KSW24}, \text{ Q. 4.1}]}"] \arrow[dl, hook] \arrow[ddr, hook] & & & \mathrm{Fact}^{\mathrm{pless}}_{X} \arrow[lll, "\simeq", "\text{Prop. 7.10}^*"'] \arrow[dl, hook] \arrow[ddr, hook] & \\
\lim\limits_{\mathfrak{B}\supset I\supsetneq\emptyset} \mathrm{Alg}^{\mathrm{m}}_{U_I} \arrow[d, "\simeq"] & & \mathrm{Alg}^{\mathrm{m}}_{\mathfrak{B}} \arrow[ll, "\simeq", "\text{Eq. (7.4)}^*"'] \arrow[d, "\simeq"] & & & \mathrm{Alg}^{\mathrm{m}}_{\mathfrak{B}_\cup} \arrow[lll, "\simeq", "\text{Prop. 7.13}^*"'] \arrow[ddr, hook] & & & \mathrm{Alg}^{\mathrm{m}}_{X} \arrow[lll] \arrow[ddr, hook] & & \\
\lim\limits_{\mathfrak{B}\supset I\supsetneq\emptyset} \mathrm{Alg}^{\mathrm{m}}_{\mathrm{open}_{\neq\emptyset}(U_I)_{\mathrm{mrk}}} \arrow[dr, hook] & & \mathrm{Alg}^{\mathrm{m}}_{\mathfrak{B}'} \arrow[ll, "\simeq", "\text{Eq. (7.3)}^*"'] \arrow[dr, hook] & & \mathrm{Alg}^{\mathrm{Weiss}}_{\mathfrak{B}} \arrow[dl] & & & \mathrm{Alg}^{\mathrm{Weiss}}_{\mathfrak{B}_\cup} \arrow[lll] \arrow[dl, hook] & & & \mathrm{Alg}^{\mathrm{Weiss}}_{X} \arrow[lll, "\simeq", "\text{Prop. 7.10}^*"'] \arrow[dl, hook] \\
 & \lim\limits_{\mathfrak{B}\supset I\supsetneq\emptyset} \mathrm{Alg}_{\mathrm{open}_{\neq\emptyset}(U_I)_{\mathrm{mrk}}} & & \mathrm{Alg}_{\mathfrak{B}'} \arrow[ll, "\simeq", "\text{Cor. 7.5}^*"'] & & & \mathrm{Alg}_{\mathfrak{B}_\cup} \arrow[lll] & & & \mathrm{Alg}_{X} \arrow[lll] &
\end{tikzcd}
\tag{1.8}
$$

1.4. Part 2: The pointless higher Morita categories. In this part we construct the pointless higher Morita categories $\mathrm{Alg}^{\mathrm{pl}}_n(\mathcal{C})$, following Scheimbauer's construction of $\mathrm{Alg}^{\mathrm{ptd}}_n(\mathcal{C})$ from [**Sch14, GS18**]. This amounts to replacing the ∞-category of constructible factorization algebras by its pointless version. The motivation for constructing a third version of the higher Morita categories which enjoys certain good properties from both Scheimbauer's pointed version $\mathrm{A{\scriptstyle LG}}_n^{\mathrm{ptd}}(\mathcal{C})$ and Haugseng's (also pointless) version is explained in detail in Section 1.2.

We briefly outline the construction of $\mathrm{A{\scriptstyle LG}}_n^{\mathrm{pl}}(\mathcal{C})$ for $n = 1$ (following the pointed version in [**GS18**, Appendix A]). One defines a suitable (ordinary) category of stratifications called $\mathrm{S{\scriptstyle TRAT}}$. From an object $S \in \mathrm{S{\scriptstyle TRAT}}$ one extracts a (maximally) marked stratification of $(0,1)$, denoted $\mathrm{A_m}(S)$. This simply corresponds to a finite number of marked points on $(0,1)$ determined by S. In Section 8.2 we establish item 3b of the list of requirements 1.1, i.e. that constructible pointless factorization algebras push forward along certain nice maps. As a consequence we get that constructible pointless factorization algebras determines a functor as in the diagram:

$$\begin{tikzcd} \mathrm{S{\scriptstyle TRAT}} \arrow[r, "\mathrm{F{\scriptstyle ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(-)}"] \arrow[d, "\pi"'] & \mathrm{C{\scriptstyle AT}}_\infty \\ \Delta^{\mathrm{op}} \arrow[ur, dashed, "(\mathrm{Alg}_1^{\mathrm{pl}})_\bullet"'] & \end{tikzcd} . \tag{1.9}$$

We then define $(\mathrm{Alg}_1^{\mathrm{pl}})_\bullet$ to be the left Kan extension of $\mathrm{F{\scriptstyle ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(-)}$, indicated by a dashed arrow above, along the localization $\pi\colon \mathrm{S{\scriptstyle TRAT}} \to \Delta^{\mathrm{op}}$.

Recall that Theorem A, i.e. that constructible pointless factorization algebras glue along open covers, corresponds to item 3a of the list of requirements 1.1. Together with having established (1.9) it follows that the proof of [**GS18**, Theorem A.11] directly carries over to the pointless setting.

THEOREM G (Corollary 9.14). The simplicial object in $\mathrm{C{\scriptstyle AT}}_\infty$ given by $(\mathrm{Alg}_1^{\mathrm{pl}})_\bullet$ from (1.9) satisfies the Segal condition. Thus, $\mathrm{Alg}_1^{\mathrm{pl}}$ is a category object in $\mathrm{C{\scriptstyle AT}}_\infty$.

With this we define the pointless higher Morita category $\mathrm{A{\scriptstyle LG}}_1^{\mathrm{pl}}(\mathcal{C})$ as the underlying $(\infty,2)$-category of $(\mathrm{Alg}_1^{\mathrm{pl}})_\bullet$. The general construction, i.e. for $n > 1$, is outlined in Section 9.1.6 and details will appear in [**Sch**]. The pointless higher Morita categories come with a symmetric monoidal structure induced by that of $\mathrm{F{\scriptstyle ACT}}_X^{\mathrm{pl,cstr}}$ from Theorem B. Explicitly, we get the pointless version of [**GS18**, Theorem A.15]:

THEOREM H (Theorem 9.19). The functor $\mathrm{Alg}_1^{\mathrm{pl},\otimes}$ satisfies the Segal condition in both $\mathrm{F{\scriptstyle IN}}_*$ and Δ^{op} and thus determines a symmetric monoidal $(\infty,2)$-category.

From Theorem C we in particular get a symmetric monoidal functor

$$\iota^*\colon \mathrm{F{\scriptstyle ACT}}^{\mathrm{cstr}}_{\mathrm{A}(S)} \longrightarrow \mathrm{F{\scriptstyle ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(S)}$$

where $\mathrm{A}(S)$ has the same stratified points as $\mathrm{A_m}(S)$ but there are *no* marked points. This functor together with the fact that the pointless higher Morita category $\mathrm{A{\scriptstyle LG}}_1^{\mathrm{pl}}(\mathcal{C})$ is constructed exactly like the pointed version $\mathrm{A{\scriptstyle LG}}_1^{\mathrm{ptd}}(\mathcal{C})$ by simply replacing $\mathrm{F{\scriptstyle ACT}}^{\mathrm{cstr}}_{\mathrm{A}(S)}$ with $\mathrm{F{\scriptstyle ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(S)}$ to get (1.9) gives rise to the theorem below for $n = 1$.

THEOREM I (Theorem 9.41). Let $\mathcal{C}$ be a symmetric monoidal $\otimes$-presentable ∞-category. We have a symmetric monoidal functor of $(\infty, n+1)$-categories

$$\iota^*\colon \mathrm{A{\scriptstyle LG}}_n^{\mathrm{ptd}}(\mathcal{C}) \longrightarrow \mathrm{A{\scriptstyle LG}}_n^{\mathrm{pl}}(\mathcal{C}) , \tag{1.10}$$

where the former refers to the pointed higher Morita category of Scheimbauer.

As a consequence of having such a symmetric monoidal functor between the two versions of the higher Morita categories we can also transfer the dualizability-data of [**GS18**] for $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ to the pointless higher Morita categories $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$.

THEOREM J (Corollary 10.1). Let $\mathcal{C}$ be a symmetric monoidal (∞, m)-category and let $\mathcal{C}^{\square}$ be $\otimes$-presentable (see Definition 9.32). The symmetric monoidal $(\infty, n+m)$-category $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ is fully n-dualizable, i.e.

(1) every object of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ has a dual; and
(2) if $1 \leq k < n$, any k-morphism in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ has both a left and a right adjoint.

In particular, this means that every object of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ is fully n-dualizable and hence ensures item 4 of the list of requirements 1.1. Having established n-dualizability we can now turn towards $(n+1)$-dualizability in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$. Recall that item 2 of the list of requirements 1.1 is tailored to get rid of the pointed structure, which by [**GS18**, Theorem 5.1] is an obstruction to interesting $(n+1)$-dualizability in $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$. The result below was first conjectured in [**Lur09b**, Remark 4.1.27] for Lurie's sketch of the higher Morita categories, where it is claimed to follow from the proof of the cobordism hypothesis. We rephrase it here for $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$.

CONJECTURE-THEOREM (Conjecture-Theorem 10.17). Let $\mathcal{C}$ be a symmetric monoidal $\otimes$-presentable ∞-category. Let $\mathcal{R}$ denote an object of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ and $\mathcal{R}$ the corresponding E_n-algebra. The object $\mathcal{R}$ is $(n+1)$-dualizable in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ if, and only if, it is dualizable over the factorization homologies

$$\int_{S^{k-1}\times\mathbb{R}^{n-k+1}} \mathcal{R} \tag{1.11}$$

for $k = 0, 1, \ldots, n$.

For $n = 1$ this is proven by Lurie, while the case $n = 2$ and $\mathcal{C} = \mathrm{Pr}$ is proven by Brochier, Jordan and Snyder in [**BJS21**]. In Section 10.2 we give a partial proof of this for $n = 2$.

1.5. Part 3: Factorization homology and enriched skein categories. This part highlights some of the results in [**KKMP**], a joint article with Corina Keller, Lukas Müller and Jan Pulmann. We define an enriched version of skein categories and prove that they compute factorization homology of surfaces and values in $\mathcal{V}$-Cat, where $\mathcal{V}$ is a suitable enriching category. This generalizes a result of Cooke which proves the analogous statement for ordinary, i.e. $\mathrm{VECT}_{\Bbbk}$-enriched, skein categories in [**Coo23**].

Let us first give the precise setting we are in for this part. We consider oriented 2-dimensional manifolds, i.e. oriented surfaces, and work with the target category $\mathcal{V}$-Cat, for $\mathcal{V}$ a complete and cocomplete symmetric monoidal closed category. Let $\mathcal{A}$ be a $\mathcal{V}$-enriched ribbon category, hence in particular a framed E_2-category in $\mathcal{V}$-Cat. In this situation the diagram (1.2) defining factorization homology as a left Kan extension reduces to

$$\begin{tikzcd} \mathcal{D}\mathrm{isks}_2^{\mathrm{or},\otimes} \arrow[r, "\mathcal{A}"] \arrow[d, hook, "\iota"'] & \mathcal{V}\text{-}\mathsf{Cat} \\ \mathcal{M}\mathrm{fld}_2^{\mathrm{or},\otimes} \arrow[ur, dashed, "\int_{\_}\mathcal{A}"'] & \end{tikzcd}.$$

Factorization homology is characterized by three properties [**AF15, AFT17a**] which in our setting becomes the following. By virtue of being a left Kan extension it follows that evaluating at an oriented disk D gives back $\mathcal{A}$, i.e. $\int_D \mathcal{A} \simeq \mathcal{A}$. Given an oriented 1-manifold

C the factorization homology of $C \times (0,1)$ comes with a canonical framed E_1-structure, i.e. in our setting a monoidal product, induced by including intervals inside a larger interval. From oriented surfaces M with collared boundaries one gets induced module structures on the factorization homology of M. If one has a collar-gluing of an oriented manifold, i.e. roughly a decomposition $\Sigma = M \cup_{C\times(0,1)} N$, it then follows that $\int_M \mathcal{A}$, respectively $\int_N \mathcal{A}$ are right, respectively left $\int_{C\times(0,1)} \mathcal{A}$-modules. With this we can state the *excision*-property of factorization homology, which says that in the above situation we have an equivalence

$$\int_\Sigma \mathcal{A} \simeq \int_M \mathcal{A} \underset{\int_{C\times(0,1)}\mathcal{A}}{\otimes} \int_N \mathcal{A} \tag{1.12}$$

The relative tensor product is given by the two-sided bar construction.

We now briefly explain how enriched skein categories are defined. The *enriched skein category* $\mathbf{Sk}_{\mathcal{A}}(\Sigma)$ associated to an oriented surface Σ has as objects finite sets of pairwise disjoint, framed and oriented $\mathcal{A}$-colored points in Σ. The ordinary version of skein categories, i.e. corresponding to $\mathcal{V} = \mathrm{VECT}_{\Bbbk}$, has as morphisms $\Bbbk$-linear combinations of so-called $\mathcal{A}$-colored ribbon tangles in $\Sigma \times [0,1]$, up to identifying two such colored ribbon tangles if they are equal outside a cube and if their evaluation on the cube are equal. In some more detail this evaluation is given by the Reshetikin-Turaev functor which evaluates ribbon graphs in a cube to morphisms in $\mathcal{A}$. For general $\mathcal{V}$ we replace this *vector space* of $\mathcal{A}$-colored ribbon tangles by a suitable $\mathcal{V}$*-object* of all possible $\mathcal{A}$-colored ribbon tangles between two fixed objects. With this we can prove the theorem below (appearing in [**KKMP**] as Theorem 4.22) which generalizes [**Coo23**, Theorem 1.22].

THEOREM K (Theorem 13.2). Let C be a 1-manifold with a thickened right embedding into the boundary of M and a thickened left embedding into the boundary of N. Then we have an induced $\mathcal{V}$-functor

$$\mathbf{Sk}_{\mathcal{A}}(M) \underset{\mathbf{Sk}_{\mathcal{A}}(C\times[0,1])}{\times} \mathbf{Sk}_{\mathcal{A}}(N) \xrightarrow{\simeq} \mathbf{Sk}_{\mathcal{A}}(M \underset{C\times[0,1]}{\sqcup} N) \tag{1.13}$$

which is an equivalence of $\mathcal{V}$-categories.

In the $\mathrm{VECT}_{\Bbbk}$-enriched setting of [**Coo23**] the excision-proof passes through the *Tambara relative tensor product* [**Tam01**]. Our proof is an adaptation of this to the enriched setting, and as such the left hand side of (1.13) is given by the $\mathcal{V}$*-enriched Tambara relative tensor product.* This is defined in [**KKMP**], where we also prove that it agrees with the two-sided bar construction which is what is used to write down excision of factorization homology, i.e. (1.12) here. As a consequence of Theorem K and the equivalence of the $\mathcal{V}$-enriched Tambara tensor product with the two-sided bar construction we get the following result ([**KKMP**, Theorem 4.14]).

THEOREM L (Theorem 13.1). Let $\mathcal{A}$ be a $\mathcal{V}$-enriched ribbon category, where $\mathcal{V}$ is a cocomplete and symmetric monoidal closed category. Then the enriched skein category computes factorization homology. Explicitly,

$$\mathbf{Sk}_{\mathcal{A}}(\Sigma) \cong \int_\Sigma \mathcal{A} \in \mathcal{V}\text{-}\mathsf{Cat} \ .$$

We now expand on the motivation for developing enriched skein categories and proving that it computes factorization homology in $\mathcal{V}$-Cat in [**KKMP**]. One main motivation of [**KKMP**] is to make the following idea more precise (in the underived setting). Heuristically, one can use factorization homology with values in some suitable object encoding local classical observables of a classical (topological) field theory to obtain the global observables of a spacetime Σ, i.e.

$$\mathrm{Obs}^{\mathrm{cl}}(\Sigma) \cong \int_{\Sigma} \mathrm{Obs}^{\mathrm{cl}}_{\mathrm{loc}} \ .$$

It would then be natural to expect that given a suitable deformation quantization $\mathrm{Obs}^{q}_{\mathrm{loc}}$ of the local classical observables one can define the global quantum observables as

$$\mathrm{Obs}^{q}(\Sigma) \cong \int_{\Sigma} \mathrm{Obs}^{q}_{\mathrm{loc}} \ .$$

To make this precise [**KKMP**] first defines suitable categories of (almost) Poisson structures (for the local classical observables) and so-called Beilinson-Drinfeld structures (for the local quantum observables). With this at hand one establishes additivity-results for both categories and prove that factorization homology commutes with taking classical limits. We refer the interested reader to [**KKMP**] for details on this.

One example of classical observables behaving as outlined above is the moduli stack of G-local systems, for G a reductive group. As in Dijkgraaf–Witten theory, the classical local observables are then encoded by the representation category $\mathrm{Rep}(G)$. One deformation quantization of $\mathrm{Rep}(G)$ is given by the representation category $\mathrm{Rep}_q(G)$ of the quantum group $U_q\mathfrak{g}$. In [**BZBJ18a, BZBJ18b**] Ben-Zvi, Brochier and Jordan compute factorization homology with coefficients in $\mathrm{Rep}_q(G)$.

Another alternative deformation quantization of $\mathrm{Rep}(G)$ is given by the Drinfeld category $U\mathfrak{g}\text{-}\mathrm{Mod}^{\Phi}[[\hbar]]$ [**Dri89a, Dri89b**]. In [**KKMP**] we show that factorization homology for the Drinfeld category recovers the quantization constructed by Li-Bland and Ševera from [**LBŠ15**]. Moreover, from the formalism in [**KKMP**] one can directly conclude from the equivalence of ribbon categories between $U\mathfrak{g}\text{-}\mathrm{Mod}^{\Phi}[[\hbar]]$ and $\mathrm{Rep}_{\hbar}(G)$ that the corresponding quantizations are also equivalent. To be able to show that one recovers known quantizations of the moduli space of flat G-bundles we have to explicitly compute factorization homology in the $(2,1)$-category of categories enriched over complete $\mathbb{C}[[\hbar]]$-modules. For these kind of computations we use Theorem L for $\mathcal{V} = \widehat{\mathbb{C}[[\hbar]]\text{-}\mathrm{Mod}}$.

Part 1

Pointless factorization algebras

Attribution. This part is a modification of the article [**KSW24**] joint with Claudia Scheimbauer and Tashi Walde. The main motivation for writing this article in the first place was to have a proof that constructible factorization algebras glue modifiable to the setting in this thesis. In particular, I was in parallel modifying the setup and proofs, and we came up with proof strategies which are adaptable. To single out some main contributions to the article [**KSW24**]: I was centrally involved in finding the proofs in Section 4 and Section 5.1. Section 4 is the toolbox for gluing techniques for ordinary (not necessarily constructible) factorization algebras. Section 5.1 is the statement that locally constructible factorization algebras are constructible. Of course I also contributed to the remaining sections and writing.

In this part of the thesis we define a version of factorization algebras called *pointless* factorization algebras, and extend the results of [**KSW24**] to this setting by mostly using the same proof-techniques. To emphasize where adaptations were necessary we use Notation 1.2 throughout.

Assumption. In this part of the thesis we fix

- a topological space X which we always assume to be Hausdorff;
- a $\otimes$-presentable symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$.

2. Marked opens and pointless prefactorization algebras

In Section 2.1 we first define what a marked space is, as well as what the corresponding notion of inclusion should be. Using this we define the category of marked opens, $\mathrm{open}(X)_{\mathrm{mrk}}$, and through examples and warnings we give the reader some intuition on how this differs from the usual category $\mathrm{open}(X)$ of opens and inclusions. Then, in Section 2.2, we turn the category of marked opens into an ∞-operad, $\mathrm{open}(X)^{\otimes}_{\mathrm{mrk}}$, and define algebras over this ∞-operad to be pointless factorization algebras. We highlight how this differs from the usual notion of prefactorization algebras through a list of examples.

2.1. The category of marked opens. In this subsection we properly introduce and define the variant of $\mathrm{open}(X)$ we will work with. We first equip our space X with a finite set of "marked points". Taking the marked points into account, we work with marked opens and marked inclusions, which assembles into a wide subcategory of $\mathrm{open}(X)$. For a thorough explanation of why we want to work with this variant we refer the reader to the motivation in Section 1.2.

Definition 2.1. We define a *marked space* (X, X^{mrk}) to be a Hausdorff space X, together with a finite subset X^{mrk} of points, which we call *marked points*. For any open subset $U \subseteq X$, the open U inherits a marked structure from that of X by $U^{\mathrm{mrk}} := U \cap X^{\mathrm{mrk}}$. In this situation we say that (U, U^{mrk}) is a *marked open* of (X, X^{mrk})

Notation 2.2. We often abuse notation and omit the finite set of marked points unless explicitly needed. That is, we simply say that X is a marked space, and U is a marked open of X.

Definition 2.3. Let (X, X^{mrk}) be a marked space. We define $\mathrm{open}(X)_{\mathrm{mrk}}$ to be the poset of marked opens of X with partial order given by inclusions $U \hookrightarrow V$ of opens that are *bijective on marked points*, i.e. such that $\pi_0(U \cap X^{\mathrm{mrk}}) = \pi_0(V \cap X^{\mathrm{mrk}})$. We call such inclusions *marked inclusions* and denote them by $U \hookrightarrow^{\mathrm{m}} V$.

Observe that $\mathrm{open}(X)_{\mathrm{mrk}}$ is a wide subcategory of $\mathrm{open}(X)$; it has the same objects, but fewer morphisms.

Example 2.4. Whenever $X^{\mathrm{mrk}} = \emptyset$ we say that X is equipped with the *trivial marked structure* or is *unmarked*. In this situation the condition making an inclusion marked is vacuous and it follows that $\mathrm{open}(X)_{\mathrm{mrk}} = \mathrm{open}(X)$.

Remark 2.5. Recall that in $\mathrm{open}(X)$ the empty set $\emptyset$ includes into any other open, and is hence an initial object in $\mathrm{open}(X)$. However, in $\mathrm{open}(X)_{\mathrm{mrk}}$ this is no longer true; we have $\emptyset \hookrightarrow^{\mathrm{m}} U$ only if U contains *no* marked points. In fact, the poset $\mathrm{open}(X)_{\mathrm{mrk}}$ is tailored to have exactly this feature.

Remark 2.6. Inclusions that are bijective on marked points are also used other places; see e.g. [**BZBJ18b**, §2.2] in the setting of (category-valued) factorization homology for surfaces with marked points.

As the notion of marked inclusions is unusual, let us consider an easy example.

Example 2.7. Let $X = \mathbb{R}$ with either one or two marked points as illustrated in Figure 1. Observe that on the left side below, where X has two marked points, none of the depicted objects $U, V, W \in \mathrm{open}(X)_{\mathrm{mrk}}$ has marked inclusions into each other, nor into X. Conversely, on the right hand side we have marked inclusions $\widetilde{U} \hookrightarrow^{\mathrm{m}} X$ and $\widetilde{W} \hookrightarrow^{\mathrm{m}} X$.

FIGURE 1. Some marked opens on $\mathbb{R}$ with two different choices of marking.

We now highlight some of the features of $\mathrm{open}(X)_{\mathrm{mrk}}$ which differs from that of $\mathrm{open}(X)$. First of all, let us make the following observation for the latter.

Observation 2.8. In the setting of $\mathrm{open}(X)$ objects and morphisms are on an equal footing. That is, to understand objects of $\mathrm{open}(X)$ one can equivalently examine morphisms into X, i.e. objects of $\mathrm{open}(X)_{/X}$. This is reflected in the fact that for any open subset $U \subseteq X$ we have an equality $\mathrm{open}(X)_{/U} = \mathrm{open}\,(U)$.

The above seems like a pretty trivial observation, however, it also turns out to be the key difference between $\mathrm{open}(X)$ and $\mathrm{open}(X)_{\mathrm{mrk}}$. We highlight this in the following warning.

Warning 2.9. Let $V \subseteq X$ be a marked open with at least one marked point. Then it is *not* true that $\mathrm{open}\,(V)_{\mathrm{mrk}}$ coincides with $\mathrm{open}(X)_{\mathrm{mrk}/V}$; the former has all open subsets of V as objects, while the latter only contains those open subsets with a marked inclusion into V.

Example 2.10. Consider $X = \mathbb{R}$ together two marked points as illustrated in the left hand side of Figure 1. Note that all of the open subsets U, V, W are objects of $\mathrm{open}(X)_{\mathrm{mrk}}$. However, since none of them have a marked inclusion into X none of them are objects in $\mathrm{open}(X)_{\mathrm{mrk}/X}$.

In a similar way we also have that marked inclusions behave differently from ordinary inclusions when it comes to intersections.

Warning 2.11. Marked inclusions are not compatible with intersections in general. Consider again the marked opens from Figure 1. For example, we see that $V \cap W$ contains no marked points and hence has neither a marked inclusion into V nor W.

However, if we work relative to a third marked open we do have such intersections; e.g. since $\widetilde{U} \hookrightarrow^{\mathrm{m}} X$ and $\widetilde{W} \hookrightarrow^{\mathrm{m}} X$ it follows that $\widetilde{U} \cap \widetilde{W} \hookrightarrow^{\mathrm{m}} \widetilde{U}$ and $\widetilde{U} \cap \widetilde{W} \hookrightarrow^{\mathrm{m}} \widetilde{W}$.

The last part of the warning gives a good clue about how one nevertheless can translate many arguments regarding ordinary opens to the marked setting. We explain this in more detail in Section 3.1 and in particular Remark 3.5 after having introduced some more terminology.

2.2. Pointless prefactorization algebras. We first explain how to obtain an ∞-operad from the ordinary category $\mathrm{open}(X)_{\mathrm{mrk}}$, or any full subposet thereof. Then we give the definition of a pointless prefactorization algebra, as well as some key examples to highlight the impact of working with $\mathrm{open}(X)_{\mathrm{mrk}}$.

Recall from Definition 2.3 that $\mathrm{open}(X)_{\mathrm{mrk}} \subset \mathrm{open}(X)$ is the wide subcategory of (marked) opens and marked inclusions. We also want to have the flexibility of working with a smaller selection of marked opens than all of $\mathrm{open}(X)_{\mathrm{mrk}}$. To that end we have the following definition:

Definition 2.12. A full subposet $\mathfrak{U} \subseteq \mathrm{open}(X)_{\mathrm{mrk}}$ is called a *poset of marked opens* (of X).

We turn this into an ∞-operad in two steps. For the reader unfamiliar with colored operads and ∞-operads we refer them to the recollection in Appendix A.1.

Construction 2.13. Let $\mathfrak{U} \subseteq \mathrm{open}(X)_{\mathrm{mrk}}$ be a poset of marked opens. It is a colored operad (in Set) by defining colors to be all marked opens $U \in \mathfrak{U}$. The set of multimorphisms is defined to be

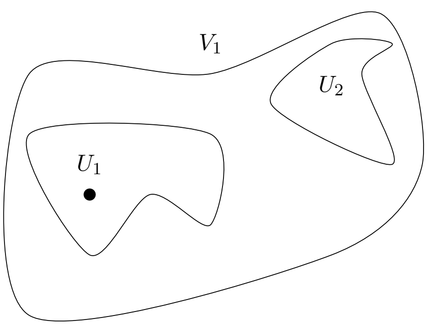
$$\mathrm{Mul}_{\mathfrak{U}}(\{U_i\}_{i\in I}, V) := \begin{cases} \star & \text{if } U_i \cap U_{i'} = \emptyset \, \forall i \neq i' \text{ and } \dot\bigcup_{i\in I} U_i \hookrightarrow^{\mathrm{m}} V \ , \\ \emptyset & \text{else} \ . \end{cases}$$

In words, it is the singleton set if all the U_i's are pairwise disjoint and their disjoint union come with a marked inclusion into V; else it is the empty set. The composition maps arise from the composition of marked inclusions and the collection of identity morphisms are exactly given by the identity inclusions.

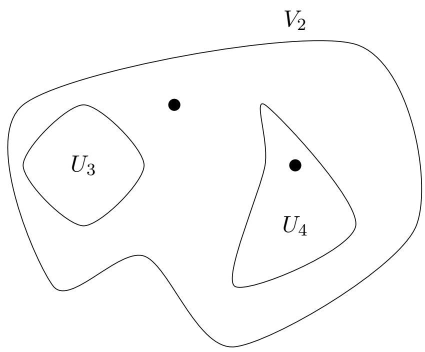
We use [**Lur17**, Construction 2.1.1.7] to turn the above colored operad into an ∞-operad. For this, recall that Fin_* is the category of finite pointed sets and point-preserving maps.

Construction 2.14. Let $\mathfrak{U}$ be a poset of marked opens. The ∞-operad $\mathfrak{U}^{\otimes}$ has as objects pairs $(I_+, (U_i))$, where $I_+ \in \mathrm{Fin}_*$ and (U_i) is an I-indexed list of marked opens $U_i \in \mathfrak{U}$. A morphism $f\colon (I_+, (U_i)) \to (J_+, (V_j))$ is a morphism $f\colon I_+ \to J_+$ in Fin_* such that for each $j \in J$ the tuple $(U_i \mid i \in f^{-1}(J))$ is a collection of pairwise disjoint opens together with a marked inclusion $\dot\bigcup_{i\in f^{-1}(j)} U_i \hookrightarrow^{\mathrm{m}} V_j$.

Example 2.15. To make this more clear, we give one example and non-example of a morphism in $\mathrm{open}(X)^{\otimes}_{\mathrm{mrk}}$. Let $\alpha\colon 2_+ \to 1_+$ be the unique active map and let the marked opens $\{U_i\}_{i\in\{1,2,3,4\}}$ and $\{V_j\}_{j\in\{1,2\}}$ be as in Figure 2. We have a morphism $(2_+, (U_1, U_2)) \xrightarrow{\alpha} (1_+, (V_1))$ in $\mathrm{open}(X)^{\otimes}_{\mathrm{mrk}}$ because the inclusion $U_1 \sqcup U_2 \hookrightarrow^{\mathrm{m}} V_1$ is bijective on marked points, hence a marked inclusion. However, there is *no* map $(2_+, (U_3, U_4)) \xrightarrow{\alpha} (1_+, (V_2))$ since the inclusion $U_3 \sqcup U_4 \hookrightarrow V_2$ is not bijective on marked points.



(A) An example of a morphism in $\mathrm{open}(X)^{\otimes}_{\mathrm{mrk}}$ over the unique active morphism $\alpha\colon 2_+ \to 1_+$.



(B) A non-example of a morphism over α since the inclusion is not bijective on marked points.

Figure 2

Remark 2.16. For $\mathfrak{U}$ any poset of marked opens the ∞-operad $\mathfrak{U}^{\otimes}$ is actually an ordinary category fibered over Fin_*, and we will not be explicit about when we apply the nerve functor to obtain an honest ∞-operad unless it is non-trivial to combine it with the arguments needed. In more detail, we are only explicit about this in Section 7.1.1 where we compute colimits of these types of ∞-operads, since taking the nerve does not commute with colimits in general.

We can now give the definition of pointless prefactorization algebras. Recall that we fix $\mathcal{C}$ to be an $\otimes$-presentable symmetric monoidal ∞-category (as in Definition A.18) in this part of the thesis. In particular, $\mathcal{C}^{\otimes}$ has the structure of an ∞-operad, see Definition A.17, which is what we use in the below definition.

Definition 2.17. Let (X, X^{mrk}) be a marked space, and let $\mathfrak{U} \subseteq \mathrm{open}(X)_{\mathrm{mrk}}$ be a poset of marked opens. A *pointless prefactorization algebra on* $\mathfrak{U}$ (with values in $\mathcal{C}$) is a morphism of ∞-operads

$$\mathcal{A} \colon \mathfrak{U}^{\otimes} \to \mathcal{C}^{\otimes}.$$

Equivalently, we say that $\mathcal{A}$ is a *pointless* $\mathfrak{U}$*-prefactorization algebra*. The ∞-category of pointless $\mathfrak{U}$-prefactorization algebras is $\mathrm{Alg}_{\mathfrak{U}} := \mathrm{Alg}_{\mathfrak{U}}(\mathcal{C})$, where we often omit the target category since it is fixed throughout.

Notation 2.18. Whenever $X^{\mathrm{mrk}} = \emptyset$ above we get back the usual definition of a prefactorization algebra. In this setting we often say *ordinary prefactorization algebras* to make the distinction between this and the pointless version more explicit.

We now give some basic examples which highlights how pointless prefactorization algebras differ from ordinary ones.

Example 2.19. Let X be a trivially marked space, i.e. $X^{\mathrm{mrk}} = \emptyset$. Since $\mathrm{open}(X)_{\mathrm{mrk}} = \mathrm{open}(X)$ in this case we get that the notions of ordinary and pointless prefactorization algebras coincide. This is also true if we work with any full subposet $\mathfrak{U}$. In this sense ordinary prefactorization algebras trivially provides examples of pointless prefactorization algebras.

Example 2.20. One instance of the above is for $X = \mathbb{R}$, $\mathfrak{U}$ the poset of nonempty open intervals and $\mathcal{C} = \mathrm{V{\small ECT}}$. Then any associative unital algebra $\mathbf{A}$ determines a $\mathfrak{U}$-prefactorization algebra $\mathcal{A}$. In particular, the inclusion $\emptyset \hookrightarrow U$ for any $U \in \mathfrak{U}$ is assigned the map $\Bbbk \to \mathbf{A}$ induced by the unit. See for example [**KSW24**, Example 3.5] for details.

There is also a second way to produce pointless prefactorization algebras from ordinary ones, namely by forgetting part of the structure.

Construction 2.21. Any ordinary prefactorization algebra is an example of a pointless prefactorization algebra. More explicitly, let (X, X^{mrk}) be a marked space. We have an inclusion of posets

$$\iota_X \colon \mathrm{open}(X)_{\mathrm{mrk}} \longrightarrow \mathrm{open}(X)$$

which induces an inclusion of the corresponding ∞-operads. By restricting algebras along ι_X we get an induced functor

$$\iota_X^* \colon \mathrm{Alg}_{\mathrm{open}(X)}(\mathcal{C}) \longrightarrow \mathrm{Alg}_{\mathrm{open}(X)_{\mathrm{mrk}}}(\mathcal{C}) \tag{2.1}$$

corresponding to forgetting the structure maps which do not come from marked inclusions. We stress that the functor ι_X^* is *not* fully faithful in general because ι_X is not fully faithful. This is a consequence of the morphisms being different in the two posets.

The following is an example of an ordinary prefactorization algebra which restricts to a pointless prefactorization algebra:

Example 2.22 (The trivial factorization algebra)**.** Let (X, X^{mrk}) be any marked space. The *trivial* pointless prefactorization algebra (on X) is given by

$$\mathcal{A}^{\mathrm{triv}}\colon \operatorname{open}(X)^{\otimes}_{\mathrm{mrk}} \to \mathcal{C}, \qquad (I_+, (U_i)) \mapsto (I_+, (\mathbb{1}_{\mathcal{C}}))\ .$$

We finish the subsection by giving the main example which perfectly showcases the difference arising from working with pointless prefactorization algebras.

Example 2.23. Let $X = \mathbb{R}$ be marked at the origin, i.e. $X^{\mathrm{mrk}} = \{0\}$. Let $\mathfrak{U}$ be the poset of nonempty marked intervals and set $\mathcal{C} = \mathrm{V{\small ECT}}$. Let $\mathbf{A}$ and $\mathbf{B}$ denote two associative, unital algebras, while $\mathbf{M}$ is an $(\mathbf{A}, \mathbf{B})$-bimodule. From this we can define a pointless $\mathfrak{U}$-prefactorization algebra $\mathcal{M}$ by

$$\mathcal{M}(U) = \begin{cases} \mathbf{A}, & \text{if } U \subset (-\infty, 0), \\ \mathbf{M}, & \text{if } 0 \in U, \\ \mathbf{B}, & \text{if } U \subset (0, +\infty). \end{cases}$$

If one instead would have considered $X = \mathbb{R}$ without any marked points one needs to ask for $\mathbf{M}$ to be *pointed*, i.e. to come with an specified object $m \in \mathbf{M}$. This is because in that setting the inclusion $\emptyset \hookrightarrow U$ for $0 \in U$ requires an additional structure map $\Bbbk \to \mathbf{M}$ corresponding to this pointing. See [**KSW24**, Example 3.7] for explicit details of this.

Remark 2.24 (Pointed versus pointless)**.** The above example and its generalizations is the reason we call our version for *pointless* prefactorization algebras. In the ordinary setting the inclusion $\emptyset \hookrightarrow U$ of the empty set into any open induces a *pointed* structure; for Example 2.20 this corresponded to starting with a *unital* algebra $\mathbf{A}$, while in Example 2.23 this corresponded to a *pointed* bimodule $\mathbf{M}$. Hence, one can argue that ordinary factorization algebras should be called pointed factorization algebras.

Since we are unable to retroactively change the literature we instead call this new version for pointless prefactorization algebras to emphasize that we are (partially) getting rid of this pointed feature. The importance of this will become apparent in the context of higher Morita categories. See Part 2 and in particular Section 9 and Section 10 for more details.

3. Background on marked Weiss covers and cosheaves

In this section we modify definitions and notation from [**KSW24**, §2] to the marked setting. That is, in Section 3.1 we recall different ways to think about subsets or subposets of $\mathrm{open}(X)_{\mathrm{mrk}}$, and constructions related to this, as well as define locality and descent for copresheaves. Section 3.2 contains the definition of marked Weiss covers, marked Weiss cosheaves and their hyper-versions. Lastly, in Section 3.3 we define multiplicative and factorizing bases, and explain why these are particularly useful. Note in particular that all of the setup and results of [**KSW24**, §2] carry over to the marked setting, and we explain the key insight behind this in Remark 3.5.

3.1. Background on marked (hyper)covers, co(pre)sheaves and descent. In this section we recall how one can think about collections of marked opens of X in different ways; namely either as subsets or as subposets inheriting the poset-structure of $\mathrm{open}(X)_{\mathrm{mrk}}$ as in [**KSW24**, §2.1 & 2.2]. We also introduce marked (hyper)refinements which are maps between such collections of marked opens. Lastly, we define copresheaves on posets of marked opens and explain how it can be evaluated on the above mentioned notions, as well as what it means for a copresheaf to satisfy descent phrased using the notion of locality.

Before giving full definitions we summarize the conventions used:

- "(Pre)covers" and "marked (pre)covers" refer to certain subsets of $\mathrm{open}(X)_{\mathrm{mrk}}$; they are denoted by mathcal-letters like $\mathcal{U}, \mathcal{V}, \mathcal{W}$.
- "Marked hyper(pre)covers" and "(pre)sieves" refer to full subposets of $\mathrm{open}(X)_{\mathrm{mrk}}$; they are denoted by mathfrak-letters like $\mathfrak{B}, \mathfrak{U}, \mathfrak{W}$.

We now turn to precise definitions.

Definition 3.1.

- A set $\mathcal{U} \subset \mathrm{open}(X)_{\mathrm{mrk}}$ of open marked subsets of X is called a *precover (of X)*.
- Let $\mathcal{V}, \mathcal{U} \subset \mathrm{open}(X)_{\mathrm{mrk}}$ be two precovers of X. We say that we have a *marked refinement* $\mathcal{V} \to^{\mathrm{m}} \mathcal{U}$ if for every $V \in \mathcal{V}$ there exists an $U \in \mathcal{U}$ such that $V \hookrightarrow^{\mathrm{m}} U$. Furthermore, marked refinements compose.
- If $\mathcal{U}$ is a precover such that we have a marked refinement $\mathcal{U} \to^{\mathrm{m}} \{V\}$, where the target is a singleton, we call $\mathcal{U}$ for a *marked precover of V*.
- We say that a marked precover $\mathcal{U} \to^{\mathrm{m}} \{V\}$ is a *marked cover* of V if $\bigcup \mathcal{U} = V$.

Notation 3.2. In the above we reserve adding "marked" to the instances involving the marked inclusions. For example, we call a set $\mathcal{U} \subset \mathrm{open}(X)_{\mathrm{mrk}}$ of open (marked) subsets of X for a *precover* (of X) because nothing in that definition 'sees' the marked structure. More explicitly, since $\mathrm{open}(X)_{\mathrm{mrk}} \subset \mathrm{open}(X)$ is a wide subcategory giving a set of objects in either is equivalent, and we could equally well have asked for $\mathcal{U} \subset \mathrm{open}(X)$ to define a precover (of X).

Notation 3.3. If we consider a trivially marked space X and hence have that marked inclusions are exactly the usual inclusions, we drop "marked" from the above notation and talk about e.g. "precovers (of V)", "refinements" and "covers".

Remark 3.4. Note that a precover and a marked precover (of some marked open V) in general contain very different types of opens. If we have a marked refinement $\mathcal{U} \to^{\mathrm{m}} \{V\}$, where V contains marked points, then every $U \in \mathcal{U}$ must contain the exact same marked points. In contrast, the opens of a precover can all have varying numbers of marked points. In this section we will mostly work with marked precovers of a fixed open.

In fact, the above remark ties into why all of the setup of [**KSW24**, §2] carries over to the marked setting. We make this a bit more explicit.

Remark 3.5 (Marked inclusions are well-behaved for marked (pre)covers)**.** All results of [**KSW24**, §2] that involve (pre)covers of a fixed open carry over to the marked setting for the following reason: Fix some $V \in \mathrm{open}(X)_{\mathrm{mrk}}$, and some marked precover $\mathcal{U} \to^{\mathrm{m}} \{V\}$ of V. Since every open $U \in \mathcal{U}$ come with a marked inclusion $U \hookrightarrow^{\mathrm{m}} V$ we have that all inclusions $U \hookrightarrow U'$, for $U, U' \in \mathcal{U}$ automatically are *marked* inclusions. Thus, because the marked refinement places a restriction on the opens appearing in $\mathcal{U}$ we, within this subset, have the same morphisms as those of $\mathrm{open}(X)$. Explicitly, this means that the opens of $\mathcal{U}$ behave nicely with respect to e.g. intersections, i.e. we automatically have $U \cap U' \hookrightarrow^{\mathrm{m}} U$ and $U \cap U' \hookrightarrow^{\mathrm{m}} U'$. This is in general not true, see for example Warning 2.11.

In summary, by choosing marked precovers of V to mean a subset with a marked refinement into V one automatically does not see the distinction between ordinary and marked inclusions for the opens of $\mathcal{U}$, which is very advantageous.

Recall from Definition 2.12 that a poset of marked opens $\mathfrak{U}$ is a full subposet $\mathfrak{U} \subseteq \mathrm{open}(X)_{\mathrm{mrk}}$.

Definition 3.6. Let $\mathfrak{U}$ be a poset of marked opens. We say that it is

- a *presieve* if it is closed under intersections, i.e. if $U, U' \in \mathfrak{U}$ it implies that $U \cap U' \in \mathfrak{U}$.
- a *sieve* if it is downward closed with respect to marked inclusions, i.e. if $U \in \mathfrak{U}$ and $U' \hookrightarrow^{\mathrm{m}} U$ imply $U' \in \mathfrak{U}$.

Remark 3.7. In [**KSW24**] the theory is set up using full subposets of $\mathrm{open}(X)$. Since $\mathrm{open}(X)_{\mathrm{mrk}}$ is *not* a full subposet of $\mathrm{open}(X)$, the results in this part of the thesis are *not* a formal consequence. However, for example the results of this section are all a straightforward modification of the analogous results for $\mathrm{open}(X)$ (once one has figured out the correct modifications to the definitions).

We can relate *marked* precovers and posets of marked opens by the following constructions

Construction 3.8.

- Every poset of marked opens $\mathfrak{U}$ has an underlying precover $\mathfrak{U}^{\diamond}$ obtained by forgetting the poset structure. [5]
- For every marked precover $\mathcal{V} \to^{\mathrm{m}} \{U\}$ we have the associated presieve
$$\mathcal{V}_{\cap} := \{V_1 \cap \cdots \cap V_n \mid n \geq 1, V_i \in \mathcal{V}\}$$
obtained by adding all finite intersections. Since all $V_i \in \mathcal{V}$ by definition come with a marked inclusion into U, so do all of the finite intersections.
- For any marked precover $\mathcal{V} \to^{\mathrm{m}} \{U\}$ we also get an associated sieve
$$\mathcal{V}_{\downarrow_{\mathrm{m}}} := \{V' \in \mathrm{open}(X)_{\mathrm{mrk}} \mid \exists V \in \mathcal{V} : V' \hookrightarrow^{\mathrm{m}} V\}$$
obtained by closing it downward with respect to marked inclusions.

We use the same notation for adding all intersections to a poset of marked opens $\mathfrak{U}$, respectively closing it downward with respect to marked inclusions.

Note that the sieve associated to a singleton precover {V} is exactly the slice category $\mathrm{open}(X)_{\mathrm{mrk}/V}$ which, as explained in Warning 2.9, is *not* equivalent to $\mathrm{open}\,(V)_{\mathrm{mrk}}$.

[5]however, forgetting the poset structure of a poset of marked opens does not in general give a *marked* precover of some fixed open.

Definition 3.9. Let $\mathfrak{U} \subseteq \mathrm{open}(X)_{\mathrm{mrk}}$ be a poset of marked opens. A ($\mathcal{C}$-valued) *copresheaf on* $\mathfrak{U}$ is a functor $\mathcal{A}\colon \mathfrak{U} \to \mathcal{C}$. If $\mathfrak{U} = \mathrm{open}(X)_{\mathrm{mrk}}$ we say that $\mathcal{A}$ is a copresheaf on X.

Precovers and marked refinements define a category, exactly like how precovers and refinements do in the ordinary (i.e. unmarked) setting. From a copresheaf on $\mathfrak{U}$ one left Kan extends along the inclusion of $\mathrm{open}(X)_{\mathrm{mrk}}$ into this category to understand how to evaluate a copresheaf on precovers and marked refinements. See [**KSW24**, Construction 2.6] for details. Explicitly, the value of a copresheaf $\mathcal{A}\colon \mathfrak{U} \to \mathcal{C}$ on a precover $\mathcal{V}$ is given by the colimit

$$\mathcal{A}^{mpc}{}_{|\mathfrak{U}}(\mathcal{V}) \coloneqq \operatorname{colim} \mathcal{A}_{|\mathfrak{U}\cap\mathcal{V}_{\downarrow_{\mathrm{m}}}}. \tag{3.1}$$

We also want to have access to the following more general notion.

Definition 3.10. A *marked hyperprecover* of X is a full subposet $\mathfrak{U} \subseteq \mathrm{open}(X)_{\mathrm{mrk}}$. Given two marked hyperprecovers $\mathfrak{U}, \mathfrak{W}$ we say that we have a *marked hyperrefinement* $\mathfrak{U} \to^{\mathrm{m}}_{\mathrm{h}} \mathfrak{W}$ if, for each $U \in \mathfrak{U}$, the poset $\mathfrak{W}_{U/} \coloneqq \{W \in \mathfrak{W} \mid U \hookrightarrow^{\mathrm{m}} W\}$ is weakly contractible.

Remark 3.11. Note that despite posets of marked opens and hyperprecovers both being full subposets of $\mathrm{open}(X)_{\mathrm{mrk}}$ we think of them very differently. The first is the *domain of definition* of a copresheaf $\mathcal{A}$, while the latter is something $\mathcal{A}$ can be *evaluated on* (see (3.2) and (3.3) below).

Definition 3.12.

- We say that $\mathfrak{W}$ is a *marked hyperprecover of* U if we have a marked hyperrefinement $\mathfrak{W} \to^{\mathrm{m}}_{\mathrm{h}} \{U\}$ of the singleton U. This corresponds to having a marked refinement $\mathfrak{W}^{\diamond} \to^{\mathrm{m}} \{U\}$, or equivalently that $\mathfrak{W}^{\diamond}$ is a marked precover of U.
- Given a marked hyperprecover $\mathfrak{W} \to^{\mathrm{m}}_{\mathrm{h}} \{U\}$, we say that it is a *marked hypercover* of U if, for each finite list $W_1, \dots, W_n \in \mathfrak{W}$, the marked precover
$$\{W \in \mathfrak{W}^{\diamond} \mid W \subseteq W_1 \cap \cdots \cap W_n\} \longrightarrow^{\mathrm{m}} \{W_1 \cap \cdots \cap W_n\}$$
is a marked cover. The intersection is here taken in U, so the $n = 0$ condition corresponds to $\mathfrak{W}^{\diamond} \to^{\mathrm{m}} \{U\}$ being a marked cover. Since we started with a marked hyperprecover $\mathfrak{W}$ we know that all opens contain the same marked points, and hence so does finite intersections thereof, so this is indeed well-defined.

To evaluate a copresheaf on a marked hyperprecover we give a pointwise formula. [6] Explicitly, let $\mathcal{A}\colon \mathfrak{U} \to \mathcal{C}$ be a copresheaf. For every marked hypercover $\mathfrak{W} \subseteq \mathfrak{U}$ we define the evaluation of $\mathcal{A}$ on $\mathfrak{W}$ to be

$$\mathcal{A}^{mhpc}(\mathfrak{W}) \coloneqq \mathcal{A}^{mhpc}{}_{|\mathfrak{U}}(\mathfrak{W}) \coloneqq \operatorname*{colim}_{W\in\mathfrak{W}} \mathcal{A}(W). \tag{3.2}$$

Since this definition does not depend on the poset of marked opens $\mathfrak{U}$ we can omit it from the notation.

Similarly, we can also evaluate copresheaf on marked hyperrefinements. Let $\mathcal{A}$ be as above and let $\mathfrak{W}, \mathfrak{V} \subseteq \mathfrak{U}$ be two hyperprecovers with a marked hyperrefinement $\mathfrak{W} \to^{\mathrm{m}}_{\mathrm{h}} \mathfrak{V}$. The evaluation of $\mathcal{A}$ on this marked hyperrefinement is defined to be the composite

$$\operatorname*{colim}_{W\in\mathfrak{W}} \mathcal{A}(W) \xleftarrow{\simeq} \operatorname*{colim}_{\{W\hookrightarrow^{\mathrm{m}}V\}} \mathcal{A}(W) = \operatorname*{colim}_{V\in\mathfrak{V}} \operatorname*{colim}_{\{W\hookrightarrow^{\mathrm{m}}V\}} \mathcal{A}(W) \longrightarrow \operatorname*{colim}_{V\in\mathfrak{V}} \mathcal{A}(V), \tag{3.3}$$

[6] The previous trick with left Kan extending used to obtain (3.1) does not work here because marked hyperprecovers and marked refinements do not assemble into a category. See [**KSW24**, Warning 2.12] and [**KSW24**, Remark 2.13] for some more details on this.

where W and V range over $\mathfrak{W}$ and $\mathfrak{V}$, respectively. The equivalence of the leftmost arrow above comes from colimit cofinality of the forgetful map $\{W \hookrightarrow^{\mathrm{m}} V\} \to \mathcal{W}$. It is colimit cofinal precisely because we have the marked hyperrefinement $\mathfrak{W} \to^{\mathrm{m}}_{\mathrm{h}} \mathfrak{V}$.

Remark 3.13. By a marked analogue of [**KSW24**, Lemma 2.17] we get that evaluating a copresheaf $\mathcal{A}\colon \mathfrak{U} \to \mathcal{C}$ on a marked precover $\mathcal{W}$ is independent of the ambient poset of marked opens $\mathfrak{U}$ as long as $\mathcal{W}_\cap \subseteq \mathfrak{U}$. In this setting we omit $\mathfrak{U}$ from the notation and simply write

$$\mathcal{A}^{mpc}(\mathcal{W}) := \mathcal{A}^{mpc}{}_{|\mathfrak{U}}(\mathcal{W}).$$

One important notion for us is when the value of a copresheaf $\mathcal{A}$ is computed by its evaluation on certain marked covers. For this we first introduce the following convenient notation:

Definition 3.14. Let $\mathfrak{U}$ be a poset of marked opens and $\mathcal{A}\colon \mathfrak{U} \to \mathcal{C}$ be a copresheaf.

- Let $\mathcal{W} \to^{\mathrm{m}} \mathcal{V}$ be a marked refinement with $\mathcal{W}, \mathcal{V} \subseteq \mathfrak{U}$. We say that the marked refinement is $\mathcal{A}$-*local* if the induced map $\mathcal{A}^{mpc}{}_{|\mathfrak{U}}(\mathcal{W}) \to \mathcal{A}^{mpc}{}_{|\mathfrak{U}}(\mathcal{V})$ is an equivalence in $\mathcal{C}$.
- Similarly, a marked hyperrefinement $\mathfrak{W} \to^{\mathrm{m}}_{\mathrm{h}} \mathfrak{V}$ with $\mathfrak{W}, \mathfrak{V} \subseteq \mathfrak{U}$ is called $\mathcal{A}$-*local* if the induced map $\mathcal{A}^{mhpc}(\mathfrak{W}) \to \mathcal{A}^{mhpc}(\mathfrak{V})$ is an equivalence.

Remark 3.15. Usually one says that a copresheaf $\mathcal{A}$ satisfies *descent* with respect to a cover $\mathcal{V} \to \{U\}$ if that cover is $\mathcal{A}$-local, and analogously for precovers and hyper(pre)covers. We adopt this lingo also in the marked setting.

3.2. Marked Weiss cosheaves and hypercosheaves. There are several ways to equip $\mathrm{open}(X)$ with a topology, and in turn also several analogous topologies for $\mathrm{open}(X)_{\mathrm{mrk}}$. Here we explain the covers relevant for pointless factorization algebras, namely marked Weiss covers, and their corresponding cosheaves. This is a straightforward adaptation of the Weiss covers used to define factorization algebras. The reader unfamiliar with Grothendieck topologies can find more details in Appendix A.7. We also define the analogous notion of marked Weiss hypercovers and marked Weiss hypercosheaves, this is a useful tool e.g. in Section 7.3.

Let us start by recalling the definition of an ordinary Weiss cover as well as giving the marked version:

Definition 3.16.

- An ordinary cover $\mathcal{W}$ of U is called a *Weiss cover* if, for each finite subset $S \subset U$, there exists a $W \in \mathcal{W}$ such that $S \subseteq W$.
- A *marked Weiss cover* $\mathcal{W}$ of U is a marked cover (Definition 3.1) which is also a Weiss cover.

We now give the simplest example of a marked Weiss cover, namely:

Example 3.17. We say that a marked cover $\mathcal{W} \to^{\mathrm{m}} \{U\}$ is *degenerate* if $U \in \mathcal{W}$. Degenerate marked covers are always marked Weiss covers.

Note that Weiss covers typically contain infinitely many elements.[7] There are two standard ways of producing interesting (ordinary) Weiss covers, and they translate to the marked setting.

[7]For example for a manifold we can always find a finite set not contained in some finite collection of proper open subsets. Hence, a Weiss cover needs to have infinitely many elements in this case.

Example 3.18. Let (U, U^{mrk}) be a marked open of X. Let $\mathcal{K}$ be a choice of an infinite set of pairwise disjoint closed subsets of $U \setminus U^{\mathrm{mrk}}$. Then $\mathcal{W}_{\mathcal{K}} := \{X \setminus K \mid K \in \mathcal{K}\}$ is a marked Weiss cover of X. For example, $\mathcal{K} := \{\{u\} \mid u \in U \setminus U^{\mathrm{mrk}}\}$ could just be the set of all singletons of U which are not marked.

Observe that e.g. for the concrete example of $\mathcal{K}$ it is crucial to *not* choose $u \in U^{\mathrm{mrk}}$; otherwise we do not obtain a marked cover of U. The second standard approach to explicitly produce a marked Weiss cover is:

Example 3.19. Let $\mathfrak{B}$ be a basis for the topology of X. Fix a marked open U of X. For any finite subset $S \subseteq U$ we set $\widetilde{S} := S \cup U^{\mathrm{mrk}}$. Since X is Hausdorff and $\mathfrak{B}$ is a basis we can choose pairwise disjoint opens $B_s \in \mathfrak{B}$ (for $s \in \widetilde{S}$) with $s \in B_s \subseteq U$ for each finite $\widetilde{S}$. We set $B_{\widetilde{S}} := \bigcup_{s \in \widetilde{S}} B_s$. By construction, the set $\mathcal{W} := \{B_{\widetilde{S}} \mid S \subseteq U \text{ finite}\}$ is a marked Weiss cover of U.

The usual construction would be to use S directly, instead of adding the marked points and working with $\widetilde{S}$. However, this extra step is needed to ensure that all the opens $B_{\widetilde{S}}$ of the cover have marked inclusions into U.

Notation 3.20. Let $\mathfrak{U}$ be a poset of marked opens. We say that a marked precover $\mathcal{W} \to^{\mathrm{m}} \{U\}$ or a marked hyperprecover $\mathfrak{W} \to^{\mathrm{m}}_{\mathrm{h}} \{U\}$ is *in* $\mathfrak{U}$ if all the involved open sets are contained in $\mathfrak{U}$. That is, if $\mathcal{W} \cup \{U\} \subseteq \mathfrak{U}^{\diamond}$ or $\mathfrak{W} \cup \{U\} \subset \mathfrak{U}$, respectively.

We are now ready for the key definition of this subsection, namely

Definition 3.21. Let $\mathfrak{U}$ be a poset of marked opens and $\mathcal{A}$ a copresheaf defined on $\mathfrak{U}$. We say that $\mathcal{A}$ is a *marked Weiss cosheaf on* $\mathfrak{U}$ if every marked Weiss cover $\mathcal{W} \to^{\mathrm{m}} \{U\}$ in $\mathfrak{U}$ is $\mathcal{A}$-local, i.e. if we have an equivalence

$$\mathcal{A}^{mpc}{}_{|\mathfrak{U}}(\mathcal{W}) := \operatorname{colim} \mathcal{A}_{|\mathfrak{U} \cap \mathcal{W}_{\downarrow_{\mathrm{m}}}} \xrightarrow{\simeq} \mathcal{A}(U). \tag{3.4}$$

We denote the full ∞-category of marked Weiss cosheaves on $\mathfrak{U}$ by

$$\mathrm{coShv}^{\mathrm{W}}_{\mathrm{mrk}}(\mathfrak{U}, \mathcal{C}) \subset \mathrm{Fun}(\mathfrak{U}, \mathcal{C}).$$

If the space X has no marked points, the above exactly reduces to the usual notion of a Weiss cosheaf. It will be convenient to also work with hyper-analogues of marked Weiss covers and cosheaves.

Definition 3.22. A marked hyperprecover $\mathfrak{W} \to^{\mathrm{m}}_{\mathrm{h}} \{U\}$ is called a *marked Weiss hypercover* if the marked precover $\mathfrak{W}^{\diamond} \to^{\mathrm{m}} \{U\}$ is a marked Weiss cover and for each $W \in \mathfrak{W}_{\cap}$ we have a marked Weiss cover $(\mathfrak{W} \cap W_{\downarrow_{\mathrm{m}}})^{\diamond} \to^{\mathrm{m}} \{W\}$. [8]

Definition 3.23. Let $\mathfrak{U}$ be a poset of marked opens, and $\mathcal{A}$ be a copresheaf defined on $\mathfrak{U}$. We say that $\mathcal{A}$ is a *marked Weiss hypercosheaf on* $\mathfrak{U}$ if every marked Weiss hypercover $\mathfrak{W} \to^{\mathrm{m}}_{\mathrm{h}} \{U\}$ in $\mathfrak{U}$ is $\mathcal{A}$-local, i.e. if

$$\mathcal{A}^{mhpc}(\mathfrak{W}) := \operatorname*{colim}_{W \in \mathfrak{W}} \mathcal{A}(W) \xrightarrow{\simeq} \mathcal{A}(U).$$

We denote the full ∞-category of marked Weiss hypercosheaves on $\mathfrak{U}$ by

$$\mathrm{coHShv}^{\mathrm{W}}_{\mathrm{mrk}}(\mathfrak{U}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathfrak{U}, \mathcal{C}).$$

[8] The definition we give here is a simplified version of [**KSW24**, Definition 2.37] which is a coinductive definition. The definition here implies that the second condition of the coinductive definition holds, c.f. [**KSW24**, Remark 2.38], and is what one actually checks in practice.

Definition 3.24. We say that a marked Weiss hypercover $\mathfrak{W} \to^{\mathrm{m}}_{\mathrm{h}} \{U\}$ is a *marked Weiss presieve* if $\mathfrak{W} = \mathfrak{W}_\cap$, i.e. $\mathfrak{W}$ is closed under intersections.

The following observation is convenient for certain computations as its allows the flexibility of working with marked Weiss presieves instead of marked Weiss covers.

Remark 3.25. Let $\mathfrak{U}$ be a presieve, and let $\mathcal{A}$ be a copresheaf defined on $\mathfrak{U}$. As explained in Remark 3.13 we know that the copresheaf $\mathcal{A}$ evaluates the same on a marked Weiss cover $\mathcal{W} \to^{\mathrm{m}} \{U\}$ as on the marked Weiss hypercover $\mathcal{W}_\cap \to^{\mathrm{m}}_{\mathrm{h}} \{U\}$. Hence, we can relate the two notions. For $\mathcal{A}$ to be a marked Weiss hypercosheaf we require *all* marked Weiss hypercovers to be $\mathcal{A}$-local, while for $\mathcal{A}$ to be a marked Weiss cosheaf only requires $\mathcal{A}$-locality for marked Weiss *presieves*. It follows that we have a fully faithful inclusion

$$\mathrm{coHShv}^{\mathrm{W}}_{\mathrm{mrk}}(\mathfrak{U}, \mathcal{C}) \subseteq \mathrm{coShv}^{\mathrm{W}}_{\mathrm{mrk}}(\mathfrak{U}, \mathcal{C}).$$

If $\mathfrak{U}$ is *not* assumed to be a presieve the above is *not* true, c.f. [**KSW24**, Warning 2.44].

3.3. Multiplicative and factorizing bases. In this subsection we introduce different types of bases; namely multiplicative and factorizing ones. The notion of factorizing basis comes from [**CG17**, §7 Definition 2.1.1], and they are particularly advantageous because of the way they interact with (marked) Weiss covers. This is leveraged when extending pointless factorization algebras in Section 7.1.2.

We split the main definitions into three steps.

Definition 3.26. Let $\mathfrak{U}$ be a poset of marked opens. A *basis of* $\mathfrak{U}$ is a subposet $\mathfrak{B} \subseteq \mathfrak{U}$ such that $\mathfrak{B}$ is a basis for the topology of $\bigcup \mathfrak{U}$, i.e. for each $x \in V \in \mathfrak{U}_{\downarrow_{\mathrm{m}}}$ there is a $B \in \mathfrak{B}$ with $x \in B \subseteq V$.

Building upon this, we have the two notions of bases we often work with:

Definition 3.27. Let $\mathfrak{U}$ be a poset of marked opens.

- A *multiplicative basis of* $\mathfrak{U}$ is a subposet $\mathfrak{B} \subseteq \mathfrak{U}$ such that $\mathfrak{B}$ is a basis of $\mathfrak{U}$, and $\mathfrak{B}$ is closed under disjoint unions subordinate to $\mathfrak{U}$, i.e. for each $B, B' \in \mathfrak{B}$ with $B \sqcup B' \in \mathfrak{U}_{\downarrow_{\mathrm{m}}}$ we have that $B \sqcup B' \in \mathfrak{B}$.
- Moreover, we say that $\mathfrak{B}$ is a *factorizing basis* if it is also closed under intersections, i.e. for each $B, B' \in \mathfrak{B}$ we have that $B \cap B' \in \mathfrak{B}$.

The only modification compared to the unmarked setting is that in the definition of a multiplicative basis we take disjoint unions subordinate to $\mathfrak{U}$ in the sense of marked inclusions.

Example 3.28. Let $\mathfrak{U}$ be the poset of marked opens consisting of all finite disjoint unions of marked disks[9] in X. This is an example of a multiplicative basis. However, it is *not* an example of a factorizing basis since the intersection of two disks might not be a disk again.

We now recall an important construction as well as a definition related to adding respectively decomposing disjoint unions.

Construction 3.29. Let $\mathfrak{U}$ be a poset of marked opens. We denote by

$$\mathfrak{U}_\sqcup := \{U_1 \sqcup \ldots \sqcup U_n \mid n \in \mathbb{N}, U_1, \ldots, U_n \in \mathfrak{U} \text{ pairwise disjoint}\}$$

the completion of $\mathfrak{U}$ under disjoint unions. This also adds the empty set since it is the disjoint union of the empty list.

[9] i.e. disks embedded in X together with their inherited (possibly trivial) marking.

Definition 3.30. We say that a poset of marked opens $\mathfrak{U}$ is *decomposable* if for any open $U \cup U' \in \mathfrak{U}$ with $U \neq \emptyset \neq U'$ it follows that $U \in \mathfrak{U}$ and $U' \in \mathfrak{U}$.

The main selling point of multiplicative bases is that they provide marked Weiss covers and hypercovers. For example, let $\mathfrak{B}$ be a multiplicative basis of $\mathfrak{U}$, a poset of marked opens. Then we have

- a marked Weiss hypercover $\mathfrak{B} \cap U_{\downarrow_\mathrm{m}} \to^{\mathrm{m}}_{\mathrm{h}} \{U\}$ for each $U \in \mathfrak{U}_{\downarrow_\mathrm{m}}$ ([**KSW24**, Lemma 2.52]).
- for each $\mathcal{W} \to^{\mathrm{m}} \{U\}$ marked Weiss cover in $\mathfrak{U}_{\downarrow_\mathrm{m}}$ we get a marked Weiss hypercover $\mathfrak{B} \cap \mathcal{W}_{\downarrow_\mathrm{m}} \to^{\mathrm{m}}_{\mathrm{h}} \{U\}$ ([**KSW24**, Lemma 2.53]).

A main feature of factorizing bases is that they interact well with the marked Weiss condition when left Kan extending. The result below is the main ingredient for proving that pointless factorization algebras extend from a factorizing basis to a larger poset of marked opens, see Section 7.1.2 for a recollection or [**KSW24**, §4.3] for the full details.

Proposition 3.31 (Unique extension from a basis)**.** [**KSW24**, Proposition 2.54] *Let $\mathcal{A}\colon \mathfrak{U} \to \mathcal{C}$ be a copresheaf on the poset of marked opens $\mathfrak{U}$. Let $\mathfrak{B} \subseteq \mathfrak{U}$ be a factorizing basis of $\mathfrak{U}$.*

(1) Every marked Weiss cosheaf on $\mathfrak{U}$ is a marked Weiss cosheaf on $\mathfrak{B}$ and a left Kan extension along $\mathfrak{B} \hookrightarrow \mathfrak{U}$.

(2) Assume that $\mathfrak{U}$ is a presieve. Left Kan extension along $\mathfrak{B} \hookrightarrow \mathfrak{U}$ from a marked Weiss cosheaf on $\mathfrak{B}$ yields a marked Weiss cosheaf on $\mathfrak{U}$.

4. Pointless factorization algebras and constructibility

We first define pointless factorization algebras in Section 4.1. In short, a pointless factorization algebra on open $(X)^{\otimes}_{\mathrm{mrk}}$ is a pointless prefactorization algebra satisfying two additional conditions analogous to those of ordinary factorization algebras. Then we review the theory of conical (smooth) stratified spaces in Section 4.2, as well as defining their marked counterparts. With a notion of stratification at hand we, in Section 4.3, define what constructible pointless factorization algebras are. In Section 4.4 we give some results regarding pushing forward and restricting (constructible) pointless factorization algebras, before explaining how to restrict ordinary factorization algebras to pointless ones in Section 4.5.

4.1. Pointless factorization algebras. In short, pointless factorization algebras are analogous to the notion of factorization algebras, but has as domain open$(X)^{\otimes}_{\mathrm{mrk}}$ instead of open(X)$^{\otimes}$. The conditions one normally impose for ordinary factorization algebras are modified accordingly.

Recall from Construction 2.13 and Construction 2.14 how one constructs an ∞-operad $\mathfrak{U}^{\otimes}$ from a poset of marked opens $\mathfrak{U}$. There is a special type of morphism in $\mathfrak{U}^{\otimes}$ which is of importance; namely the cocartesian ones.

Observation 4.1. The cocartesian morphisms in $\mathfrak{U}^{\otimes}$ are of the form

$$(I_+, (U_i)) \xrightarrow{f} \left(J_+, \left(\dot{\bigcup_{i \in f^{-1}(j)}} U_i \right) \right) .$$

Given $I_+, (U_i)$), J_+ and the map $I_+ \to J_+$ in $\textsc{Fin}_*$ we have a cocartesian map as above exactly when, for each $j \in J$, the opens $(U_i)_{i \in f^{-1}(j)}$ are all pairwise disjoint.

Note that cocartesian maps are automatically marked inclusions. Now recall that we assume our fixed target category $(\mathcal{C}, \otimes)$ to be a $\otimes$-presentable symmetric monoidal category.

Remark 4.2. We view the symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$ as encoded in a cocartesian fibration $\mathcal{C}^{\otimes} \to \textsc{Fin}_*$ as in Definition A.17. The tensor product of $\mathcal{C}^{\otimes}$ is defined uniquely (up to contractible choice) by cocartesian transfer. Thus, by definition, the cocartesian morphisms in $\mathcal{C}^{\otimes}$ are of the form

$$(I_+, (V_i)) \xrightarrow{f} \left(J_+, \left(\bigotimes_{i \in f^{-1}(j)} V_i \right) \right) .$$

Analogous to how ordinary factorization algebras are defined we need to impose two conditions on a pointless prefactorization algebra (Definition 2.17) to get a pointless factorization algebra. The first is that of multiplicativity, which says that disjoint unions of (marked) opens should be sent to tensor products.

Definition 4.3. A pointless $\mathfrak{U}$-prefactorization algebra is called *multiplicative* if it sends cocartesian morphisms in $\mathfrak{U}^{\otimes}$ to cocartesian morphisms in $\mathcal{C}^{\otimes}$. We denote the full ∞-subcategory of multiplicative $\mathfrak{U}$-prefactorization algebras by

$$\mathrm{Alg}^{\mathrm{m}}_{\mathfrak{U}} \hookrightarrow \mathrm{Alg}_{\mathfrak{U}}.$$

Remark 4.4. Note that it is enough to know multiplicativity for cocartesian arrows of the form $(2_+, (U_1, U_2)) \to (1_+, (U_1 \,\dot{\cup}\, U_2))$ and $(+, (\emptyset)) \to (+, (\emptyset))$. The general case follows from these by a simple induction-argument.

Example 4.5. We can extend the pointless $\mathfrak{U}$-prefactorization algebras from Example 2.20 and Example 2.23 to multiplicative algebras on $\mathfrak{U}^{\otimes}_{\sqcup}$, i.e. the disjoint union completion of $\mathfrak{U}$. That is, set $\mathcal{F} = \mathcal{A}$ and $\mathcal{F} = \mathcal{M}$, respectively. Using Remark 4.4, the assignment

$$\mathcal{F}(U_1 \sqcup U_2) \coloneqq \mathcal{F}(U_1) \otimes \mathcal{F}(U_2),$$

gives pointless multiplicative prefactorization algebras on $\mathfrak{B}^{\otimes}_{\sqcup}$. Of course, this assignment immediately gives $\mathcal{F}(U_1 \sqcup \ldots \sqcup U_n) = \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n)$ for a general multiplicative morphism.

Secondly, factorization algebras also satisfy a *local-to-global* condition. This is formulated using *marked Weiss covers* $\mathcal{W} \to^{\mathrm{m}} \{U\}$ (Definition 3.16), or rather the corresponding cosheaves which go by the name of *marked Weiss cosheaves* (Definition 3.21).

Definition 4.6. Let $\mathfrak{U}$ be a poset of marked opens. A *marked Weiss algebra* on $\mathfrak{U}$ is a pointless $\mathfrak{U}$-prefactorization algebra whose underlying copresheaf $\mathfrak{U} \to \mathcal{C}$ is a marked Weiss cosheaf. We define the full ∞-subcategory of marked Weiss algebras on $\mathfrak{U}$ to be the pullback

$$\begin{tikzcd}
\mathrm{Alg}^{\mathrm{Weiss}}_{\mathfrak{U}}(\mathcal{C}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Alg}_{\mathfrak{U}}(\mathcal{C}) \arrow[d] \\
\mathrm{coShv}^{\mathrm{W}}_{\mathrm{mrk}}(\mathfrak{U}, \mathcal{C}) \arrow[r, hook] & \mathrm{Fun}(\mathfrak{U}, \mathcal{C})
\end{tikzcd} .$$

Now we can finally give our main definition.

Definition 4.7. Let (X, X^{mrk}) be a marked space, and let $\mathfrak{U}$ be a poset of marked opens, i.e. $\mathfrak{U} \subseteq \mathrm{open}(X)_{\mathrm{mrk}}$. The ∞*-category of pointless* $\mathfrak{U}$*-factorization algebras* (with values in $\mathcal{C}$) is the pullback

$$\begin{tikzcd}
\textsc{Fact}^{\mathrm{pless}}_{\mathfrak{U}}(\mathcal{C}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Alg}^{\mathrm{m}}_{\mathfrak{U}}(\mathcal{C}) \arrow[d] \\
\mathrm{Alg}^{\mathrm{Weiss}}_{\mathfrak{U}}(\mathcal{C}) \arrow[r, hook] & \mathrm{Alg}_{\mathfrak{U}}(\mathcal{C})
\end{tikzcd} .$$

Explicitly, a pointless $\mathfrak{U}$-factorization algebra is a morphism of ∞-operads $\mathcal{A}\colon \mathfrak{U}^{\otimes} \to \mathcal{C}^{\otimes}$ that sends cocartesian morphisms in $\mathfrak{U}^{\otimes}$ to cocartesian morphisms in $\mathcal{C}^{\otimes}$ and restricts to a marked Weiss cosheaf.

Notation 4.8. Whenever $\mathfrak{U} = \mathrm{open}(X)_{\mathrm{mrk}}$ we abbreviate the above notation and simply talk about pointless factorization algebras (on X) and similarly denote the corresponding category by $\textsc{Fact}^{\mathrm{pless}}_X$.

Remark 4.9. For a detailed comparison of the slightly different formulations of the ordinary Weiss cosheaf condition in the literature on factorization algebras we refer the reader to [**KSW24**, Section 2.5 & Remark 3.16]. Note that when X is trivially marked our Weiss condition, and hence also the notion of pointless factorization algebras, coincides with the standard literature on factorization algebras.

Example 4.10. Recall the $\mathfrak{U}$-prefactorization algebra $\mathcal{A}$ from Example 2.20. This satisfies both multiplicativity and the Weiss condition, and is hence an example of an ordinary $\mathfrak{U}$-factorization algebra. See e.g. [**KSW24**, Example 3.14] for details.

Example 4.11. Let $\mathcal{M}$ be a pointless $\mathfrak{U}$-prefactorization algebra as in Example 2.23. Recall that $X = \mathbb{R}$ with the origin as a marked point, i.e. $X^{\mathrm{mrk}} = \{0\}$. Then we have that $\mathcal{M}$ is a marked Weiss algebra by a straightforward modification of the argument in [**KSW24**, Example 3.15]. [10]

4.2. Conical manifolds and their marked version. In this section we recall the notions of conical manifolds and smooth conical manifolds from [**AFT17b**], where they are named "C_0 stratified spaces" respectively "conically smooth stratified spaces". These play the roles of topological and smooth manifolds when working with stratified spaces. We also define their marked counterparts, which we call conical marked manifolds respectively smooth conical marked manifolds.

Definition 4.12. [**Lur17**, Definition A.5.1] Let P be a partially ordered set. We regard P as a topological space by defining opens to be the subsets that are closed upwards, i.e. $U \subset P$ is open if $u \leq p$ and $u \in U$ implies that $p \in U$.

Let X be a topological space. We define a *P-stratification of X* to be a continuous map $S\colon X \to P$. In this situation we say that X is a *stratified space*.

Asking for a map of partially ordered sets to be continuous exactly means that it has to be a map of partially ordered sets.

Example 4.13. Any (non-empty) topological space X can be seen as a stratified space by giving it the *trivial stratification*, i.e. $X = (X \to \{\star\})$. The empty set is trivially stratified by itself, i.e. $\emptyset = (\emptyset \to \emptyset)$.

Example 4.14. Figure 3 depicts two different stratifications of the form $(0,1)^2 \to P \coloneqq \{0 < 1 < 2\}$. In both cases all points are sent to 0, the line segments are sent to 1 and everything else is sent to 2.

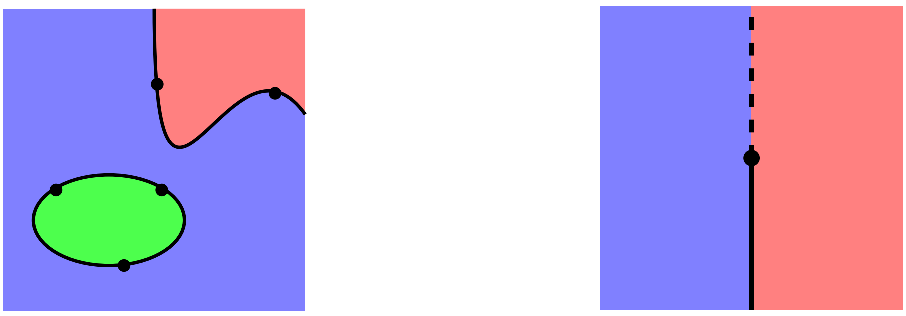


FIGURE 3. Two examples of $\mathbb{R}^2 \cong (0,1)^2$ with non-trivial stratification.

Example 4.15. Let $(X \xrightarrow{S} P)$ and $(Y \xrightarrow{T} Q)$ be two stratified spaces. Then their product $(X \times Y \xrightarrow{S\times T} P \times Q)$ is another stratified space. The product poset $P \times Q$ has the partial order $(p,q) \leq (p',q') \iff (p \leq p') \wedge (q \leq q')$.

Having seen some examples of stratified spaces we now turn to the maps between them.

[10]For example, for opens of the form $(-\infty, a)$ such that $0 < a < 1$ the argument can be modified by setting $S_i = \{-i, a - \frac{1}{ki}\}$ for some suitable factor $k > 0$.

Definition 4.16. A *map of stratified spaces* $(X \to P) \to (Y \to Q)$ is a continuous map $f\colon X \to Y$ such that the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ P \arrow[r] & Q \end{tikzcd}$$

commutes.

A type of stratified map which is of particular importance to us is the following.

Definition 4.17. A stratified map $f\colon X \to Y$ is a *stratified open embedding* if

- $f\colon X \to Y$ is an embedding of topological spaces, and
- the restriction $f_|\colon X_p \to Y_{f(p)}$ is an open embedding of topological manifolds for each $p \in P$, where P is the stratifying poset of X.

With these stratified maps at hand we can also define the marked analogue of a stratified space. Recall from Definition 2.1 the definition of a marked space.

Definition 4.18. Let P be a partially ordered set and let (X, X^{mrk}) be a marked space. We define a *marked P-stratification of* X to be a stratified open embedding from $X^{\mathrm{mrk}} \to \{\star\}$ to $X \to P$. We often omit the poset and the marked points and simply say that X is a *marked stratified space.*

Remark 4.19. Observe that for a non-trivial collection of marked points, i.e. $X^{\mathrm{mrk}} \neq \emptyset$, asking for a stratified open embedding as in Definition 4.18 corresponds to sending X^{mrk} to the (necessarily non-trivial) 0-dimensional strata of X. If X has no 0-dimensional strata we can only have $X^{\mathrm{mrk}} = \emptyset$.

Example 4.20. Any choice of a (possibly empty) subset of the 0-dimensional strata of the stratified spaces in Figure 3 gives an example of a marked stratified space.

Example 4.21. Let (X^{mrk}, X) and (Y^{mrk}, Y) be two marked stratified spaces. Then the product stratified space $X \times Y$ from Example 4.15 is a marked stratified space with marked points $(X \times Y)^{\mathrm{mrk}} := X^{\mathrm{mrk}} \times Y^{\mathrm{mrk}}$. Note that if $X^{\mathrm{mrk}} = \emptyset$ or $Y^{\mathrm{mrk}} = \emptyset$ it follows that $X \times Y$ is trivially marked, i.e. $(X \times Y)^{\mathrm{mrk}} = \emptyset$.

One particularly important choice of marking which will be used a lot throughout this thesis is the following:

Definition 4.22. Let X be a marked stratified space with *finite* 0-dimensional strata. We say that X is *maximally marked* if all points that can be marked are marked. This corresponds to letting X^{mrk} be the maximal choice of marked points, i.e. X^{mrk} is all of the 0-dimensional strata of X.

Example 4.23. Consider for example the (conically smooth) stratified manifolds of Figure 3. If the leftmost space is maximally marked it has a total of 5 marked points, while the rightmost space is maximally marked exactly when the one 0-dimensional strata is marked.

We now recall a very important way of producing stratified spaces.

Construction 4.24. Let $S\colon X \to P$ be a stratified space. We define a new stratified space, called the *cone of X of radius t*, by

$$\mathbf{C}_t X := \{0\} \amalg_{\{0\} \times X} [0, t) \times X \ .$$

In the special case of infinite radius, we abbreviate $\mathbf{C}X \coloneqq \mathbf{C}_\infty X$. The stratification on the cone is given by

$$\mathbf{C}_t X \to \{-\infty\} \star P, \qquad (t,x) \mapsto \begin{cases} -\infty, & \text{for } t = 0 \\ S(x), & \text{for } t > 0 \end{cases},$$

where $\{-\infty\} \times P$ comes from adding a new minimal element to P.

Example 4.25. The stratified space from Example 4.14 corresponding to the rightmost picture in Figure 3 is a cone. More explicitly, it is the cone of $\mathbf{S}^1 \to \{0 < 1\}$, where the circle is stratified by two points (directly opposite to each other) which are both sent to 0 while the rest of the circle is sent to 1.

Following [**AFT17b**, Definition 2.1.15 and Lemma 2.2.2], we define conical manifolds which is the stratified analogue of a topological manifold. Note that in [**AFT17b**] the below corresponds to what they call "C^0 stratified spaces" and "C^0 basics", respectively.

Definition 4.26. Conical manifolds and conical disks are defined by mutual induction:

(1) A *conical manifold* is a stratified space X which is second countable Hausdorff, and the collection of embedded conical disks forms a basis for the topology of X. If X is empty we declare its dimension to be -1. Otherwise we say that X is of dimension N or $\leq N$ if all these embedded conical disks are of dimension N or $\leq N$, respectively.

(2) A *conical disk* or *basic* is a stratified space isomorphic to

$$\mathbf{D}(X,i) \coloneqq \mathbf{C}X \times \mathbb{R}^i,$$

where X is a compact conical manifold, $i \in \mathbb{N}$, and $\mathbb{R}^i$ is equipped with the trivial stratification. If X is of dimension N or $\leq N$, we declare $\mathbf{D}(X,i)$ to be of dimension $N+i+1$ or $\leq N+i+1$, respectively.

Conical manifolds behave a bit differently than ordinary manifolds when it comes to dimensions, which might not be well-defined globally. For example, the cone on $\mathbf{S}^1 \sqcup \mathbf{S}^0$ is a conical disk of dimension ≤ 2, because it has points of local dimension both 1 and 2.

Example 4.27. A conical manifold with trivial stratification corresponds exactly to an ordinary topological manifold because the only local disk that can appear is $\mathbf{D}(\emptyset, n) = \mathbb{R}^n$.

The marked version of a conical manifold is, similar to before, defined using stratified open embeddings.

Definition 4.28. Let X be a conical manifold. We say that X is a *conical marked manifold* if it comes equipped with a stratified open embedding $f\colon X^{\mathrm{mrk}} \to X$, where X^{mrk} is a finite subset of points in X together with the trivial stratification. If the conical manifold X happens to be a conical disk we say that X is a *conical marked disk* or a *marked basic*.

Remark 4.29. If X happens to be a basic, i.e. $X \cong \mathbb{R}^i \times \mathbf{C}Z$ for some compact stratified space Z, the data of marked points is very limited. If $i > 0$ we can only have $X^{\mathrm{mrk}} = \emptyset$ since there is no 0-dimensional strata. If $i = 0$ we can either choose $X^{\mathrm{mrk}} = \{*\}$ or $X^{\mathrm{mrk}} = \emptyset$, i.e. either the empty set or the singleton set corresponding to choosing the origin to be a marked point or not.

Remark 4.30 (Inherited marked structure)**.** Given a conical marked manifold $X^{\mathrm{mrk}} \to X$, any conical disk embedded into X inherits a marked structure. Explicitly, given $\iota\colon \mathbf{D} \to X$, where $\mathbf{D}$ is some conical disk, define $\mathbf{D}^{\mathrm{mrk}} := X^{\mathrm{mrk}} \cap \iota(\mathbf{D})$. This intersection will, by definition, either be empty or the singleton (corresponding to the origin of $\mathbf{D}$) and comes with an open embedding into $\mathbf{D}$ guaranteed by the open stratified embedding ι. Put differently, we are simply pulling back the marked structure along ι.

Since the embedded conical disks forms a basis for the topology of X, it follows immediately that the embedded conical disks with inherited marking also forms a basis for the topology of X.

Remark 4.31. It follows from the previous remark and the argument in Example 3.19 that every conical marked manifold admits a marked Weiss cover by (finite disjoint unions) of conical disks with inherited marking. The special case where the conical manifold has trivial marking gives existence of an ordinary Weiss cover.

We also need smooth analogues of Definition 4.26, which is significantly more subtle to define since it requires conditions on the transition maps appearing between the charts by conical disks. Here we only sketch the definition, and refer to [**AFT17b**, §3.2] for the details. Note that what we call smooth conical manifolds here corresponds to their "conically smooth stratified spaces".

Definition 4.32 (Sketch of [**AFT17b**, §3.2])**.** The following notions are defined by mutual recursion:

- A *smooth conical manifold* X is a paracompact and Hausdorff stratified space equipped with a *conically smooth atlas*. That is, an inclusion-maximal collection $\{\psi_i \colon \mathbf{D}_i \hookrightarrow X\}_{i \in I}$ of open embeddings of smooth conical disks $\mathbf{D}_i$, such that for each $i, j \in I$ and each $x \in \psi_i(\mathbf{D}_i) \cap \psi_j(\mathbf{D}_j)$ there is a $k \in I$ with $x \in \mathbf{D}_k$ and a commutative square
$$\begin{tikzcd} \mathbf{D}_k \arrow[r, hook] \arrow[d, hook] \arrow[dr, hook, "\psi_k"] & \mathbf{D}_i \arrow[d, hook, "\psi_i"] \\ \mathbf{D}_j \arrow[r, hook, "\psi_j"'] & X \end{tikzcd}$$
where the left vertical and upper horizontal maps are conically smooth embeddings.
- A *conically smooth map* $f \colon X' \to X$ between smooth conical manifolds with smooth conical atlas $\{\psi'_i\}$ and $\{\psi_j\}$, respectively, is a map of stratified spaces which induces conically smooth maps $\psi_j^{-1} \circ f \circ \psi'_i$ on all charts.
- A *smooth conical disk* is a stratified space of the form
$$\mathbf{D}(X, n) := \mathbf{C}X \times \mathbb{R}^n \ ,$$
where X is a compact smooth conical manifold.
- *Conical smoothness* for a map $\mathbf{D}(X, n) \to \mathbf{D}(Y, m)$ *of smooth conical disks* is defined in terms of *conical smoothness* of maps $X \to Y$, plus the ordinary notion of smoothness between Euclidean spaces.

Example 4.33. Every smooth manifold is an example of a conically smooth manifold with trivial stratification. In this situation the conically smooth maps are exactly ordinary smooth maps.

Example 4.34. In [**NV23**], Nocera–Volpe proves that any Whitney stratified space admits a canonical smooth conical structure. Thus, all Whitney stratified spaces are examples of smooth conical manifolds.

Example 4.35. The stratified spaces given in Example 4.14 admits smooth conical structures and are hence examples of conically smooth manifolds. For more interesting examples we refer the reader to [**AFT17b**, §3.5].

We can assemble smooth conical manifolds and embeddings into an ordinary category.

Definition 4.36. [**AFT17b**, Definition 3.2.21 & Proposition 3.3.7] We define the (ordinary) category Mfld of *smooth conical manifolds* and embeddings as follows: An object is a pair $(X, \mathcal{A})$ consisting of a conical manifold together with a (maximal) conically smooth atlas $\mathcal{A}$. A morphism $(X, \mathcal{A}) \to (X', \mathcal{A}')$ is a *conically smooth embedding*, i.e. a stratified open embedding $f\colon X \hookrightarrow X'$ such that $\mathcal{A}_X = f^\star \mathcal{A}_{X'}$.

Notation 4.37. We write

$$\mathrm{Bsc} \subset \mathrm{Mfld} \qquad \text{and} \qquad \mathrm{Disks} \subset \mathrm{Mfld}$$

for the full subcategories whose objects are smooth conical disks ("basics"), respectively finite disjoint unions of smooth conical disks.

We now introduce a marked version of a conically smooth embedding before explaining how smooth conical marked manifold also assemble into an ordinary category.

Definition 4.38. Let (X, X^{mrk}) and (Y, Y^{mrk}) be two smooth conical marked manifolds. We say that a conically smooth embedding $f\colon X \to Y$ is *bijective on marked points* if it induces a bijection $f_{|X^{\mathrm{mrk}}}\colon X^{\mathrm{mrk}} \xrightarrow{\cong} Y^{\mathrm{mrk}}$. Explicitly, f has to induce a commutative square of the form

$$\begin{tikzcd} X^{\mathrm{mrk}} \arrow[r, "\cong"] \arrow[d, hook] & Y^{\mathrm{mrk}} \arrow[d, hook] \\ X \arrow[r, hook, "f"] & Y \end{tikzcd} .$$

Since conically smooth embeddings compose, so do the conically smooth embeddings which are bijective on marked points.

Definition 4.39. Let $\mathrm{Mfld}^{\mathrm{mrk,bij}}$ be the ordinary category of smooth conical marked manifolds and conically smooth embeddings which are bijective on marked points. That is, objects are smooth conical marked manifold, i.e. a smooth conical manifold $(X, \mathcal{A})$ together with a stratified open embedding $X^{\mathrm{mrk}} \to X$ from a finite collection of marked points (of X). Morphisms are conically smooth embeddings which are bijective on marked points.

Remark 4.40. To be more precise, X^{mrk} also has the structure of a smooth conical manifold in the above definitions. We always choose to equip it with the atlas coming from the atlas on X, i.e. for $\iota\colon X^{\mathrm{mrk}} \hookrightarrow X$ and $X = (X, \mathcal{A}_X)$ we set $\mathcal{A}_{X^{\mathrm{mrk}}} := \iota^* \mathcal{A}_X$ making ι a conically smooth embedding by construction.

Notation 4.41. We write

$$\mathrm{Bsc}^{\mathrm{mrk,bij}} \subset \mathrm{Mfld}^{\mathrm{mrk,bij}} \qquad \text{and} \qquad \mathrm{Disks}^{\mathrm{mrk,bij}} \subset \mathrm{Mfld}^{\mathrm{mrk,bij}}$$

for the full subcategories whose objects are smooth conical marked disks ("marked basics") respectively finite disjoint unions of smooth conical marked disks.

All of the ordinary categories introduced in this section can also be assembled into ∞-categories. This is explained in detail in Section 6.

4.3. Constructible pointless factorization algebras. In this subsection we introduce the notion of *constructible* pointless factorization algebras. Informally, these are pointless factorization algebras which are locally constant with respect to the stratification of the space. Here, the stratifications we use are those of Section 4.2, namely we work with marked conical manifolds or conically smooth marked manifolds depending on the situation.

We start by setting up some notation to ensure that we have enough disks in our manifold to avoid checking a vacuous condition.

Notation 4.42. We write $\mathrm{disk}(X)_{\mathrm{mrk}} \subset \mathrm{open}\,(X)_{\mathrm{mrk}}$ for the full subposet of finite disjoint unions of embedded disks with inherited marking, see e.g. Remark 4.30. Here, "embedded disks" needs to be interpreted in terms of what type of object we view X as. Explicitly, $\mathrm{disk}(X)_{\mathrm{mrk}}$ refers to:

- the full subposet of finite disjoint unions of embedded conical disks (with inherited marking) if X is (viewed as) a conical marked manifold.
- the full subposet of finite disjoint unions of conically smoothly embedded conically smooth disks (with inherited marking) if X is (viewed as) a smooth conical manifold.

In the case where X has trivial stratification (and hence also no marked points) the above reduces to the standard notions of continuously or smoothly embedded disks in a topological or smooth manifold.

Definition 4.43. We say that $\mathfrak{B}$ is a *disk-basis of* $\mathfrak{U}$ if $\mathfrak{B} \subseteq \mathrm{disk}(X)_{\mathrm{mrk}}$ and it is a basis for the topology as in Definition 3.26.

Following [**KSW24**, Definition 3.34] we introduce the following notion.

Definition 4.44. A poset of marked opens $\mathfrak{U} \subseteq \mathrm{open}(X)_{\mathrm{mrk}}$ is said to have *enough good marked disks* if there exists a factorizing disk-basis $\mathfrak{B}$ of $\mathfrak{U}$. We say that $\mathfrak{B}$ is a *factorizing disk-basis of* $\mathfrak{U}$.

Similarly, we say that X *has enough good marked disks* if the poset of marked opens $\mathrm{open}(X)_{\mathrm{mrk}}$ has enough good marked disks witnessed by some factorizing disk-basis $\mathfrak{B} \subset \mathrm{disk}(X)_{\mathrm{mrk}}$.

Remark 4.45. Let X be a conical marked manifold with enough good marked disks witnessed by a factorizing disk-basis $\mathfrak{B}$. For any marked open $U \in \mathrm{open}\,(X)_{\mathrm{mrk}}$ it follows that $\mathfrak{B} \cap \mathrm{open}\,(U)_{\mathrm{mrk}}$ is a factorizing disk-basis of $\mathrm{open}\,(U)_{\mathrm{mrk}}$, which hence also has enough good marked disks.

Remark 4.46 (Existence of enough good marked disks)**.** Note that $\mathfrak{B} = \mathrm{disk}(X)_{\mathrm{mrk}}$ is *not* an example of a factorizing disk-basis because it is not necessarily closed under intersections. Hence, it is not clear that X has enough good marked disks for X some conically (smooth) marked manifold. In general, we will simply "define away" this problem by restricting to those conically (smooth) marked manifolds which has enough good marked disks when needed.

We are now ready for the main definition of this subsection, namely that of constructible pointless factorization algebras.

Definition 4.47. Let X be a conical marked manifold, and let $\mathfrak{U} \subseteq \mathrm{open}(X)_{\mathrm{mrk}}$ be a poset of marked opens with enough good marked disks. Let $\mathcal{A}\colon \mathfrak{U}^{\otimes} \to \mathcal{C}^{\otimes}$ be a pointless $\mathfrak{U}$-factorization algebra. We say that $\mathcal{A}$ is *constructible* or *locally constant with respect to the stratification* if every marked inclusion $U \hookrightarrow^{\mathrm{m}} V$ in $\mathfrak{U}$ of (abstractly) isomorphic marked conical disks is $\mathcal{A}$-local. Let

$$\mathrm{F\small{ACT}}_{\mathfrak{U}}^{\mathrm{pl,cstr}} \subset \mathrm{F\small{ACT}}_{\mathfrak{U}}^{\mathrm{pless}}$$

denote the full ∞-subcategory of constructible pointless factorization algebras on $\mathfrak{U}$.

Note that if we do not assume $\mathfrak{U}$ to have enough good marked disks one might get an entirely vacuous condition; e.g. if $\mathfrak{U}$ is some poset of marked opens containing no marked disks. If the space is trivially stratified (and hence also trivially marked) the notion of constructibility has a special name.

Definition 4.48. Let X be a manifold and $\mathfrak{U} \subseteq \mathrm{open}(X)$ a poset of opens with enough good disks. A $\mathfrak{U}$-factorization algebra is called *locally constant* if it is constructible with respect to the trivial stratification.

Example 4.49. Firstly, the trivial factorization algebra from Example 2.22 is (trivially) seen to be constructible for any stratification. Recall the $\mathfrak{U}$-factorization algebra $\mathcal{A}$ from Example 2.20 and Example 4.10. We see that $\mathcal{A}$ is locally constant because any inclusion of intervals is sent to the identity of the algebra $\mathbf{A}$.

Example 4.50. Recall the pointless $\mathfrak{U}$-factorization algebra $\mathcal{M}$ from Example 2.23 and Example 4.11. Since we have established that marked points can only be part of some 0-dimensional strata it follows that we have to consider $\mathbb{R}$ with some non-trivial stratification. Let $\mathbb{R} \cong \mathbf{CS}^0$ as smooth conical marked manifolds, i.e. $\mathbb{R}$ is given the stratification $\{0\} \subset \mathbb{R}$. With this we see that $\mathcal{M}$ is a constructible pointless $\mathfrak{U}$-factorization algebra; each marked inclusion between conical disks both containing 0 is sent to the identity on $\mathbf{M}$, and inclusions between intervals on either side of 0 is sent to the identity on $\mathbf{A}$, respectively $\mathbf{B}$.

4.4. Restriction and pushforward. Here we establish that, just like ordinary factorization algebras, pointless factorization algebras can be transferred from one marked space to another by pushforward and restriction.

Observation 4.51 (Restriction and pushforward)**.** Let (X, X^{mrk}) and (Y, Y^{mrk}) be two marked spaces as in Definition 2.1.

- Given an open (marked) subspace $U \subseteq X$, the inclusion of operads $\mathrm{open}\,(U)^{\otimes}_{\mathrm{mrk}} \hookrightarrow \mathrm{open}\,(X)^{\otimes}_{\mathrm{mrk}}$ induces the *restriction functor*
$$-_{|U} \colon \mathrm{Alg}_X \to \mathrm{Alg}_U$$
of algebras by precomposition.
- For each continuous map $f \colon X \to Y$ that sends marked points to marked points, i.e. $f(X^{\mathrm{mrk}}) \subseteq Y^{\mathrm{mrk}}$, we have an induced functor
$$f^{-1} \colon \mathrm{open}\,(Y)^{\otimes}_{\mathrm{mrk}} \to \mathrm{open}(X)^{\otimes}_{\mathrm{mrk}}$$
of ∞-operads which induces the *pushforward* functor
$$f_{\star} \colon \mathrm{Alg}_X \to \mathrm{Alg}_Y$$
of algebras by precomposition.

Remark 4.52. Since the subspace U inherits its marked points from those of X the precomposition is indeed well-defined. In the second item we need the continuous map f to send marked points to marked points, otherwise $f_{\star}$ will not be a well-defined functor. This is because we would have inclusions in $\mathrm{open}\,(Y)_{\mathrm{mrk}}$ where the corresponding preimages are *not* marked inclusions. A simple illustration of this is given in Figure 4 where X is $\mathbb{R}$ with one marked point, and Y is $\mathbb{R}$ with no marked points. Let $\mathcal{A}$ be a pointless factorization algebra on X. The preimage of the inclusion $U \hookrightarrow V$ in Y is not a marked inclusion (because $f^{-1}(U)$ does not contain the marked point but $f^{-1}(V)$ does), making the pushforward $f_\star\mathcal{A}$ ill-defined.

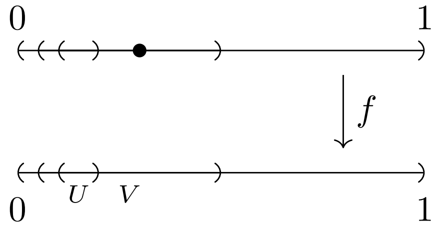


FIGURE 4. An example of how pushing forward along a map $f\colon X \to Y$ which does not send marked points to marked points is ill-defined.

Lemma 4.53. *Restriction along $U \subseteq X$ preserves (separately) multiplicative algebras, Weiss cosheaves and constructibility. In particular, it induces functors*

$$-_{|U}\colon \mathrm{FACT}_X^{\mathrm{pless}} \longrightarrow \mathrm{FACT}_U^{\mathrm{pless}} \quad \text{and} \quad -_{|U}\colon \mathrm{FACT}_X^{\mathrm{pl,cstr}} \longrightarrow \mathrm{FACT}_U^{\mathrm{pl,cstr}}$$

on (constructible) pointless factorization algebras.

PROOF. Note that all instances of multiplicativity or constructibility in U is also an instance of multiplicativity or constructibility in X. This is also true for the Weiss condition: By Remark 3.13 we get that marked Weiss covers in U are evaluated the same as marked Weiss covers in X since $\mathrm{open}(U)_{\mathrm{mrk}}$ is a presieve. □

Lemma 4.54. *Pushforward along a continuous map $f\colon X \to Y$ that sends marked points to marked points preserves (separately) multiplicative algebras and Weiss cosheaves. In particular, it induces a pushforward functor*

$$f_\star\colon \mathrm{FACT}_X^{\mathrm{pless}} \longrightarrow \mathrm{FACT}_Y^{\mathrm{pless}}$$

on pointless factorization algebras.

PROOF. Note that for any finite disjoint union $V = V_1 \sqcup \ldots \sqcup V_n$ in Y we get a corresponding finite disjoint union $f^{-1}(V) = f^{-1}(V_1) \sqcup \ldots \sqcup f^{-1}(V_n)$ in X. Thus, the multiplicativity condition translates directly and $f_\star$ preserves multiplicative algebras.

For the marked Weiss condition we first observe that f^{-1} preserves intersections, so $f_\star$ is compatible with the evaluation of marked (pre-)covers. We next need that the preimage of a marked Weiss cover $\mathcal{W} \to^{\mathrm{m}} V$ in Y is a marked Weiss cover in X. Since f sends marked points to marked points we get that $f^{-1}(\mathcal{W}) \to^{\mathrm{m}} f^{-1}(V)$ is a marked cover. For the Weiss condition, let $S \subseteq f^{-1}(V)$ be any finite set. There exists an open $W \in \mathcal{W}$ covering the finite set $f(S)$, thus $f^{-1}(W) \in f^{-1}(\mathcal{W})$ covers S. □

Observe that the pushforward along a continuous map does *not* preserve constructibility in general. For this one needs to also ensure that the map "interacts" well with the stratification.

We refer the reader to Section 8.2 for some examples of maps whose pushforward also preserves constructibility.

Lastly, we recall a straightforward lemma allowing to combine pushforwards on disjoint unions:

Lemma 4.55. [**KSW24**, Lemma 3.48] *Let $f_i \colon X_i \to Y_i$ be a map of marked spaces for each $i \in I$. If the pushforward along each f_i preserves constructible pointless factorization algebras, then so does the pushforward along their disjoint union $\coprod_{i \in I} f_i$*

PROOF. Follows immediately from the observation that any marked inclusion of abstractly isomorphic conical disks $U \hookrightarrow^{\mathrm{m}} V$ in the disjoint union $\coprod_{i \in I} Y_i$ already lies in one of the Y_i's. $\square$

4.5. Restricting ordinary to pointless factorization algebras. Here we explain how to restrict from ordinary (constructible) factorization algebras to pointless ones. We also give two results showing that restricting from (constructible) ordinary factorization algebras to pointless ones commutes with restrictions and push forwards along sufficiently nice maps.

Let X be a conical marked manifold. Recall the inclusion

$$\iota_X \colon \mathrm{open}(X)_{\mathrm{mrk}} \to \mathrm{open}(X) \tag{4.1}$$

of posets from Construction 2.21. This induces a restriction functor (2.1) at the level of algebras.

Lemma 4.56. *Let $\mathcal{C}$ be a symmetric monoidal $\otimes$-presentable ∞-category, and let X be a conical marked manifold. Let ι_X be the inclusion* (4.1)*. Restriction along ι_X induces functors*

$$\iota_X^* \colon \mathrm{F{\small ACT}}_X(\mathcal{C}) \longrightarrow \mathrm{F{\small ACT}}_X^{\mathrm{pless}}(\mathcal{C}) \qquad \text{and} \qquad \iota_X^* \colon \mathrm{F{\small ACT}}_X^{\mathrm{cstr}}(\mathcal{C}) \longrightarrow \mathrm{F{\small ACT}}_X^{\mathrm{pl,cstr}}(\mathcal{C}) \tag{4.2}$$

where on the left hand sides we consider X with its trivial marking.

PROOF. We need to check that the restriction functor

$$\iota_X^* \colon \mathrm{Alg}_{\mathrm{open}(X)}(\mathcal{C}) \longrightarrow \mathrm{Alg}_{\mathrm{open}(X)_{\mathrm{mrk}}}(\mathcal{C}) \tag{4.3}$$

given by precomposing with ι_X restricts to (constructible) pointless factorization algebras. This follows from ι_X^* (separately) preserving multiplicativity, constructibility and the marked Weiss condition. Fix some object $\mathcal{F} \in \mathrm{Alg}_{\mathrm{open}(X)}$. If $\mathcal{F}$ is multiplicative it follows that $\iota_X^*\mathcal{F}$ also is, since all of the multiplicative morphisms are automatically marked. In the exact same manner it follows that $\iota_X^*\mathcal{F}$ is constructible if $\mathcal{F}$ is so, because for each inclusion $U \hookrightarrow V$ such that U and V are stratified isotopy equivalent it follows that the inclusion automatically is a marked inclusion. Lastly, let $\mathcal{F}$ be a Weiss cosheaf. The marked Weiss covers are in particular ordinary Weiss covers, ensuring that $\iota_X^*\mathcal{F}$ is a marked Weiss cosheaf. $\square$

Remark 4.57. At the level of algebras the restriction functor (4.3) has a left adjoint given by left Kan extension. An interesting question to ask is if the left adjoint also restricts to (pointless) factorization algebras. We do not investigate this further in this thesis.

We now explain how the restriction from pointed to pointless factorization algebras is compatible with the restricting to open subspaces, as well as pushing forward along sufficiently nice maps. This is used in Section 5.2 when proving that the second functor of (4.2) is even a symmetric monoidal functor. Moreover, we also need these lemmas in Section 9.3 where we compare the pointed and pointless higher Morita categories constructed from (pointless) factorization algebras.

Lemma 4.58. *Let X be a conical marked manifold and let $j\colon U \hookrightarrow X$ be an inclusion of a subspace (with inherited marking). Then the diagrams*

$$\begin{tikzcd} \mathrm{Fact}_X \arrow[r, "j^*"] \arrow[d, "\iota_X^*"'] & \mathrm{Fact}_U \arrow[d, "\iota_U^*"] \\ \mathrm{Fact}_X^{\mathrm{pless}} \arrow[r, "j^*"'] & \mathrm{Fact}_U^{\mathrm{pless}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Fact}_X^{\mathrm{cstr}} \arrow[r, "j^*"] \arrow[d, "\iota_X^*"'] & \mathrm{Fact}_U^{\mathrm{cstr}} \arrow[d, "\iota_U^*"] \\ \mathrm{Fact}_X^{\mathrm{pl,cstr}} \arrow[r, "j^*"'] & \mathrm{Fact}_U^{\mathrm{pl,cstr}} \end{tikzcd} \tag{4.4}$$

commute. Here, the copies of U and X on the top row come with the trivial marking.

Proof. At the level of ∞-operads we have the commuting square

$$\begin{tikzcd} \mathrm{open}\,(U)^{\otimes} \arrow[r, "j"] & \mathrm{open}(\mathrm{X})^{\otimes} \\ \mathrm{open}\,(U)^{\otimes}_{\mathrm{mrk}} \arrow[r, "j"'] \arrow[u, "\iota_X"] & \mathrm{open}(X)^{\otimes}_{\mathrm{mrk}} \arrow[u, "\iota_U"'] \end{tikzcd} . \tag{4.5}$$

Now apply Alg_ to (4.5) and observe that the commuting square of algebras restricts to the subcategories of (constructible) pointless factorization algebras by Lemma 4.53 and Lemma 4.56. We indeed get $\mathrm{Fact}_{\_}$, respectively $\mathrm{Fact}_{\_}^{\mathrm{cstr}}$ in the top row of the diagrams (4.4) because the corresponding copies of U and X are trivially marked. □

We now also consider the analogous result for push forwards along sufficiently nice maps.

Lemma 4.59. *Let $f\colon X \to Y$ be a continuous map which sends marked points to marked points, i.e. $f(X^{\mathrm{mrk}}) \subseteq Y^{\mathrm{mrk}}$. Then we have that the square*

$$\begin{tikzcd} \mathrm{Fact}_X \arrow[r, "f_\star"] \arrow[d, "\iota_X^*"'] & \mathrm{Fact}_Y \arrow[d, "\iota_Y^*"] \\ \mathrm{Fact}_X^{\mathrm{pless}} \arrow[r, "f_\star"'] & \mathrm{Fact}_Y^{\mathrm{pless}} \end{tikzcd} \tag{4.6}$$

commutes, where the copies of U and X on the top row come with the trivial marking.

Assume that f is a map which constructible pointless factorization algebras push forward along.[11] *In this situation the above restricts to the commuting square*

$$\begin{tikzcd} \mathrm{Fact}_X^{\mathrm{cstr}} \arrow[r, "f_\star"] \arrow[d, "\iota_X^*"'] & \mathrm{Fact}_Y^{\mathrm{cstr}} \arrow[d, "\iota_Y^*"] \\ \mathrm{Fact}_X^{\mathrm{pl,cstr}} \arrow[r, "f_\star"'] & \mathrm{Fact}_Y^{\mathrm{pl,cstr}} \end{tikzcd} . \tag{4.7}$$

Proof. Consider the commuting square of ∞-operads

$$\begin{tikzcd} \mathrm{open}\,(Y)^{\otimes} \arrow[r, "f^{-1}"] & \mathrm{open}(\mathrm{X})^{\otimes} \\ \mathrm{open}\,(Y)^{\otimes}_{\mathrm{mrk}} \arrow[r, "f^{-1}"'] \arrow[u, "\iota_Y"] & \mathrm{open}(X)^{\otimes}_{\mathrm{mrk}} \arrow[u, "\iota_X"'] \end{tikzcd} .$$

[11]For example, if f is an adequately stratified map (between marked stratified space) this condition is satisfied as proven in Lemma 8.14. See also Section 8.2 for more examples of such maps.

Again, by applying Alg_ we obtain a commuting square of algebras. We know that ι restricts to the corresponding subcategories of pointless factorization algebras, and by Lemma 4.54 f also restricts which gives the square (4.6). Moreover, ι preserves constructibility, and by assumption on f so does $f_\star$ yielding the second square (4.7). $\square$

5. Main results

In this section we explain that constructible pointless factorization algebras have the same key features as ordinary factorization algebras. In particular, we start by proving that they glue, or more precisely satisfy descent, in Section 5.1. Using this the ∞-category $\text{Fact}_X^{\text{pl,cstr}}$ of constructible pointless factorization algebras is given a symmetric monoidal structure in Section 5.2, and we also promote the restriction functor from ordinary constructible to pointless factorization algebras to a symmetric monoidal functor. Lastly, in Section 5.3 we explain that constructible pointless factorization algebras on a marked cone $\mathbf{C}Z$ can be assembled from constructible pointless factorization algebras away from the cone point and on $[0,\infty)$. The results here are analogous to those of [**KSW24**, §6] for ordinary factorization algebras.

Remark 5.1. The results here are straightforward adaptations of those of [**KSW24**, §6] for constructible ordinary factorization algebras. However, they rely on results for (constructible) pointless factorization algebras established in Section 7. Some of those follow directly from the corresponding statements for (constructible) ordinary factorization algebras, while some require additional work to adapt to the pointless setting. We highlight this using Notation 1.2, i.e. adding an asterisk to results which follow straightforwardly from [**KSW24**].

5.1. Gluing of constructible pointless factorization algebras. In this section we prove that constructible pointless factorization algebras glue, following the same strategy as in [**KSW24**, §6.1] where this is proven for ordinary constructible factorization algebras.

We mean the following when we say that constructible pointless factorization algebras "glue". Given an open cover of some (maximally marked) space X and constructible pointless factorization algebras on the opens of the cover that are compatible, one can produce a constructible pointless factorization algebra on X. In other words, constructible pointless factorization algebras form a sheaf of ∞-categories. This passes through several intermediate posets of marked opens, one of which is:

Definition 5.2. Let $\mathcal{U} = \{U_i\}_{i<\beta}$ be an open cover of X. We define the full subposet $\text{open}\,(\mathcal{U})_{\text{mrk}} \subset \text{open}(X)_{\text{mrk}}$ to have objects those marked opens that are objects in $\text{open}\,(U_i)_{\text{mrk}}$ for some $U_i \in \mathcal{U}$, and morphisms are all marked inclusions between such opens.

It was proven that locally constant factorization algebras form a sheaf of ∞-categories in [**Mat17**, Theorem 1.3]. For ordinary constructible factorization algebras this was proven in [**KSW24**, Theorem 6.1], and the proof also adapts to the pointless setting as we will see here.

Theorem 5.3. *Let $\mathcal{C}^{\otimes}$ be a $\otimes$-presentable symmetric monoidal ∞-category. Let X be a smooth conical maximally marked manifold with enough good marked disks. Let $\mathcal{U} = \{U_i\}_{i<\beta}$ be an open cover of X. For any finite subset $I \subset \beta$ denote $U_I \coloneqq \bigcap_{i\in I} U_i$. Then we have an equivalence of ∞-categories*

$$\text{Fact}_X^{\text{pl,cstr}}(\mathcal{C}) \xrightarrow{\simeq} \lim_{\beta\supset I\supsetneq\emptyset} \text{Fact}_{U_I}^{\text{pl,cstr}}(\mathcal{C}) \ .$$

In other words the assignment $U \mapsto \text{Fact}_U^{\text{pl,cstr}}(\mathcal{C})$ is a sheaf of ∞-categories.

Proof. The wanted equivalence factors as the following composite of equivalences:

$$\text{Fact}_X^{\text{pl,cstr}} \xrightarrow[\text{Prop 5.4}^*]{\simeq} \text{Fact}_{\mathfrak{B}_\cup}^{\text{pl,cstr}} \xrightarrow[\text{Prop 5.5}]{\simeq} \text{Fact}_{\mathfrak{B}}^{\text{pl,cstr}} \xrightarrow[\text{Cor 7.7}^*]{\simeq} \lim_{\beta\supset I\supsetneq\emptyset} \text{Fact}_{U_I}^{\text{pl,cstr}}$$

where $\mathfrak{B} := \mathrm{open}\,(\mathcal{U})_{\mathrm{mrk}}$. The leftmost equivalence follows from Proposition 5.4 and the rightmost equivalence is due to Proposition 7.7. For the middle equivalence we use that X is maximally marked to apply Proposition 5.5; note that $\mathfrak{B} = \mathrm{open}\,(\mathcal{U})_{\mathrm{mrk}}$ has enough good marked disks because X does (see e.g. Remark 4.45). $\square$

We now prove the two intermediate results used in the above proof. The first one follows directly from having established Proposition 7.10* and Theorem 7.20* as in [**KSW24**].

Proposition 5.4. [**KSW24**, Proposition 6.2] *Assume that X has enough good marked disks. Let $\mathcal{U}$ be an open cover of X. Then the restriction functor*

$$\iota^* \colon \mathrm{F{\small ACT}}_X^{\mathrm{pl,cstr}} \xrightarrow{\simeq} \mathrm{F{\small ACT}}_{\mathfrak{B}_\sqcup}^{\mathrm{pl,cstr}} \ ,$$

where $\mathfrak{B} := \mathrm{open}\,(\mathcal{U})_{\mathrm{mrk}}$, is an equivalence of ∞-categories.

The following proposition is proven exactly as [**KSW24**, Proposition 6.3], but we stress that part of the results this relies on are obtained through non-trivial modifications compared to the ordinary setting. To make this explicit we include the proof. Note also that this relies on assuming X to be maximally marked.

Proposition 5.5. *Let $\mathfrak{P} \subseteq \mathrm{open}\,(X)_{\mathrm{mrk}}$ be a decomposable presieve with enough good marked disks, and let X be maximally marked. Then the restriction*

$$\iota^* \colon \mathrm{F{\small ACT}}_{\mathfrak{P}_\sqcup}^{\mathrm{pl,cstr}} \xrightarrow{\simeq} \mathrm{F{\small ACT}}_{\mathfrak{P}}^{\mathrm{pl,cstr}} \tag{5.1}$$

is an equivalence of ∞-categories.

PROOF. By Corollary 7.15* we have the equivalence

$$\iota^* \colon \mathrm{Alg}_{\mathfrak{P}_\sqcup}^{\mathrm{m,cstr}} \xrightarrow{\simeq} \mathrm{Alg}_{\mathfrak{P}}^{\mathrm{m,cstr}} \ .$$

We claim that this equivalence restricts to that of Equation (5.1). For this we need to prove that (the copresheaf underlying) any given multiplicative constructible pointless prefactorization algebra $\mathcal{A}$ on $\mathfrak{P}_\sqcup$ is a marked Weiss cosheaf if and only if its restriction to $\mathfrak{P}$ is a marked Weiss cosheaf.

"Only if" follows from $\mathfrak{P}$ being a presieve; this ensures that evaluating $\mathcal{A}$ on a marked Weiss cover in $\mathfrak{P}$ gives the same value as the evaluation in $\mathfrak{P}_\sqcup$.

For the converse, let $\mathcal{A} \in \mathrm{Alg}_{\mathfrak{P}_\sqcup}^{\mathrm{m,cstr}}$ such that $\mathcal{A}_{|\mathfrak{P}}$ is a marked Weiss cosheaf. Since $\mathfrak{P}$ has enough good marked disks there exists a factorizing disk-basis $\mathfrak{D}$ of $\mathfrak{P}$. Without loss of generality we can further assume $\mathfrak{D}$ to be decomposable. We recall and apply two facts to the current situation:

- Since $\mathcal{A}_{|\mathfrak{P}}$ is a marked Weiss cosheaf it is a left Kan extension from the factorizing basis $\mathfrak{D}$ by Proposition 3.31 1. In addition, $\mathfrak{D}$ is closed under disjoint unions subordinate to $\mathfrak{P}$ so by Lemma A.44 it follows that $\mathcal{A}_{|\mathfrak{P}^\otimes}$ is an *operadic* left Kan extension from $\mathfrak{D}^\otimes$.
- Applying Proposition 7.13* to $\mathfrak{B} := \mathfrak{D}$ and $\mathfrak{B} := \mathfrak{P}$ (both decomposable) gives that the multiplicative prefactorization algebras $\mathcal{A}_{|\mathfrak{D}_\sqcup^\otimes}$ and $\mathcal{A}_{|\mathfrak{P}_\sqcup^\otimes}$ are operadic left Kan extensions from $\mathcal{A}_{|\mathfrak{D}^\otimes}$ and $\mathcal{A}_{|\mathfrak{P}^\otimes}$.

By transitivity and the 2-out-of-3-property for operadic left Kan extensions as in the commuting diagram

$$\begin{tikzcd} \mathfrak{D}^{\otimes} \arrow[r, hook] \arrow[d, hook] & \mathfrak{P}^{\otimes} \arrow[d, hook] \\ \mathfrak{D}_{\sqcup}^{\otimes} \arrow[r, hook] & \mathfrak{P}_{\sqcup}^{\otimes} \end{tikzcd}$$

it follows that $\mathcal{A}$, which is defined on $\mathfrak{P}_{\sqcup}^{\otimes}$, is an operadic left Kan extension from $\mathfrak{D}_{\sqcup}^{\otimes}$. Again using Lemma A.44 we note that the underlying copresheaf $\mathcal{A}_{|\mathfrak{P}_{\sqcup}}$ is a left Kan extension from $\mathcal{A}_{|\mathfrak{D}_{\sqcup}}$. Moreover, $\mathcal{A}_{|\mathfrak{D}_{\sqcup}}$ is constructible since $\mathcal{A}$ is constructible. Together with the fact that $\mathfrak{D}_{\sqcup}$ is a decomposable multiplicative disk-basis of $\mathfrak{P}_{\sqcup}$ and X is maximally marked this puts us in the situation of Corollary 7.34; hence we conclude that $\mathcal{A}$ is a marked Weiss cosheaf on $\mathfrak{P}_{\sqcup}$ as wanted. □

We now explain how one can use the marked Weiss condition to compute the value of a glued constructible pointless factorization algebra.

Example 5.6. Let U and V be two intervals with non-trivial overlap, i.e. $U \cap V \neq \emptyset$, and set $X = U \cup V$. Moreover, let U and V each have a marked point away from the overlap, i.e. $x_1 \in U \setminus V$ and $x_2 \in V \setminus U$. To be explicit, let $\mathcal{F}_U$ and $\mathcal{F}_V$ be two constructible pointless factorization algebras on U respectively V as in Example 2.23, Example 4.11 (and Example 7.12). That is, arising from an $\mathbf{A_1}$-$\mathbf{A_2}$-bimodule $\mathbf{M_1}$ and an $\mathbf{A_2}$-$\mathbf{A_3}$ -bimodule $\mathbf{M_2}$. By Theorem 5.3 we can glue $\mathcal{F}_U$ and $\mathcal{F}_V$ to obtain a constructible pointless factorization algebra on X (because ${\mathcal{F}_U}_{|U\cap V} \simeq \mathbf{A_2} \simeq {\mathcal{F}_V}_{|U\cap V}$).

We claim that the value of the glued pointless factorization algebra $\mathcal{F}$ on X is given by the relative tensor product $\mathbf{M_1} \otimes_{\mathbf{A_2}} \mathbf{M_2}$, which additionally has the structure of a $\mathbf{A_1}$-$\mathbf{A_3}$-bimodule. The computation relies on choosing a convenient marked Weiss cover. Let $\mathcal{W}$ be as in [**KSW24**, Example 6.4].[12] With this we get

$$\begin{aligned} \mathcal{F}(X) &\simeq \operatorname*{colim}_{\mathcal{W}} \mathcal{F} \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \mathcal{F}(X \setminus (K_0 \sqcup \ldots \sqcup K_n)) \\ &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \mathcal{F}(U_0) \otimes \cdots \otimes \mathcal{F}(U_{n+1}) \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \mathbf{M_1} \otimes \mathbf{A_2}^{\otimes n} \otimes \mathbf{M_2}\ , \end{aligned}$$

where from left to right we used that $\mathcal{F}$ is a marked Weiss cosheaf, that it factors through the localization which moreover is colimit cofinal, multiplicativity of $\mathcal{F}$ and finally that $\{x_1\} \subset U_0$ and $\{x_2\} \subset U_{n+1}$, respectively. The final colimit is exactly the usual expression for the relative tensor product, so we indeed get $\mathcal{F}(X) \simeq \mathbf{M_1} \otimes_{\mathbf{A_2}} \mathbf{M_2}$ as claimed.

Remark 5.7. The difference between the above example and the analogous one for constructible ordinary factorization algebras ([**KSW24**, Example 6.4]) is that in the latter both bimodules are also equipped with a pointing. This trivially equips the relative tensor product with a pointing as well.

Remark 5.8. Theorem 5.3 is a key ingredient when constructing the pointless higher Morita category in Section 9. In short, gluing is crucial to define composition (up to also pushing forward along certain maps); see Section 9.1 for more details.

[12] That is, from the analogous example for constructible ordinary factorization algebras using pointed bimodules. However, the Weiss cover chosen there happens to be (what we here call) a marked Weiss cover so the computation directly carries over.

5.2. Symmetric monoidal structure on $\mathrm{FACT}_X^{\mathrm{pl,cstr}}$. We will now explain how the ∞-category of constructible pointless factorization algebras is given the structure of a symmetric monoidal ∞-category. This is a direct modification of the results in [**KSW24**, §6.2] where the symmetric monoidal structure of the ∞-category of constructible factorization algebras is constructed. With this we can upgrade the functor from Lemma 4.56 to a symmetric monoidal functor.

Recall the inclusion map $\iota_X\colon \mathrm{open}(X)_{\mathrm{mrk}} \longrightarrow \mathrm{open}(X)$. As explained in Lemma 4.56 this gives rise to a functor

$$\iota_X^*\colon \mathrm{FACT}_X^{\mathrm{cstr}} \longrightarrow \mathrm{FACT}_X^{\mathrm{pl,cstr}} \ , \tag{5.2}$$

where on the left we only use the underlying conical manifold X of the marked conical manifold (X, X^{mrk}). We already know that the left hand side has a symmetric monoidal structure from [**KSW24**, §6.2]. Hence, by exploiting the same setup to provide the right hand side with a symmetric monoidal structure here we can upgrade the functor to a *symmetric monoidal* functor. This is done in Proposition 5.12 below.

First, let us briefly explain how the symmetric monoidal structure on $\mathrm{FACT}_X^{\mathrm{pl,cstr}}$ is constructed. Here OPD denotes the $(2,1)$-category of operads and operad maps as in Notation A.20.

Construction 5.9. [**KSW24**, Construction 6.6] Let X be a marked space. Consider the functor of 1-categories

$$\mathrm{FIN}_* \longrightarrow \mathrm{OPD}^{\mathrm{op}}, \qquad I_+ \mapsto \mathrm{open}\left(X^{\amalg I}\right)^{\otimes}_{\mathrm{mrk}} \ .$$

On morphisms this sends $f\colon I_+ \to J_+$ to

$$\mathrm{open}\left(X^{\amalg J}\right)^{\otimes}_{\mathrm{mrk}} \longrightarrow \mathrm{open}\left(X^{\amalg I}\right)^{\otimes}_{\mathrm{mrk}}, \qquad \coprod_{j\in J} U_j \mapsto \coprod_{i \in I} U_{f(i)} \ .$$

We use the convention that for the base point $+$ of J_+ we have $U_+ := \emptyset$. Postcomposing with the functor

$$\mathrm{Alg}_{\_}(\mathcal{C})\colon \mathrm{OPD}^{\mathrm{op}} \longrightarrow \mathrm{CAT}_\infty$$

we get

$$\mathrm{Alg}_X^{\otimes}(\mathcal{C})\colon \mathrm{FIN}_* \longrightarrow \mathrm{CAT}_\infty, \qquad I_+ \mapsto \mathrm{Alg}_{X^{\amalg I}}(\mathcal{C}) \ .$$

This functor exhibits the symmetric monoidal structure on the ∞-category $\mathrm{Alg}_X(\mathcal{C})$ of pointless prefactorization algebras.

Remark 5.10. Recall that to push forward a pointless prefactorization algebra along a continuous map we additionally require the map to send marked points to marked points. We verify this in two minimal examples, the general cases are analogous as explained in the proof of Proposition 5.11 below. Recall from Definition A.2 the notion of active and inert maps in FIN_*. Let $f\colon 2_+ \to 1_+$ be the unique active map in FIN_*; then $\mathrm{Alg}_X^{\otimes}(f)\colon X \coprod X \to X$ is the fold map which indeed sends marked points to marked points. For $f\colon 2_+ \to 1_+$ the inert map of FIN_* sending $+, 1 \mapsto +$ and $2 \mapsto 1$ we have that $\mathrm{Alg}_X^{\otimes}(f)\colon X \hookrightarrow X \coprod X$ is the embedding of X into the second copy of X, which also manifestly sends marked points to marked points.

We have the following, which is a pointless version of [**KSW24**, Proposition 6.7 & Corollary 6.8].

Proposition 5.11. *We have a subfunctor*

$$\mathrm{F\small{ACT}}_X^{\mathrm{pl,cstr},\otimes} \subset \mathrm{Alg}_X^{\otimes} \colon \mathrm{F\small{IN}}_* \longrightarrow \mathrm{C\small{AT}}_\infty$$

by pointwise passing to the full ∞-subcategories of constructible pointless factorization algebras. For every smooth conical maximally marked manifold X with enough good marked disks, the functor

$$\mathrm{F\small{ACT}}_X^{\mathrm{pl,cstr},\otimes} \colon \mathrm{F\small{IN}}_* \longrightarrow \mathrm{C\small{AT}}_\infty, \qquad I_+ \mapsto \mathrm{F\small{ACT}}_{X\amalg I}^{\mathrm{pl,cstr}} \tag{5.3}$$

exhibits a monoid-object in $(\mathrm{C\small{AT}}_\infty, \times)$. That is, a symmetric monoidal structure on the ∞-category $\mathrm{F\small{ACT}}_X^{\mathrm{pl,cstr}}$ of constructible pointless factorization algebras.

Proof. For the first part one uses that constructible pointless factorization algebras push forward along disjoint unions of fold maps by Lemma 8.14 and Lemma 4.55 , as well as the fact that they are preserved by restriction along some open embedding by Lemma 4.53 .

For the second part one needs to establish the Segal condition. Here we use the assumption that X is maximally marked to employ Theorem 5.3 in the exact same way as in [**KSW24**, Corollary 6.8]. □

The symmetric monoidal structure of $\mathrm{F\small{ACT}}_X^{\mathrm{cstr}}$, i.e. constructible factorization algebras on X is constructed in the exact same way. We are now ready to upgrade the functor (5.2) established in Lemma 4.56 to a symmetric monoidal functor.

Proposition 5.12. *Let $\mathcal{C}$ be a symmetric monoidal $\otimes$-presentable ∞-category. Assume that X is a smooth conical manifold which is maximally marked, and let ι_X be the map* (5.2)*. Restriction along ι_X induces a symmetric monoidal functor*

$$\iota_X^* \colon \mathrm{F\small{ACT}}_X^{\mathrm{cstr}}(\mathcal{C}) \longrightarrow \mathrm{F\small{ACT}}_X^{\mathrm{pl,cstr}}(\mathcal{C}) ,$$

where on the left hand side we forget about the marking of X and only remember the underlying smooth conical manifold.

Proof. We need to ensure that the functor from Lemma 4.56 is compatible with the symmetric monoidal structure. Explicitly, we need it to induce a natural transformation between the two functors $\mathrm{F\small{ACT}}_X^{\mathrm{cstr},\otimes}$ and $\mathrm{F\small{ACT}}_X^{\mathrm{pl,cstr},\otimes}$ defined as in (5.3). Let $f \colon I_+ \to J_+$ be a morphism in $\mathrm{F\small{IN}}_*$ and consider the square

$$\begin{tikzcd}
\mathrm{F\small{ACT}}_{X\amalg I}^{\mathrm{cstr}} \arrow[r, "\mathrm{F\small{ACT}}_X^{\mathrm{cstr},\otimes}(f)"] \arrow[d, "\iota_{X\amalg I}^*"'] & \mathrm{F\small{ACT}}_{X\amalg J}^{\mathrm{cstr}} \arrow[d, "\iota_{X\amalg J}^*"] \\
\mathrm{F\small{ACT}}_{X\amalg I}^{\mathrm{pl,cstr}} \arrow[r, "\mathrm{F\small{ACT}}_X^{\mathrm{pl,cstr},\otimes}(f)"'] & \mathrm{F\small{ACT}}_{X\amalg J}^{\mathrm{pl,cstr}}
\end{tikzcd} .$$

Using the factorization system on $\mathrm{F\small{IN}}_*$ we consider active and inert maps separately. When f is active both $\mathrm{F\small{ACT}}_X^{\mathrm{cstr},\otimes}(f)$ and $\mathrm{F\small{ACT}}_X^{\mathrm{pl,cstr},\otimes}(f)$ are given by pushing forward along a disjoint union of fold maps, while if f is inert $\mathrm{F\small{ACT}}_X^{\mathrm{cstr},\otimes}(f)$ and $\mathrm{F\small{ACT}}_X^{\mathrm{pl,cstr},\otimes}(f)$ are given by restriction along some open embedding. In the former case we use that the fold map is adequately stratified so by Lemma 8.14 and Lemma 4.55 we are in the setting of Lemma 4.59. In the latter case the square commutes by Lemma 4.58. □

Given the symmetric monoidal structure on $\mathrm{F\small{ACT}}_X^{\mathrm{pl,cstr}}$ there is also an improvement of the statement in Theorem 5.3 . Namely, constructible pointless factorization algebras is not just a sheaf of ∞-categories but of *symmetric monoidal* ∞-categories. Let $\mathrm{Mfld}^{\mathrm{mrk}}$ denote the 1-category of smooth conical marked manifolds and open embeddings which send marked points to marked points. Upon replacing (the 1-category) Mfld with $\mathrm{Mfld}^{\mathrm{mrk}}$ Construction 6.9 and Proposition 6.10 of [**KSW24**] immediately carries over to the setting of constructible *pointless* factorization algebras. We summarize this by giving the precise statement.

Proposition 5.13. [**KSW24**, Proposition 6.10] *The functor* $X \mapsto \mathrm{FACT}_X^{\mathrm{pl,cstr},\otimes}(\mathcal{C})$ *of* [**KSW24**, Construction 6.9] *is a sheaf of symmetric monoidal* ∞*-categories.*

5.3. Link decomposition of constructible pointless factorization algebras. In this section we briefly explain how the results in [**KSW24**, §6.3] carry over to the pointless setting. The motivation is to answer the following question posed by David Ayala. In the setting of ordinary constructible factorization algebras one has to replace 'module' by 'pointed module' below, however, the pointless version spelled out here is what one actually is interested in.

Question 5.14. Is the data of a constructible pointless factorization algebra on a marked cone $\mathbf{C}Z$ the same as the following?

(1) An associative algebra $\mathcal{A}$ in constructible pointless factorization algebras on Z, and
(2) a module for the associative algebra $A(Z) = \int_Z \mathcal{A}$.

This question is reinterpreted as follows:

- Associative algebras are interpreted as locally constant factorization algebras on $(0,\infty)$, as justified by results of Lurie. Moreover, we interpret a module for the associative algebra $A(Z)$ as a constructible *pointless* factorization algebra on $[0,\infty)$.[13]
- By a version of Dunn's additivity theorem we can think of the data in 1 as a constructible pointless factorization algebra $\mathcal{A}$ on $Z \times (0,\infty)$.

Remark 5.15. For locally constant factorization algebras (i.e. in the unstratified and unmarked setting) such a version of Dunn's additivity theorem was proven in [**Ber21**]. The general result for smooth conical manifolds, i.e. corresponding to constructible factorization algebras, is work in progress by Anja Švraka. We expect these results to also carry over to the pointless setting.

We will now spell out the main theorem of this section, which under the reinterpretation gives a positive answer to Question 5.14. This is the pointless analogue of [**KSW24**, Theorem 6.12], and we explain how the proof indeed carries over.

Theorem 5.16. [**KSW24**, Theorem 6.12] *Let* $\mathcal{C}^{\otimes}$ *be a* $\otimes$*-presentable symmetric monoidal* ∞*-category. Let* Z *be a compact smooth conical manifold with enough good disks, and let* $\mathbf{C}Z$ *be the corresponding cone which moreover is marked at the cone point. We have a pullback square of* ∞*-categories*

$$\begin{tikzcd} \mathrm{FACT}^{\mathrm{pl,cstr}}_{\mathbf{C}Z} \arrow[r] \arrow[d, "p_\star"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{FACT}^{\mathrm{pl,cstr}}_{Z\times(0,\infty)} \arrow[d, "p_\star"] & \mathcal{A} \arrow[d, mapsto] \\ \mathrm{FACT}^{\mathrm{pl,cstr}}_{[0,\infty)} \arrow[r] & \mathrm{FACT}^{\mathrm{pl,cstr}}_{(0,\infty)} & \int_Z \mathcal{A} \end{tikzcd} . \tag{5.4}$$

[13]In the ordinary setting the latter is made precise in [**KSW24**, Example 5.27] (for pointed modules) using [**KSW24**, Corollary 5.26]. Despite not having proven an analogous result to this (see also Remark 7.32) we nevertheless expect this kind of comparison to hold also in the pointless setting.

Here, the vertical maps are given by pushforward along the quotient map $p\colon \mathbf{C}Z \to [0,\infty)$, while the horizontal functors are given by restriction to open subspaces.

Since the cone point of $\mathbf{C}Z$ is marked, it follows that $\{0\} \subseteq [0,\infty)$ also is a marked point. Hence, the change compared to the ordinary unmarked setting are in those two entries, i.e. the ∞-operads $\mathrm{open}\,(\mathbf{C}Z)^{\otimes}_{\mathrm{mrk}}$ and $\mathrm{open}\,([0,\infty))^{\otimes}_{\mathrm{mrk}}$. The two entries corresponding to $Z \times (0,\infty)$ and $(0,\infty)$ contains *no* marked points, which is why the right vertical map is indeed given by factorization homology.

Remark 5.17. At the level of (discrete, colored) operads we have the following square:

$$\begin{tikzcd} \mathrm{open}\,((0,\infty))^{\otimes}_{\mathrm{mrk}} \arrow[r, "p^{-1}"] \arrow[d, hook] & \mathrm{open}\,(Z \times (0,\infty))^{\otimes}_{\mathrm{mrk}} \arrow[d, hook] \\ \mathrm{open}\,([0,\infty))^{\otimes}_{\mathrm{mrk}} \arrow[r, "p^{-1}"'] & \mathrm{open}\,(\mathbf{C}Z)^{\otimes}_{\mathrm{mrk}} \end{tikzcd}$$

The pullback square in Equation (5.4) is induced by applying $\mathrm{Alg}_{\_}(\mathcal{C})$ to the above square and then passing to the ∞-subcategories of constructible pointless factorization algebras.

Remark 5.18. In the ordinary setting the statement of [**KSW24**, Theorem 6.12] also follows from results of Brav–Rozenblyum [**BR23**, §3.2] by using [**KSW24**, Corollary 5.26] to relate their definition of a constructible factorization algebra to the one used here.

Remark 5.19. The result of Theorem 5.16 can be globalized in a way where one can inductively assemble data of algebras and modules in locally constant factorization algebras on the individual strata to construct a global constructible factorization algebra. Since we do not need these results in this thesis, and the constructions directly carry over to constructible pointless factorization algebras we do not go into detail about this. We refer the interested reader to Construction 6.16, Corollary 6.17 and Remark 6.18 in [**KSW24**].

We now turn towards the proof of Theorem 5.16. For this we first need to introduce some notation and a preliminary result.

Definition 5.20. Let U be a marked space. We define $\mathrm{open}_{\neq\emptyset}\,(U)_{\mathrm{mrk}} \subset \mathrm{open}\,(U)_{\mathrm{mrk}}$ to be the full subposet whose opens are all *non-empty* marked opens of U.

Construction 5.21. [**KSW24**, Construction 6.19] Denote by

$$\mathfrak{N} := \mathrm{open}_{\neq\emptyset}\,(Z \times (0,\infty))_{\mathrm{mrk}} \qquad \text{and} \qquad \mathfrak{H} := p^{-1}\Big(\mathrm{open}_{\neq\emptyset}\,([0,\infty))_{\mathrm{mrk}}\Big)$$

the posets of *non-empty* marked opens that

- do *not* contain the cone point, or
- are *horizontal*, i.e. of the form $p^{-1}(U)$ for some marked open $U \in \mathrm{open}\,([0,\infty))_{\mathrm{mrk}}$,

respectively. Define $\mathfrak{U} := \mathfrak{N} \cup \mathfrak{H}$ to be their union. By construction, the associated operad $\mathfrak{U}^{\otimes}$ fits into a pullback square

$$\begin{tikzcd} \mathfrak{N}^{\otimes} \cap \mathfrak{H}^{\otimes} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{N}^{\otimes} \arrow[d, hook] \\ \mathfrak{H}^{\otimes} \arrow[r, hook] & \mathfrak{U}^{\otimes} \end{tikzcd} \tag{5.5}$$

of (discrete, colored) operads and fully faithful embeddings. Moreover, we denote $\mathfrak{U}_{\emptyset} = \mathfrak{U} \cup \{\emptyset\}$, i.e. the poset of marked opens obtained by adding the empty set to $\mathfrak{U}$.

The proposition below follows from [**KSW24**, Proposition D.1] by observing that the proof also applies to the setting where (using the notation there) $O^{\otimes}$ (and hence also $U^{\otimes}$) is not necessarily unital. More precisely, by a careful inspection of the proof, the proposition still holds when modifying assumption (A3) to be

(A3') every color $x \in U^{\otimes} \setminus O^{\otimes}$ admits a unique 0-ary operation.

In our setting this corresponds exactly to saying that we have an inclusion of the empty set into any open which does not contain the cone point, which we indeed have. Thus, we get:

Proposition 5.22. [**KSW24**, Proposition 6.20] *The square* (5.5) *is a pushout of operads and even of* ∞*-operads.*

The proof of [**KSW24**, Proposition D.1] is rather technical. However, the key idea of the proof is still instructive, and we briefly recall it here. First of all, the main hurdle is that we have operations in the $\mathfrak{U}^{\otimes}$ which exists neither in $\mathfrak{N}^{\otimes}$ nor in $\mathfrak{H}^{\otimes}$; e.g. if the output contains the cone point but the input contains at least one non-empty open which is not horizontal. See [**KSW24**, Figure 6] for a simple example. What one needs to show is that such an operation will nevertheless lie in the pushout. This follows by showing that each such operation f can be factored canonically into a composition of an operation $b \in \mathfrak{N}^{\otimes}$ away from the cone point, and a horizontal operation $a \in \mathfrak{H}^{\otimes}$. We refer the reader to [**KSW24**] for more details.

We now explain that the proof of [**KSW24**, Proposition 6.12] indeed carries over to the pointless setting.

Sketch of proof of Theorem 5.16. By Proposition 5.22 we have a pullback square of ∞-categories of algebras

$$\begin{tikzcd} \mathrm{Alg}_{\mathfrak{U}} \arrow[r] \arrow[d, "p_\star"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Alg}_{Z\times(0,\infty)} \arrow[d, "p_\star"] \\ \mathrm{Alg}_{\mathrm{open}_{\neq\emptyset}([0,\infty))_{\mathrm{mrk}}} \arrow[r] & \mathrm{Alg}_{(0,\infty)} \end{tikzcd} .$$

This square stays a pullback square upon restricting to the ∞-subcategories of multiplicative algebras. Using Lemma 7.6 for each entry in the square we get that this pullback square is equivalent to the pullback square

$$\begin{tikzcd} \mathrm{Alg}^{\mathrm{m}}_{\mathfrak{U}_\emptyset} \arrow[r] \arrow[d, "p_\star"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Alg}^{\mathrm{m}}_{Z\times(0,\infty)} \arrow[d, "p_\star"] \\ \mathrm{Alg}^{\mathrm{m}}_{[0,\infty)} \arrow[r] & \mathrm{Alg}^{\mathrm{m}}_{(0,\infty)} \end{tikzcd} .$$

Using that $\mathfrak{U}_\emptyset$ is a presieve one gets that this square stays a pullback square upon restricting to the ∞-subcategories of constructible pointless factorization algebras in the same way as for ordinary constructible factorization algebras.

The next step is to show that the inclusion $\mathfrak{U}_\emptyset \hookrightarrow \mathrm{open}\,(\mathbf{C}Z)_{\mathrm{mrk}}$ of posets of marked opens induces an equivalence at the level of constructible pointless factorization algebras. We factor this inclusion through the disjoint union completion of $\mathfrak{U}_\emptyset$, i.e. as $\mathfrak{U}_\emptyset \hookrightarrow (\mathfrak{U}_\emptyset)_{\sqcup} = \mathfrak{U}_{\sqcup} \hookrightarrow \mathbf{C}Z$, and claim that the corresponding two restriction functors

$$\mathrm{FACT}^{\mathrm{pl,cstr}}_{\mathbf{C}Z} \longrightarrow \mathrm{FACT}^{\mathrm{pl,cstr}}_{\mathfrak{U}_{\sqcup}} \longrightarrow \mathrm{FACT}^{\mathrm{pl,cstr}}_{\mathfrak{U}} \tag{5.6}$$

are equivalences. For the leftmost arrow of (5.6) observe that $\mathfrak{U}_{\cup}$ is a factorizing basis of the presieve $\mathbf{C}Z$ so by Proposition 7.10* and Corollary 7.21* we get that the map is an equivalence. For the rightmost arrow of (5.6) one gets an equivalence

$$\mathrm{Alg}^{\mathrm{m,cstr}}_{\mathfrak{U}_{\cup}} \xrightarrow{\simeq} \mathrm{Alg}^{\mathrm{m,cstr}}_{\mathfrak{U}_{\emptyset}} .$$

by Corollary 7.15* (using that $\mathfrak{U}_{\emptyset}$ is decomposable). To see that this restricts to an equivalence of constructible pointless factorization algebras one uses Corollary 7.34 in a similar way to how it is used in the proof of Proposition 5.5. That is, to show the marked Weiss condition one instead shows that the copresheaf is a left Kan extensions from some suitable collection of disks. This part is a straightforward adaptation of the ordinary (unmarked) proof by simply restricting to the marked Weiss covers. $\square$

6. The ∞-category of smooth conical marked manifolds

In this section we first define the ∞-category $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}$ in Section 6.1, together with its ∞-subcategories $\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}$ and $\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}$. Then we go on to consider marked basics in more detail in Section 6.2. Lastly, in Section 6.3 we establish a marked version of [**AFT17a**, Lemma 2.21], which is needed in Section 7.2. This section leverages a lot of set up and results from the unmarked setting as developed in [**AFT17b**] and [**AFT17a**].

6.1. The ∞-category $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}$.

In this section we first recall from [**AFT17b**, §4.1] how the ordinary category Mfld is enriched in KAN-complexes. Upon applying the simplicial nerve this gives an honest ∞-category $\mathcal{M}\mathrm{fld}$. Building upon these constructions we also equip $\mathrm{Mfld}^{\mathrm{mrk,bij}}$ with a KAN-enriched structure. This defines an ∞-category $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}$ of conically smooth marked manifolds and embeddings which, in a suitable sense, are bijective on marked points.

We first recall some basic definitions and setup from [**AFT17b**, §3 & 4].

Definition 6.1. Let STRAT denote the category whose objects are smooth conical manifolds and morphisms are conically smooth maps.

Definition 6.2. Let X, Y and Z be smooth conical manifolds. We define

$$\mathcal{S}\mathrm{trat}_Z(X,Y) := \left\{ f\colon Z\times X \to Z\times Y \mid \mathrm{pr}_Z \circ f = \mathrm{pr}_Z \right\} ,$$

i.e. it is the set of commutative diagrams of conically smooth maps

$$\begin{tikzcd} Z\times X \arrow[rr, "f"] \arrow[dr, "\mathrm{pr}_Z"'] & & Z\times Y \arrow[dl, "\mathrm{pr}_Z"] \\ & Z & \end{tikzcd} \tag{6.1}$$

where pr_Z is the projection onto Z.

Notation 6.3. Recall that to give a map f into a product as in Definition 6.2 we can equivalently give two maps

$$f_1\colon Z\times X \to Z \quad \text{and} \quad f_2\colon Z\times X\to Y \ .$$

With this decomposition of f, the condition that the diagram (6.1) commutes exactly corresponds to $f_1 = \mathrm{pr}_Z$.

The category STRAT has finite products ([**AFT17b**, Corollary 3.4.10]) and conically smooth maps compose. This makes the following definition well-defined.

Definition 6.4. [**AFT17b**, Definition 4.1.2] Let $\mathcal{S}\mathrm{trat}$ define the following category enriched over $\mathrm{Fun}(\mathrm{STRAT}^{\mathrm{op}}, \mathrm{SET})$. Objects are smooth conical manifolds. For two such objects X and Y, the presheaf of maps $\mathcal{S}\mathrm{trat}(X,Y) \in \mathrm{Fun}(\mathrm{STRAT}^{\mathrm{op}},\mathrm{SET})$ assigns values

$$\mathcal{S}\mathrm{trat}(X,Y) := \left(Z \mapsto \mathcal{S}\mathrm{trat}_Z(X,Y)\right) \ .$$

Similarly, let $\mathcal{M}\mathrm{fld}$ denote the category enriched over $\mathrm{Fun}(\mathrm{STRAT}^{\mathrm{op}},\mathrm{SET})$ whose objects are smooth conical manifolds. For two such objects X and Y, the presheaf of maps is

$$\mathcal{M}\mathrm{fld}(X,Y) := \Big(Z \mapsto \mathcal{M}\mathrm{fld}_Z(X,Y)\Big)$$

where

$$\mathcal{M}\mathrm{fld}_Z(X,Y) \coloneqq \{f \in \mathcal{S}\mathrm{trat}_Z(X,Y) \mid f \text{ is an open embedding}\}\Big) .$$

To understand how the above presheaves of maps actually are KAN-complexes we also need the following definition.

Definition 6.5. Define the *standard cosimplicial manifold* as the functor

$$\Delta_e^\bullet \colon \Delta \longrightarrow \mathrm{STRAT}$$

given by assigning

$$[p] \mapsto \Delta_e^p \coloneqq \left\{ t \colon \{0,\ldots,p\} \to \mathbb{R} \;\middle|\; \sum_{i=0}^{p} t_i = 1 \right\} .$$

With the above definition we see that restriction along $\Delta \xrightarrow{\Delta_e^\bullet} \mathrm{STRAT}$ defines a product-preserving functor

$$(-)_{|_\Delta} \colon \operatorname{Fun}(\mathrm{STRAT}^{\mathrm{op}}, \mathrm{SET}) \longrightarrow \mathrm{sSET} , \tag{6.2}$$

where $\mathrm{sSET} \coloneqq \operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{SET})$ is the category of simplicial set.

Lemma 6.6. [**AFT17b**, Lemma 4.1.4] *Let X and Y be smooth conical manifolds. Both of the simplicial sets* $\mathcal{S}\mathrm{trat}(X,Y)_{|_\Delta} \colon [p] \mapsto \mathcal{S}\mathrm{trat}_{\Delta_e^p}(X,Y)$ *and* $\mathcal{M}\mathrm{fld}(X,Y)_{|_\Delta} \colon [p] \mapsto \mathcal{M}\mathrm{fld}_{\Delta_e^p}(X,Y)$ *are* KAN *complexes.*

From the above lemma and the (product-preserving) functor (6.2) it follows that $\mathcal{S}\mathrm{trat}$ and $\mathcal{M}\mathrm{fld}$ are KAN-enriched categories, i.e. that the $\operatorname{Fun}(\mathrm{STRAT}^{\mathrm{op}}, \mathrm{SET})$-enrichments restrict to KAN-enrichments.

Notation 6.7. Upon applying the simplicial nerve functor (see [**Lur09a**, §1.1.5]) we regard the KAN-enriched categories here as ∞-categories. That is, we suppress the simplicial nerve from the notation and write $\mathcal{M}\mathrm{fld}$ for the ∞-category of smooth conical manifolds and embeddings. Here, the homotopies are given by conically smooth stratified isotopies. Similarly, we also write

$$\mathcal{B}\mathrm{sc} \subset \mathcal{M}\mathrm{fld} \qquad \text{and} \qquad \mathcal{D}\mathrm{isks} \subset \mathcal{M}\mathrm{fld}$$

for the full ∞-subcategories whose objects are those smooth conical manifolds which are conical disks ("basics") respectively finite disjoint unions of conical disks.

We now turn towards defining marked variants of the above ∞-categories. For this we first need marked variants of Definition 6.1 and Definition 6.5.

Definition 6.8. Let $\mathrm{STRAT}_{\mathrm{mrk}}$ denote the following category. Objects are smooth conical marked manifolds. Let (X^{mrk}, X) and (Y^{mrk}, Y) be two objects. A morphism between them is a conically smooth map $f \colon X \to Y$ which sends marked points to marked points, i.e. such that we have a commuting diagram

$$\begin{tikzcd} X^{\mathrm{mrk}} \arrow[r, "f_|"] \arrow[d, hook] & Y^{\mathrm{mrk}} \arrow[d, hook] \\ X \arrow[r, "f"] & Y \end{tikzcd} ,$$

where $f_| = f_{|_{X^{\mathrm{mrk}}}}$.

Observation 6.9. Observe that the category $\mathrm{S{\scriptstyle TRAT}}_{\mathrm{mrk}}$ have products induced by the products of $\mathrm{S{\scriptstyle TRAT}}$. The marked structure of the smooth conical product manifold is as in Example 4.21.

Definition 6.10. Let X, Y and Z be smooth conical marked manifolds. Define

$$\mathcal{S}\mathrm{trat}_Z^{\mathrm{mrk}}(X,Y) := \left\{ f \in \mathrm{S{\scriptstyle TRAT}}_Z(X,Y) \,\middle|\, f_| \colon (Z\times X)^{\mathrm{mrk}} \to (Z\times Y)^{\mathrm{mrk}} \right\} ,$$

where $f_| = f_{|_{(Z\times X)^{\mathrm{mrk}}}}$. More explicitly, it is the set of commutative diagrams (6.1) of conically smooth maps that moreover sends marked points to marked points.

We are now ready to define the marked variant of $\mathcal{M}\mathrm{fld}$ we are interested in. Recall from Definition 4.38 what it means for an embedding to be bijective on marked points.

Definition 6.11. Let $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}$ be the following $\mathrm{Fun}(\mathrm{S{\scriptstyle TRAT}}^{\mathrm{op}}_{\mathrm{mrk}}, \mathrm{S{\scriptstyle ET}})$-enriched category. Objects are smooth conical marked manifolds. Given two such objects (X^{mrk}, X) and (Y^{mrk}, Y), the presheaf of morphisms is

$$\begin{aligned}&\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}\left(\left(X^{\mathrm{mrk}}, X\right), \left(Y^{\mathrm{mrk}}, Y\right)\right) :=\\ &\qquad\left((Z^{\mathrm{mrk}}, Z) \mapsto \{ f \in \mathcal{M}\mathrm{fld}_Z(X,Y) \,|\, f \text{ is bijective on marked points}\}\right) .\end{aligned}$$

For notational simplicity we will often omit the marked points when writing out the presheaf of morphisms, i.e. simply abbreviate it by $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}(X,Y)$.

Remark 6.12. If $Z^{\mathrm{mrk}} = \emptyset$ the bijective on marked points condition of the embedding $f \colon Z\times X \to Z\times Y$ as above is vacuous. The same is true whenever $X^{\mathrm{mrk}} = \emptyset$ *and* $Y^{\mathrm{mrk}} = \emptyset$. Now consider $Z^{\mathrm{mrk}} \neq \emptyset$. To check that f is bijective on marked points it is a priori necessary to check that for each $z \in Z^{\mathrm{mrk}}$ the map $f_{2|_{\{z\}\times X}} \colon \{z\}\times X \to Y$ is bijective on marked points, where f_2 is as in Notation 6.3. However, since f is a stratified embedding it automatically "preserves" marked points. See also Remark 6.17 for a more explicit example of this. Thus it is sufficient to check the condition for only one marked point $z \in Z^{\mathrm{mrk}}$.

We also need to give the standard cosimplicial manifold a marked structure.

Definition 6.13. Define the *marked standard cosimplicial manifold* as the functor

$$\Delta^\bullet_{\mathrm{mrk}} \colon \Delta \longrightarrow \mathrm{S{\scriptstyle TRAT}}_{\mathrm{mrk}}$$

given by assigning

$$[p] \mapsto \Delta^p_{\mathrm{mrk}} := \left\{ t \colon \{0,\dots,p\} \to \mathbb{R} \,\middle|\, \sum_{i=0}^{p} t_i = 1 \right\} .$$

Moreover, Δ^p_{mrk} has p marked points placed at $t_i = 1$ (and hence $t_{j\neq i} = 0$) for each $i \in \{0,\dots,p\}$.

Note that in the above definition we implicitly equip Δ^p_{mrk} with a 0-dimensional strata which is equal to the marked points. As before, restriction along $\Delta^\bullet_{\mathrm{mrk}}$ defines a product-preserving functor

$$(-)_{|_\Delta} \colon \mathrm{Fun}(\mathrm{S{\scriptstyle TRAT}}^{\mathrm{op}}_{\mathrm{mrk}}, \mathrm{S{\scriptstyle ET}}) \longrightarrow \mathrm{sS{\scriptstyle ET}} . \tag{6.3}$$

We now want to prove that the presheaf of morphisms in Definition 6.11 restricts to $\mathrm{K{\scriptstyle AN}}$ complexes.

Lemma 6.14. *Let (X^{mrk}, X) and (Y^{mrk}, Y) be two smooth conical marked manifolds. The simplicial set $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}(X,Y)_{|\Delta}\colon \Delta^{\mathrm{op}} \to \mathrm{SET}$ is a* KAN *complex.*

PROOF. Since $\mathcal{M}\mathrm{fld}(X,Y)_{|\Delta}$ is a Kan complex, the claim follows if we can show that

$$\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}\left(X,Y\right)_{|\Delta} \longrightarrow \mathcal{M}\mathrm{fld}(X,Y)_{|\Delta}$$

is a Kan fibration. Let $p \geq 1$ and fix some $0 \leq i \leq p$. We need to find a dashed lift for any map which makes the outer square below commute

$$\begin{tikzcd} \Lambda_i^p \arrow[r] \arrow[d, hook] & \mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}\left(X,Y\right)_{|\Delta} \arrow[d] \\ \Delta^p \arrow[r, "f"'] \arrow[ur, dashed] & \mathcal{M}\mathrm{fld}\left(X,Y\right)_{|\Delta} \end{tikzcd} . \tag{6.4}$$

Explicitly, we have a p-simplex $f\colon X \times \Delta^p_{\mathrm{mrk}} \to Y \times \Delta^p_{\mathrm{mrk}}$ in $\mathcal{M}\mathrm{fld}\left(X,Y\right)_{|\Delta}$ such that for each subset $i \notin S \subset [p]$ the composite $\Delta[S] \hookrightarrow \Delta[p] \to \mathcal{M}\mathrm{fld}\left(X,Y\right)_{|\Delta}$ factors through $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}\left(X,Y\right)_{|\Delta}$. This simply corresponds to knowing that f is bijective on marked points for all marked points apart from $t_i = 1$. Since it is sufficient to check this for exactly one marked point of Δ^p_{mrk} (Remark 6.12) it immediately follows that f lifts to be a p-simplex in $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}\left(X,Y\right)_{|\Delta}$, and we are done. □

Remark 6.15. When the cardinality of X^{mrk} and Y^{mrk} do *not* agree we immediately have that $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}\left(\left(X^{\mathrm{mrk}}, X\right), \left(Y^{\mathrm{mrk}}, Y\right)\right)_{|\Delta} = \emptyset$. In that case there is no non-trivial f as in (6.4) and the above proof reduces to the observation that the trivial fibration is a Kan fibration.

From Lemma 6.14 and the product-preserving functor in Equation (6.3) it follows that $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}$ is also a KAN-enriched category.

Notation 6.16. Following Notation 6.7 we double-book notation and also write $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}$ for the ∞-category of smooth conical marked manifolds. Homotopies are again given by conically smooth stratified isotopies. Similarly, we write

$$\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}} \subset \mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}} \qquad \text{and} \qquad \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}} \subset \mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}$$

for the full ∞-subcategories whose objects are those smooth conical marked manifolds which are conical marked disks ("marked basics"), respectively finite disjoint unions of conical marked disks.

Remark 6.17. Let us explain in some more words how to think about the ∞-category $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}$, or rather its simplicial set of morphisms $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}\left(X,Y\right)_{|\Delta}$. For $[p] = [0]$ we have that $\Delta^0_{\mathrm{mrk}} = \{1\} \subset \mathbb{R}$, and that single point is also marked. Hence, an 0-simplex of the simplicial hom-set is given by a map $f_2\colon X \times \Delta^0_{\mathrm{mrk}} \to Y$ which induces a bijection on marked points of X and Y.

For $[p] = [1]$ we have the line $\Delta^0_{\mathrm{mrk}} = \{t_0 + t_1 = 1\} \subset \mathbb{R}^2$ marked at its two endpoints $(1,0)$ and $(0,1)$. A 1-simplex corresponds to an embedding $f_2\colon X \times \Delta^1_{\mathrm{mrk}} \to Y$ giving rise to a stratified isotopy. Moreover, at either of the marked points we need to ensure that f_2 induces a bijection on marked points of X and Y. Once we have this the bijection at the other marked point is automatic because f_2 is a stratified embedding and hence is forced to respect this condition. This kind of observation also holds for all higher simplicies.

6.2. Marked basics are also easy. In this subsection we want to understand the ∞-category $\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}$ in more detail. In particular, we will establish results analogous to those of [**AFT17b**, §4.3] for $\mathcal{B}\mathrm{sc}$. The main result is Theorem 6.25 which explains in what sense we can think of the ∞-category $\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}$ as a poset of (marked) singularity types.

Before we start with the main results we recall some basic setup from [**AFT17b**]. For the following definition we use the notion of *Lebesgue covering dimension*. For example, in the case of an n-dimensional manifold the Lebesgue covering dimension is n as long as $n \geq -1$.

Definition 6.18 (Dimension)**.** [**AFT17b**, Definition 2.4.1] Let $X = (X \to P)$ be a non-empty conical manifold. The *local dimension* of X at x, denoted as $\dim_x(X)$, is the covering dimension of X at x. We define the *dimension* of X to be

$$\dim(X) \coloneqq \sup_{x\in X} \dim_x(X) \ .$$

If the local dimension of X is constantly n we say that X has *pure dimension* n. Moreover, we use the convention that the dimension of $\emptyset$ is -1.

Definition 6.19 (Depth)**.** [**AFT17b**, Definition 2.4.4] Let $X = (X \xrightarrow{S} P)$ be a non-empty conical manifold. The *local depth* of X at x is

$$\mathrm{depth}_x(X) \coloneqq \dim_x(X) - \dim_x(X_{S(x)}) \ ,$$

i.e. it is the difference between the local dimension of X at x and the local dimension at x of the stratum of X in which x belongs. The *depth of* X, denoted $\mathrm{depth}(X)$, is the supremum over the local depths of X. We use the convention that the depth of $\emptyset$ is -1.

Example 6.20. Consider the leftmost conical manifold X of Figure 3 from Example 4.14. The local depth of X at x chosen to be any of the points, i.e. 0-dimensional strata, is $2-0=2$. If we choose x to be any point of the lines (i.e. the 1-dimensional strata) we instead get that the local depth is $2-1=1$, while for x any point in the bulk the local depth is 0. In total we learn that the depth of X is 2.

Example 6.21. Let $U = \mathbb{R}^i \times \mathbf{C}Z$ be a (marked) basic. It follows that $\mathrm{depth}(U) = \dim(Z)+1$. If we have an open embedding $X \hookrightarrow Y$ of (marked) conical manifolds then it follows that

$$\dim(X) \leq \dim(Y), \qquad \text{and} \qquad \mathrm{depth}(X) \leq \mathrm{depth}(Y) \ .$$

We now introduce some notation for the set of isomorphism classes of the ordinary (un-enriched) category $\mathrm{Bsc}^{\mathrm{mrk,bij}}$:

Notation 6.22. Write

$$[\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}] \coloneqq \Big\{ \big[(U^{\mathrm{mrk}}, U)\big] \,\big|\, (U^{\mathrm{mrk}}, U) \in \mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}} \Big\}$$

for the set of isomorphism classes of objects of $\mathrm{Bsc}^{\mathrm{mrk,bij}}$, the ordinary (un-enriched) ∞-category of marked basics.

The above set comes equipped with the relation

$$\Big\{ \big([(U^{\mathrm{mrk}}, U)], [(V^{\mathrm{mrk}}, V)]\big) \,\big|\, \exists\, (U^{\mathrm{mrk}}, U) \to (V^{\mathrm{mrk}}, V) \Big\} \tag{6.5}$$

consisting of those pairs of isomorphism classes for which there is a morphism between representatives. Recall that morphisms here are embeddings which are bijective on marked points.

Observation 6.23. For any object $(U^{\mathrm{mrk}}, U) \in \mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}$ we have

$$\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}\big((U^{\mathrm{mrk}}, U), (U^{\mathrm{mrk}}, U)\big) \simeq \mathcal{B}\mathrm{sc}(U, U)$$

because the bijectivity on marked points is automatic for any open embedding in this situation.

Notation 6.24. Let $\mathcal{I}$ denote the (discrete) category with two objects $\emptyset$ and 1 and *no* non-trivial morphisms.

The theorem below is a modification of [**AFT17b**, Theorem 4.3.1] to the marked setting.

Theorem 6.25 (Marked basics are also easy).

(1) *For each object $U = (U^{\mathrm{mrk}}, U) \in \mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}$, with $U = \mathbb{R}^i \times \mathbf{C}Z$, the inclusion of Kan monoids*

$$\mathrm{O}(\mathbb{R}^i) \times \mathrm{Aut}(Z) \hookrightarrow \mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}((U^{\mathrm{mrk}}, U), (U^{\mathrm{mrk}}, U))$$

is a homotopy equivalence. In particular, $\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}((U^{\mathrm{mrk}}, U), (U^{\mathrm{mrk}}, U))$ is group-like.

(2) *Let $(U^{\mathrm{mrk}}, U) \xrightarrow{f} (V^{\mathrm{mrk}}, V)$ be a morphism in $\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}$. Then exactly one of the following are true:*
   (a) *f is an equivalence in the ∞-category $\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}$.* [14]
   (b) *The depth of U is strictly less than the depth of V.*

(3) *The relation* (6.5) *on $[\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}]$ is a partial order.*

(4) *The canonical functor between ∞-categories $[-]\colon \mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}} \to [\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}]$ is conservative. In other words, a morphism $f\colon (U^{\mathrm{mrk}}, U) \to (V^{\mathrm{mrk}}, V)$ in the ∞-category $\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}$ is an equivalence if and only if (U^{mrk}, U) and (V^{mrk}, V) are isomorphic in* $\mathrm{Bsc}^{\mathrm{mrk,bij}}$.

(5) *The map of sets $\pi_0 \times \mathrm{depth}\colon [\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}] \to \mathcal{I} \times \mathbb{Z}_{\geq 0}$ is a map of posets, where the latter is given the product order. In other words, the existence of a morphism $(U^{\mathrm{mrk}}, U) \to (V^{\mathrm{mrk}}, V)$ implies $\mathrm{depth}(U) \leq \mathrm{depth}(V)$ and $\pi_0(U) = \pi_0(V)$.*

Proof.

(1) Because of Observation 6.23 this follows immediately from [**AFT17b**, Theorem 4.3.1 (1)].

(2) Let $U = \mathbb{R}^i \times \mathbf{C}Z$ and $V = \mathbb{R}^j \times \mathbf{C}Y$ be two marked basics. The only cases not covered by [**AFT17b**, Theorem 4.3.1 (2)] are embeddings $f\colon U \to V$ where both U and V have a marked point, which forces $i = 0 = j$. The embedding f then satisfies the condition of [**AFT17b**, Lemma 4.3.7 (2)] making it an equivalence.

(3) Again, the only embeddings $U \to V$ not already covered by [**AFT17b**, Theorem 4.3.1 (3)] are those where both U and V have one marked point. As just explained U and V are then equivalent, so $[\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}]$ is indeed also a poset.

(4) Follows immediately from [**AFT17b**, Lemma 4.3.7] and the special case where $U \to V$ both have a marked point as explained above.

(5) Given a map $f\colon U \to V$ in $\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}$ it follows that $\pi_0(U^{\mathrm{mrk}}) = \pi_0(V^{\mathrm{mrk}})$ because f has to induce a bijection on the marked points. If $U^{\mathrm{mrk}} = V^{\mathrm{mrk}} \neq \emptyset$ we know that $\mathrm{depth}(U) = \mathrm{depth}(V)$, otherwise we are in the situation of [**AFT17b**, Theorem 4.3.1 (5)]. Hence, we indeed have a map of posets when taking into account the marked points.

□

[14] Note that this implies that the depth of U and V are equal.

Remark 6.26. We summarize how $[\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}]$ from Notation 6.22 differs from the analogous notion for $\mathcal{B}\mathrm{sc}$. First recall that only basics with a 0-dimensional strata can be marked, c.f. Remark 4.29. Hence, each of those basics give rise to two isomorphism classes corresponding to either marking the cone point or not. There is *no* morphism between these two choices as the number of marked points are different. To summarize, we can think of $[\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}]$ as (the poset) $[\mathcal{B}\mathrm{sc}]$ plus an additional isomorphism class for each "pure cone", i.e. basic with 0-dimensional strata.

We now recall the following result from the unmarked setting which tells us that points in stratified spaces have a unique local type.

Corollary 6.27 (Unique local types)**.** [**AFT17b**, Corollary 4.3.2] *Let X be a stratified space and let $x \in X$ be a point. Let $(U, 0) \hookrightarrow (X, x) \hookleftarrow (V, 0)$ be two coordinate charts about x. Then there is an isomorphism $U \cong V$. In particular, writing $U = \mathbb{R}^i \times \mathbf{C}Z$ and $V = \mathbb{R}^j \times \mathbf{C}Y$, we have $i = j$ and $Z \cong Y$.*

Remark 6.28. If in the above setting we consider X to also be marked, and give U and V the inherited marking (see Remark 4.30), it immediately follows that U and V also are isomorphic as marked basics. Hence, we conclude that every point in a stratified marked space comes with a unique marked local type as well.

We now recall what the stratum of a specific local type of an ordinary, i.e. unmarked, conical manifold is.

Definition 6.29. [**AFT17b**, Definition 4.4.5] Let $U = \mathbb{R}^i \times \mathbf{C}Z$ be a basic, and let X be a conical manifold. Denote the pullback topological space

$$\begin{tikzcd} X_{[U]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ \{[U]\} \arrow[r] & {[\mathcal{B}\mathrm{sc}]} \end{tikzcd}$$

and refer to it as the $[U]$*-stratum of* X. In words it is the locus of those points in X about which embeddings $U \hookrightarrow X$ form a local base. The topological space $X_{[U]}$ is moreover a smooth manifold of dimension $\dim(U) - \mathrm{depth}(U)$.

We now want a marked version of the above. Let (X^{mrk}, X) be a marked conical manifold and let (U^{mrk}, U), with $U = \mathbb{R}^i \times \mathbf{C}Z$, be a marked basic. Let $X_{[U]}$ be as in Definition 6.29. We can then equip $X_{[U]}$ with an inherited marked structure by pulling back that of X. Explicitly, we have

$$\begin{tikzcd} {X_{[U]}}^{\mathrm{mrk}} \arrow[r] \arrow[d, "\iota"'] \arrow[dr, phantom, "\lrcorner"] & X^{\mathrm{mrk}} \arrow[d] \\ X_{[U]} \arrow[r] & X \end{tikzcd} \tag{6.6}$$

Definition 6.30. Assume we are in the above situation and let $\iota\colon {X_{[U]}}^{\mathrm{mrk}} \hookrightarrow X_{[U]}$ be the map from (6.6).

(1) If ${X_{[U]}}^{\mathrm{mrk}} = \emptyset$ we define $X_{[\emptyset,U]} \coloneqq X_{[U]}$.
(2) If ${X_{[U]}}^{\mathrm{mrk}} \neq \emptyset$ we define $X_{[*,U]} \coloneqq \mathrm{im}(\iota) \subset X_{[U]}$ and $X_{[\emptyset,U]} \coloneqq X_{[U]} \setminus \mathrm{im}(\iota)$.

In words this corresponds to thinking about $X_{[\emptyset,U]}$ as the $[\emptyset,U]$-stratum of (X^{mrk},X), while $X_{[*,U]}$ is the $(*,U)$-stratum of (X^{mrk},X), i.e. it takes into account whether the basic U is marked or not. We will refer to this as the *marked* $[U]$*-stratum of* X.

Remark 6.31. It is an immediate consequence of the definition that $X_{[U]} = X_{[\emptyset,U]} \coprod X_{[*,U]}$ whenever U is a basic with 0-dimensional strata.

Remark 6.32. Let X be maximally marked (Definition 4.22) and let $U = \mathbf{C}Z$ be a basic which appears in X. A consequence of X being maximally marked is that $X_{[*,U]} = X_{[U]}$, where the former refers to the marked $[U]$-strata of the marked space X and the latter is the ordinary $[U]$-strata of X after forgetting about the marked structure.

6.3. Computing the marked slice $\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/X}$. We now want to understand the underlying ∞-groupoid of the slice $\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/X}$, for X some conically smooth maximally marked manifold. This will be needed in Section 7.2 where we aim to understand what the localization of certain poset of marked opens is. We start by considering the (unmarked) result from [**AFT17a**], before building upon that proof to get a marked (and bijective on marked point) variant.

Below we use B for the classifying space functor. The following is an immediate variation of the first part of [**AFT17a**, Lemma 2.21]. The only difference is that $\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}$ contains more isomorphism classes than $\mathcal{B}\mathrm{sc}$ exactly corresponding to counting every basic which is a "pure cone" twice as explained in Remark 6.26. However, their automorphisms are the same regardless of working with marked points or not, which gives that the following lemma follows immediately from the equivalent unmarked statement.

Lemma 6.33. [**AFT17a**, Lemma 2.21] *The maximal ∞-subgroupoid of $\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}$ is*

$$\left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}\right)^{\simeq} \simeq \coprod_{[U]} \prod_{i\in I} \mathsf{B}\left(\Sigma_{k_i} \wr \mathrm{End}_{\mathcal{B}}(U_i)\right) \ ,$$

where $\mathcal{B} := \mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}$. The coproduct is indexed by the isomorphism classes of finite disjoint unions of objects of $\mathcal{B}$, whose connected components are grouped according to isomorphism type.

Notation 6.34. For a topological space Y we write $\mathrm{Conf}_k(Y)$ for the *unordered* configuration space of k points in Y.

Lemma 6.33 above is in fact only the first half of [**AFT17a**, Lemma 2.21]. We now recall the second half of the statement. This is what we will work to generalize to the marked setting in the rest of this section.

Lemma 6.35. [**AFT17a**, Lemma 2.21] *The underlying ∞-groupoid of $\mathcal{D}\mathrm{isks}_{/X}$ is identified with the space*

$$\left(\mathcal{D}\mathrm{isks}_{/X}\right)^{\simeq} \xrightarrow{\simeq} \coprod_{[U]} \prod_{i\in I} \mathrm{Conf}_{k_i}(X_{[U_i]}) \ , \tag{6.7}$$

where the coproduct ranges over the isomorphism classes of disjoint unions of conical disks.

Remark 6.36. From the proof of [**AFT17a**, Lemma 2.21] one can extract an explicit description of the equivalence (6.7): An embedding ($\phi\colon U = \coprod_i k_i \times U_i \hookrightarrow X$) is sent to the tuple of configurations $(\phi(S_i))_i$, where each $S_i = \{x_1,\ldots,x_{k_i}\}$ contains precisely one point $x_j \in (U_i)_{[U_i]}$ in the deepest stratum of each of the k_i copies of U_i. This is well-defined because the space of such configurations S_i is contractible.

Before turning to the marked version of the above we need to introduce some terminology.

Definition 6.37. Let $D \in \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}$ and $X \in \mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}$. We say that D has *the same marked points as* X if there exists at least one embedding $D \to X$ that is bijective on marked points.

Note that we can only have an embedding as above which is bijective on marked points if D contains all the marked basics of X which have a marked points. We make this more explicit through an example.

Example 6.38. Let us keep considering the (conically smooth) stratified manifolds of Figure 3, where both manifolds are now maximally marked. Recall that the rightmost space of Figure 3 is in fact a marked basic, which we denote by B. An object $D \in \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}$ has the same marked points as B if it is of the form $D = B \amalg D'$, where $D' \in \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}$ is any other disk appearing in B.

Let X denote the leftmost space of Figure 3. For an object $D \in \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}$ to have the same marked points as X it has to be of the form $D = 5 \times B \coprod D'$, where $D' \in \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}$ is any finite disjoint union of basics without 0-dimensional strata which appears in X.

Let $U \in \mathcal{B}\mathrm{sc}$ and let $\mathrm{Aut}_0(U)$ denote the sub-Kan-monoid of $\mathcal{B}\mathrm{sc}(U,U)$ from [**AFT17b**, Definition 4.3.3]. In words it is given by the simplicies which fix "the origin" of U. For example if $U = \mathbf{C}Z$ "the origin" of U corresponds to the cone point. It is moreover shown in [**AFT17b**, Lemma 4.3.6] that $\mathrm{Aut}_0(U)$ and $\mathcal{B}\mathrm{sc}(U,U)$ have equivalent underlying Kan complexes, which we use in the proof below.

We are now ready to consider the marked version of Lemma 6.35.

Lemma 6.39. *Let X be a smooth conical marked manifold that is maximally marked. Then the underlying ∞-groupoid of the overcategory $\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/X}$ is identified as the space*

$$\left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/X}\right)^{\simeq} \simeq \coprod_{[\widetilde{U}]} \prod_{i \in I} \mathrm{Conf}_{k_i}(X_{[\widetilde{U}_i]}) \tag{6.8}$$

where the coproduct runs over all isomorphism classes with the same marked points as X.

Proof. Consider

$$\begin{aligned}\left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/X}\right)^{\simeq} &= \coprod_{[U]} \mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}(U,X)_{\mathrm{Aut}_{\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}}(U)} \\ &\simeq \coprod_{[\widetilde{U}]} \mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}(\widetilde{U},X)_{\mathrm{Aut}_{\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}}(\widetilde{U})} \\ &\simeq \coprod_{[\widetilde{U}]} \mathcal{M}\mathrm{fld}(\widetilde{U},X)_{\mathrm{Aut}_{\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}}(\widetilde{U})} \\ &\simeq \coprod_{[\widetilde{U}]} \prod_{i \in I} \mathcal{M}\mathrm{fld}(U_i^{k_i},X)_{\mathrm{Aut}_{\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}(\widetilde{U})}}\,,\end{aligned}$$

where $[\widetilde{U}] = [\coprod_{i \in I} k_i \times \widetilde{U}_i]$ runs over all isomorphism classes with the same marked points as X. The first equivalence is Lemma A.37 for $\mathcal{D} := \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}$ and $\mathcal{C} := \mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}$. The second equivalence follows from $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}(U,X)_{|\Delta}$ being trivial whenever U is not an isomorphism type with the same marked points as X, c.f. Remark 6.15. The third equivalence is a consequence of X being maximally marked and having restricted the isomorphism types; in this case the bijective on marked points condition is automatically satisfied for any stratified embedding. The last equivalence corresponds to the unmarked setting as in [**AFT17a**, Lemma 2.21], where it is used that $\mathcal{M}\mathrm{fld}$ comes from a KAN-enriched category.

Since endomorphisms of a marked basic automatically are bijective on marked points we have

$$\mathrm{Aut}_{\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}}(\widetilde{U}) = \mathsf{B}\left(\Sigma_{k_i} \wr \mathrm{End}_{\mathcal{B}\mathrm{sc}^{\mathrm{mrk,bij}}}(\widetilde{U}_i)\right) = \mathsf{B}\left(\Sigma_{k_i} \wr \mathrm{End}_{\mathcal{B}\mathrm{sc}}(\widetilde{U}_i)\right) .$$

Moreover, by [**AFT17b**, Lemma 4.3.6] the inclusion of KAN-groups $\mathrm{Aut}_0(\widetilde{U}_i) \to \mathrm{End}_{\mathcal{B}\mathrm{sc}}(\widetilde{U}_i)$ is an equivalence of underlying KAN complexes.

In total, this means that the statement of the lemma follows if we can show that, for each finite set k_i and easy basic $\widetilde{U}_i$ as above, the Σ_{k_i}-equivariant map which is evaluation at each center of $\widetilde{U}_i$

$$\mathrm{ev}_{(0)_{j\in k_i}} \colon \mathrm{Map}_{\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}}\left(\widetilde{U}_i^{\amalg k_i}, X\right)_{\mathrm{Aut}_0(\widetilde{U}_i)^{k_i}} \xrightarrow{\simeq} \mathrm{Conf}_{k_i}(X_{[\widetilde{U}_i]})$$

is an equivalence of Σ_{k_i}-spaces. This is exactly the second half of the proof of [**AFT17a**, Lemma 2.21], so we are done. □

Remark 6.40. Note that since we use the proof of [**AFT17a**, Lemma 2.21] above the same remark as in Remark 6.36 regarding an explicit description of the equivalence (6.8) holds. The only difference is in the marked setting we restrict to isomorphism classes $\widetilde{U}$ with the same marked points as X.

7. Toolbox for (constructible) pointless factorization algebras

In this section we explain how the toolbox of gluing-techniques for (constructible) ordinary factorization algebras from [**KSW24**, §4 & §5] translates to yield similar results for (constructible) pointless factorization algebras. We have gathered the results which are (more or less) straightforward to adapt to the pointless setting in Section 7.1. This includes the computation of a (homotopy) colimit of ∞-operads of marked opens coming for an open cover, extending pointless factorization algebras from factorizing bases, extending multiplicative pointless prefactorization algebras to disjoint union completions as well as constructibility being a local property.

Then in Section 7.2 we compute the localization of a certain poset of marked opens at its isotopy equivalences. This uses the ∞-category $\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}$ (as well as the slice $\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/V}$) from Section 6, so we here include full details as they differ from those in the ordinary setting ([**KSW24**, §5.2]). For the same reason we also include all details in Section 7.3 where we establish that constructibility implies the marked Weiss cosheaf condition on marked disks.

7.1. Adaptation of tools for ordinary (constructible) factorization algebras. In this section we explain how many of the tools for ordinary (constructible) factorization algebras from [**KSW24**, §4 & §5.1] carry over to the pointless setting. In more detail, we first recall how to compute a (homotopy) colimit of ∞-operads in Section 7.1.1 coming from an open cover. Then we explain how pointless factorization algebras can be extended from factorizing bases in Section 7.1.2. In Section 7.1.3 we explain that, just like ordinary multiplicative algebras, the pointless multiplicative algebras extend to disjoint union completions. Lastly, in Section 7.1.4 we recall that constructibility is a property that can be checked locally, i.e. for some open cover.

7.1.1. A colimit of ∞-operads of marked opens. Here we explain that the results of [**KSW24**, §4.2] translate to the pointless setting. In particular, this corresponds to understanding a certain (homotopy) colimit of ∞-operads of marked opens coming from an open cover. This section relies on the model of dendroidal sets, and the reader not familiar with this theory can get a brief introduction in Appendix A.2.

First recall that the *category of dendroidal sets* is defined to be the presheaf category $\mathrm{dS{\small ET}} \coloneqq \mathrm{Fun}(\Omega^{\mathrm{op}}, \mathrm{S{\small ET}})$. Here Ω is the (full subcategory) of symmetric trees $\Omega \hookrightarrow \mathrm{O{\small PD}}'$, where $\mathrm{O{\small PD}}'$ denotes the (1,1)-category of operads and isomorphisms (c.f. Notation A.20). The corresponding *dendroidal nerve functor*

$$\mathrm{N_d} \colon \mathrm{O{\small PD}}' \to \mathrm{dS{\small ET}}$$

is defined by $\mathrm{N_d}(\mathcal{O})(T) = \mathrm{O{\small PD}}'(T, \mathcal{O})$, i.e. it selects operations of shape T in the operad $\mathcal{O}$. See Example A.23 for an example of this coming from a poset of marked opens. Before giving the colimit cube arising from an open cover that we are working with in this section we need to introduce some new notation.

Definition 7.1. Let $\mathcal{U}$ be a precover (of X). Define $\mathrm{open}_{\neq\emptyset}(\mathcal{U})_{\mathrm{mrk}} \subset \mathrm{open}(X)_{\mathrm{mrk}}$ to be the full subposet whose objects are *non-empty* marked opens that are objects in $\mathrm{open}\,(U_i)_{\mathrm{mrk}}$ for some $U_i \in \mathcal{U}$, and morphisms are all marked inclusions between such opens.

Note that $\mathrm{open}_{\neq\emptyset}(\mathcal{U})_{\mathrm{mrk}} = \mathrm{open}\,(\mathcal{U})_{\mathrm{mrk}} \setminus \{\emptyset\}$, where $\mathrm{open}\,(\mathcal{U})_{\mathrm{mrk}}$ is that of Definition 5.2. Moreover, the above definition reduces to Definition 5.20 in the special case of the precover $\mathcal{U}$ consisting of a single open.

Remark 7.2. We stress that $\mathrm{open}_{\neq\emptyset}(\mathcal{U})_{\mathrm{mrk}}$ is the marked analogue of the restricted sieve, denoted $\mathcal{U}_{\underline{\downarrow}}$, in the ordinary unmarked setting (c.f. [**KSW24**, Definition 4.6]). Working with $\mathcal{U}_{\downarrow_{\mathrm{m}}} \setminus \{\emptyset\}$ is, because of the marked inclusions, too restrictive on objects.

Let $\mathcal{U} = \{U_i\}_{i<\beta}$ be an open cover of X, and let $U_I := \bigcap_{i\in I} U_i$ for any finite subset $I \subset \beta$. The first goal of this subsection is to "glue" pointless prefactorization algebras F_I on $\mathrm{open}_{\neq\emptyset}(U_I)_{\mathrm{mrk}}$ together to give a pointless prefactorization algebra on the bigger full subposet $\mathrm{open}_{\neq\emptyset}(\mathcal{U})_{\mathrm{mrk}}$. For this we work with the following β-cube in dSet:

$$\begin{aligned} \mathrm{N_d}(\mathrm{open}_{\neq\emptyset}(-)^{\otimes}_{\mathrm{mrk}})\colon \mathcal{P}^{\mathrm{op}}_{\mathrm{fin}}(\beta) &\longrightarrow \mathrm{dSet} \\ I &\mapsto \mathrm{N_d}(\mathrm{open}_{\neq\emptyset}(U_I)^{\otimes}_{\mathrm{mrk}}) \ . \end{aligned} \tag{7.1}$$

In more detail, the colimit coming from an open cover $\mathcal{U}$ takes the shape of a (punctured) cube. Moreover, to avoid finiteness assumptions on the open covers we work with, we will even work with transfinite cubes. For details regarding these transfinite cubes and when they compute the homotopy colimit we refer the reader to [**KSW24**, Appendix C].

Lemma 7.3. [**KSW24**, Lemma 4.9] *The cube* (7.1) *induces an isomorphism*

$$\operatorname*{colim}_{\beta\supset I\supsetneq} \mathrm{N_d}(\mathrm{open}_{\neq\emptyset}(U_I)^{\otimes}_{\mathrm{mrk}}) \xrightarrow{\cong} \mathrm{N_d}(\mathrm{open}_{\neq\emptyset}(\mathcal{U})^{\otimes}_{\mathrm{mrk}})$$

of dendroidal sets, and the strict colimit coincides with the homotopy colimit.

Remark 7.4. The main reason the above lemma holds both when working with ordinary opens as well as the marked version here is that all the maps in the cube (7.1) are cofibrations in the model structure on dendroidal sets. The way one sees that the maps are cofibrations is by first noticing that all the maps in the cube are monomorphisms. Moreover, since we have excluded the empty set they are even monomorphisms between normal dendroidal sets, making them normal monomorphisms (i.e. cofibrations). See Theorem A.24, Example A.26 and Remark A.28 for details on this.

We now want to move away from the model category of dendroidal sets. There are a string of comparison results between the different models of ∞-operads as recalled in Remark A.29. In particular, the homotopy theory of Lurie's ∞-operads is (Quillen) equivalent to that of dendroidal sets which immediately gives that the inclusions $U_I \subset \bigcup_{i\in\beta} U_i$ induce an equivalence of ∞-operads

$$\operatorname*{colim}_{I\supsetneq\emptyset} \mathrm{open}_{\neq\emptyset}(U_I)^{\otimes}_{\mathrm{mrk}} \xrightarrow{\cong} \mathrm{open}_{\neq\emptyset}(\mathcal{U})^{\otimes}_{\mathrm{mrk}} \ .$$

At the level of algebras, or in this setting pointless prefactorization algebras, we then get:

Corollary 7.5. [**KSW24**, Corollary 4.11] *Restriction induces an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathrm{open}_{\neq\emptyset}(\mathcal{U})_{\mathrm{mrk}}} \xrightarrow{\simeq} \lim_{\beta\supset I\supsetneq\emptyset} \mathrm{Alg}_{\mathrm{open}_{\neq\emptyset}(U_I)_{\mathrm{mrk}}} \ . \tag{7.2}$$

So far in this section we have worked with posets of marked opens, like $\mathrm{open}_{\neq\emptyset}(\mathcal{U})_{\mathrm{mrk}}$, that do not contain the empty set. However, ultimately the results we want should be given for the version of the posets of marked opens also containing the empty set like $\mathrm{open}(\mathcal{U})_{\mathrm{mrk}}$. For this we first restrict the equivalence (7.2) to the multiplicative subcategories giving an equivalence ([**KSW24**, Lemma 4.12]):

$$\mathrm{Alg}^{\mathrm{m}}_{\mathrm{open}_{\neq\emptyset}(\mathcal{U})_{\mathrm{mrk}}} \xrightarrow{\simeq} \lim_{\beta\supset I\supsetneq\emptyset} \mathrm{Alg}^{\mathrm{m}}_{\mathrm{open}_{\neq\emptyset}(U_I)_{\mathrm{mrk}}} \ . \tag{7.3}$$

Then we note that the proof of the lemma below immediately carries over to the marked setting.

Lemma 7.6. [**KSW24**, Lemma 4.13] *Let $\mathfrak{B}$ be a poset of marked opens. Restriction induces an equivalence of ∞-categories*

$$\mathrm{Alg}^{\mathrm{m}}_{\mathfrak{B}} \xrightarrow{\simeq} \mathrm{Alg}^{\mathrm{m}}_{\mathfrak{B}\setminus\{\emptyset\}} .$$

Heuristically, this corresponds to the observation that the value of a multiplicative algebra on the empty set is always the monoidal unit, and hence carries no non-trivial information. Combining the equivalence (7.3) with the above lemma one in turn obtains an equivalence ([**KSW24**, Corollary 4.14]):

$$\mathrm{Alg}^{\mathrm{m}}_{\mathrm{open}(\mathcal{U})_{\mathrm{mrk}}} \xrightarrow{\simeq} \lim_{\beta\supset I\supsetneq\emptyset} \mathrm{Alg}^{\mathrm{m}}_{\mathrm{open}(U_I)_{\mathrm{mrk}}} . \tag{7.4}$$

That is, we have added in the empty set to the poset of marked opens on both sides of (7.3) using multiplicativity. By a rather straightforward argument we then get the analogous equivalence at the level of (constructible) pointless factorization algebras:

Proposition 7.7. [**KSW24**, Proposition 4.15 & Corollary 4.16] *The equivalence* (7.4) *restricts to equivalences of pointless factorization algebras, respectively constructible pointless factorization algebras. Explicitly, we have*

$$\mathrm{F{\small ACT}}^{\mathrm{pless}}_{\mathrm{open}(\mathcal{U})_{\mathrm{mrk}}} \xrightarrow{\simeq} \lim_{\beta\supset I\supsetneq\emptyset} \mathrm{F{\small ACT}}^{\mathrm{pless}}_{U_I} \quad \text{and} \quad \mathrm{F{\small ACT}}^{\mathrm{pl,cstr}}_{\mathrm{open}(\mathcal{U})_{\mathrm{mrk}}} \xrightarrow{\simeq} \lim_{\beta\supset I\supsetneq\emptyset} \mathrm{F{\small ACT}}^{\mathrm{pl,cstr}}_{\mathrm{open}(U_I)_{\mathrm{mrk}}} .$$

7.1.2. Extending from factorizing bases. Let $\mathfrak{B}$ be a factorizing basis of the poset of marked opens $\mathfrak{U}$. Then we can extend a pointless $\mathfrak{B}$-factorization algebra to a pointless $\mathfrak{U}$-factorization algebra. This is afforded by an operadic left Kan extension, and even gives rise to an equivalence between the corresponding ∞-categories of pointless factorization algebras. This extension result is well-known for ordinary factorization algebras, c.f. [**CG17**, §3 Chapter 7.2] in the setting of $\mathcal{C} = \mathrm{C{\small HAIN}}$, or [**KSW24**, §4.3] for a general target category. We here explain how the latter proof directly translates to the pointless setting.

Consider $\mathfrak{U}$ and $\mathfrak{B}$ two posets of marked opens. Any map $\iota\colon \mathfrak{B} \to \mathfrak{U}$ gives rise to an adjunction

$$\iota_!\colon \mathrm{Alg}_{\mathfrak{B}}(\mathcal{C}) \leftrightarrows \mathrm{Alg}_{\mathfrak{U}}(\mathcal{C})\colon \iota^* \tag{7.5}$$

where the right adjoint ι^* is given by restriction, and the left adjoint $\iota_!$ is given by operadic left Kan extension. The reader unfamiliar with the theory of operadic left Kan extensions can find a brief recollection of important definitions and facts in Appendix A.6.

In other words, if we have a factorizing basis $\mathfrak{B}$ of $\mathfrak{U}$ we immediately get an adjunction at the level of pointless prefactorization algebras. Moreover, this restricts to an equivalence at the level of marked Weiss algebras.

Proposition 7.8. [**KSW24**, Proposition 4.17] *Let $\mathfrak{U}$ and $\mathfrak{B}$ be two posets of marked opens, where $\mathfrak{U}$ is a presieve and $\mathfrak{B}$ is a factorizing basis of $\mathfrak{U}$. Then restriction induces an equivalence*

$$\iota^*\colon \mathrm{Alg}^{\mathrm{Weiss}}_{\mathfrak{U}} \xrightarrow{\simeq} \mathrm{Alg}^{\mathrm{Weiss}}_{\mathfrak{B}}.$$

Remark 7.9. The main ingredients for proving this proposition are Proposition 3.31 and [**KSW24**, Lemma B.3]. The former is a marked version of [**KSW24**, Proposition 2.54] which essentially captures how well left Kan extensions from factorizing bases interact with the (marked) Weiss cosheaf condition. The latter is a lemma giving a condition on the ∞-operads ensuring that ordinary and operadic left Kan extensions are compatible.

We also have that the established equivalence restricts to pointless factorization algebras in the same way as for ordinary factorization algebras.

Proposition 7.10. [**KSW24**, Proposition 4.18] *Let $\mathfrak{U}$ and $\mathfrak{B}$ be two posets of marked opens, where $\mathfrak{U}$ is a presieve and $\mathfrak{B}$ is a factorizing basis of $\mathfrak{U}$. Then the equivalence in Proposition 7.8 restricts to an equivalence*

$$\iota^* \colon \mathrm{F\small{ACT}}_{\mathfrak{U}}^{\mathrm{pless}} \xrightarrow{\simeq} \mathrm{F\small{ACT}}_{\mathfrak{B}}^{\mathrm{pless}}.$$

Remark 7.11. Just like for ordinary factorization algebras, the marked Weiss condition is crucial to make the above argument work. We refer the reader to [**KSW24**, Example 4.20] for a concrete example of how just having a multiplicative algebra on $\mathfrak{B}$ does not give a multiplicative structure on the (operadic) left Kan extension without the extra condition of working with Weiss algebras.

Example 7.12. Recall Example 4.5. There $\mathfrak{U}_{\sqcup}$ is a factorizing basis of $\mathrm{open}\,(\mathbb{R})_{\mathrm{mrk}}$, so by Proposition 7.10 we can extend $\mathcal{A}$ and $\mathcal{M}$ to pointless factorization algebras on all of $\mathbb{R}$.

7.1.3. Extending multiplicative algebras to disjoint union completion. Let $\mathfrak{B}$ be a poset of marked opens that moreover is decomposable. In this section we explain that multiplicative pointless prefactorization algebras can be extended to the disjoint union completion $\mathfrak{B}_{\sqcup}$ just like in the ordinary setting from [**KSW24**, §4.4].

We start by giving the main statement. For this, recall from Definition 3.30 that a poset of marked opens $\mathfrak{B}$ is called *decomposable* if for any open $b \sqcup b' \in \mathfrak{B}$ with $b \neq \emptyset \neq b'$ it follows that $b \in \mathfrak{B}$ and $b' \in \mathfrak{B}$, i.e. one can "decompose" disjoint unions.

Proposition 7.13. [**KSW24**, Proposition 4.21] *Let $\mathfrak{B} \hookrightarrow \mathfrak{B}_{\sqcup}$ be the inclusion of a decomposable poset of marked opens into its disjoint union completion. Then the adjunction* (7.5) *restricts to an equivalence of ∞-categories*

$$\mathrm{Alg}^{\mathrm{m}}_{\mathfrak{B}_{\sqcup}} \xrightarrow{\simeq} \mathrm{Alg}^{\mathrm{m}}_{\mathfrak{B}}\ . \tag{7.6}$$

Proof. Recall that the pointless version, i.e. working with marked opens, is *not* a formal consequence of the corresponding statements for ordinary opens because $\mathrm{open}(X)_{\mathrm{mrk}}$ is *not* a full subposet of $\mathrm{open}(X)$. However, the same arguments work ad verbatim apart from that of [**KSW24**, Corollary 4.37] where it is used that $\emptyset \in \mathrm{open}(X)$ is an initial element. Hence, we require a new argument which does not rely on the empty set being initial. This is the content of Lemma 7.18. □

Remark 7.14. The intuition for why all of the arguments of [**KSW24**] carry over to the marked setting is first of all that all cocartesian morphisms inherently correspond to marked inclusions. Moreover, since we always start with a morphism of $\mathfrak{B}^{\otimes}$ we already have the corresponding marked inclusions, and hence arguments which manipulate how active morphisms can be decomposed (e.g. into an active followed by a cocartesian morphism) inherently work also in the marked setting.

A special instance of Proposition 7.13 is if $\mathfrak{B}$ is a poset of marked opens all of whose objects are connected. Then restriction induces an equivalence ([**KSW24**, Corollary 4.22])

$$\mathrm{Alg}^{\mathrm{m}}_{\mathfrak{B}_{\sqcup}} \xrightarrow{\simeq} \mathrm{Alg}_{\mathfrak{B}}\ .$$

This is immediate because $\mathfrak{B}$ contains no non-trivial cocartesian arrows. It also follows directly that the equivalence restricts to constructible pointless prefactorization algebras, since constructibility is something that is checked at the level of connected disks.

Corollary 7.15. [**KSW24**, Corollary 4.24] *Assume that X is a conical manifold. Let $\mathfrak{B}$ be a decomposable poset of marked opens. Then the equivalence in* (7.6) *restricts to an equivalence*

$$\mathrm{Alg}^{\mathrm{m,cstr}}_{\mathfrak{B}_{\sqcup}} \xrightarrow{\simeq} \mathrm{Alg}^{\mathrm{m,cstr}}_{\mathfrak{B}} \ .$$

Before we turn towards the one intermediate result we need to provide a separate proof for, let us comment on the key steps for the proof in [**KSW24**]. The first step addresses the main difficultly, namely to understand why the operadic left Kan extension of a multiplicative pointless prefactorization algebra on $\mathfrak{B}$ is again multiplicative. For this the key tool is to leverage the (operadic) colimit computing the operadic left Kan extension (as given in Equation (A.8)), and simplify the corresponding diagram in a way that allows for using the multiplicativity condition. Hence, many of the preliminary results of [**KSW24**, §4.4] involves colimit cofinality statements between categories of active maps, and factorizations of active maps into an active map followed by a cocartesian one.

Being more precise, the previous step is actually proven for $\mathfrak{B}' = \mathfrak{B} \setminus \{\emptyset\}$, i.e. a decomposable poset of marked opens which does *not* contain the empty set. This is to avoid anomalies of some of the constructions used. For an explicit example of this see e.g. Construction 7.17 below. Hence, the second step is to add the empty set back in using Lemma 7.6.

We now turn towards the argument that needs to be modified to work in the marked/pointless setting. To be able to give the precise statement we first need to recall some notation and basic constructions from [**KSW24**].

Notation 7.16. Let $\mathfrak{R}$ be a poset of marked opens, with corresponding ∞-operad $\mathfrak{R}^{\otimes}$. Let $\pi_+ \colon \mathfrak{R}^{\otimes} \to \mathrm{F{\scriptsize IN}}_*$ denote the forgetful functor down to finite pointed sets.

- Objects of $\mathfrak{R}^{\otimes}$ are abbreviated by only the finite tuple, i.e. $\bar{r} := (I_+, \bar{r})$, where $\bar{r} = (r_i)_{i \in I}$. A 1-tuple is simply denoted by $r := (1_+, (r))$.
- For an object $\bar{r}$ as above we denote by $\pi(\bar{r}) = I$ the subset of $\pi_+(\bar{r}) = I_+$ where the base point, $+$, is discarded.
- If an edge is cocartesian we label the arrow by $\xrightarrow{!}$.
- Let $\mathfrak{B} \hookrightarrow \mathfrak{R}$ be an inclusion of posets of marked opens, and fix some $\bar{r} \in \mathfrak{R}^{\otimes}$. We denote the category of active-cocartesian arrows from $\mathfrak{B}^{\otimes}$ into $\bar{r}$ by
$$\{\bar{t} \xrightarrow{!} \bar{r}\} \subset \mathfrak{B}^{\otimes}_{/^{\mathrm{act}}_{\bar{r}}} \ ,$$
where it is implicit that $\bar{t} \in \mathfrak{B}^{\otimes}$ varies. The morphisms are active-cocartesian maps $\bar{t} \xrightarrow{!} \bar{t}'$ such that the obvious triangle commutes.

One of the key tricks to get a better handle on the opens and inclusions is to think of them in terms of the equivalence classes they induce on (the fundamental group of) their target open.

Construction 7.17. Let $\mathfrak{R}$ be a poset of marked opens. All objects and morphisms here are in $\mathfrak{R}^{\otimes}$. From an active-cocartesian map $\alpha \colon \bar{t} \to r$ one gets an equivalence relation $\sim_\alpha$ on $\pi_0(r)$ of the form[15]

$$\gamma \colon \bar{t} \xrightarrow{!} r \qquad \leftrightsquigarrow \qquad \pi_0(r) = \coprod_{i \in \pi(\bar{t})} \pi_0(t_i) \twoheadrightarrow \mathcal{J}_\gamma = \{i \in \pi(\bar{t}) \mid t_i \neq \emptyset\} \ .$$

[15] If α is just active we also get a similar equivalence relation after transitively completing. See [**KSW24**, Construction 4.28 (1)] for details.

That is, the equivalence relation corresponds to the "partitioning" of $\pi_0(r)$ into at most $\pi(\bar{t})$ parts. Moreover, if all entries of $\bar{t}$ are non-empty the data of this partitioning of $\pi_0(r)$ is equivalent to the data of the active-cocartesian arrow γ.

Let $\mathfrak{B} \hookrightarrow \mathfrak{R}$ be an inclusion of posets of marked opens with $\emptyset \notin \mathfrak{B}$, and fix $r \in \mathfrak{R}$. From [**KSW24**, Lemma 4.31] we get that the assignment $\gamma \mapsto \sim_\gamma$ of Construction 7.17 yields a fully faithful embedding

$$\{\bar{t} \xrightarrow{!} r\} \hookrightarrow \left\{\sim \text{ on } \pi_0(r) \;\middle|\; |\pi_0(r)_{/\sim}| < \infty\right\} \ , \tag{7.7}$$

where the category on the left is that of Notation 7.16, and on the right we have the poset of equivalence relations on $\pi_0(r)$ with *finitely* many equivalence classes. The inverse functor sends an equivalence relation $\sim$ on $\pi_0(r)$ to the cocartesian arrow

$$\left(I_+, \left(\bigcup\!\!\!\!\cdot\; V\right)_{V \in I}\right) \longrightarrow (1_+, (b)), \tag{7.8}$$

where $I = \pi_0(r)_{/\sim}$ is the finite set of equivalence classes of $\sim$.[16]

We can now give the lemma which requires a separate argument in the marked setting.

Lemma 7.18. *Let $\mathfrak{B} \hookrightarrow \mathfrak{R}$ be an inclusion of posets of marked opens, where $\mathfrak{B}$ is decomposable and $\emptyset \notin \mathfrak{B}$. Fix some object $\bar{r} \in \mathfrak{R}^\otimes$. The category of active-cocartesian maps from $\mathfrak{B}^\otimes$ into $\bar{r}$, i.e. $\{\bar{t} \xrightarrow{!} \bar{r}\}$ from Notation 7.16, is either empty or weakly contractible.*

Proof. Assuming the category to be non-empty, let $\bar{t} \xrightarrow{\gamma} r$ and $\bar{t}' \xrightarrow{\gamma'} r$ be two objects. By the fully faithful embedding of (7.7) this corresponds to two equivalence classes $\sim_\gamma$ and $\sim_{\gamma'}$ on $\pi_0(r)$. The intersection $\sim_\gamma \cap \sim_{\gamma'}$ defines a third equivalence relation which we denote by $\sim_{\gamma''}$. Since we assumed $\mathfrak{B}$ to be decomposable, it follows that the inverse functor assignment from Equation (7.8) is well-defined on the equivalence relation $\sim_{\gamma''}$. Hence, we see that the category of cocartesian maps into r is cofiltered and thus weakly contractible. □

7.1.4. Constructibility is local. In this section we recall that constructibility is a property of a pointless factorization algebra that can be checked locally, i.e. for some open cover. The idea of this proof comes from [**Gin15**, Proposition 13 & 24], and was adapted to conically stratified spaces in [**KSW24**, §5.1]. The latter directly modifies to the pointless setting.

Definition 7.19. Let $\mathcal{A}$ be a pointless factorization algebra on a marked space X. We say that $\mathcal{A}$ is *locally constructible* if there exists some open cover $\mathcal{U}$ of X such that for each $U \in \mathcal{U}$ the restriction $\mathcal{A}_{|_U}$ is a constructible pointless factorization algebra on U.

Theorem 7.20 (Constructibility is local). [**KSW24**, Theorem 5.2] *Let M be a conical marked manifold with enough good marked disks, and let $\mathcal{A}$ be a pointless factorization algebra on M. Then $\mathcal{A}$ is constructible if and only if it is locally constructible.*

We briefly explain the key part of the proof corresponding to [**KSW24**, Lemma 5.5] and then explain why this translates to the marked setting. Let $\mathcal{A}$ be a pointless factorization algebra on the stratified space $\mathbf{D} := \mathbf{C}X \times \mathbb{R}^n$, for some $n \in \mathbb{N}$. Further assume that $\mathcal{A}$ is constructible away from the cone point, i.e. on $\mathbf{D} \setminus \{0\}$. Then, for every $0 < t \leq \infty$, the marked inclusion

$$\mathbf{D}_t := \mathbf{C}_t X \times \mathbf{B}_t(0) \overset{\mathrm{m}}{\hookrightarrow} \mathbf{C}X \times \mathbb{R}^n := \mathbf{D}$$

[16] Note that this is well-defined only on the full subposet of those equivalence relations $\sim$ such that for each equivalence class $V \in \pi_0(r)_{/\sim}$ the disjoint unions $\bigcup\!\!\!\!\cdot\; V$ lie in $\mathfrak{B}$.

is $\mathcal{A}$-local. To prove this one picks a convenient Weiss cover $\mathcal{W}$ consisting of sectors and disks; an example of two such sectors appearing for the cone of $\mathbf{S}^1$ is illustrated in Figure 5. In essence, $\mathcal{W}$ is tailored to be able to use the constructibility away from the cone point to iteratively get $\mathcal{A}$-locality for inclusions $\mathbf{D}_t \hookrightarrow^{\mathrm{m}} \mathbf{D}_{t+\varepsilon}$. This adapts straightforwardly to the marked setting by modifying $\mathcal{W}$ to get a marked Weiss cover $\widetilde{\mathcal{W}}$. Briefly, this only corresponds to ensuring that we actually have a marked cover, i.e. that all opens of the marked Weiss cover contains the cone point (in the case where it is marked).

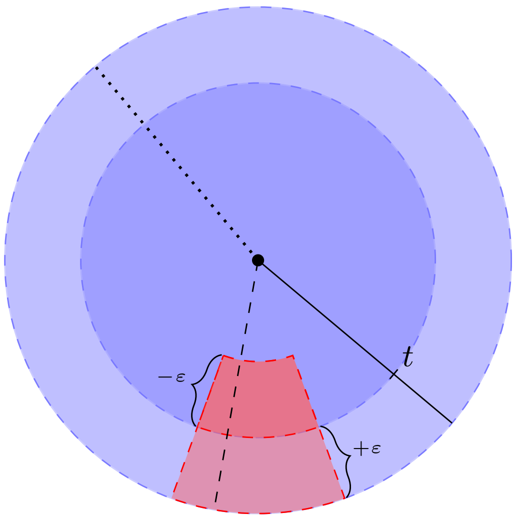


FIGURE 5. An illustration of the (blue) cones of $\mathbf{S}^1$, with two stratified points, of radius t and $t+\varepsilon$. Additionally, the inclusion of two (red) "sectors" $S_{t-\varepsilon,t}(U) \hookrightarrow S_{t-\varepsilon,t+\varepsilon}(U)$ as in the proof of [**KSW24**, Lemma 5.5].

The following consequence of Theorem 7.20 states that constructibility is only a non-vacuous condition away from the 0-dimensional strata. The modifications to account for marked points here are the same as for Theorem 7.20.

Corollary 7.21. [**KSW24**, Corollary 5.7] *Let M be a conical marked manifold with enough good marked disks and let $\mathcal{A}$ be a pointless factorization algebra on M. Let S be the union of all 0-dimensional strata of M and assume $\mathcal{A}$ is constructible on $M \setminus S$. Then $\mathcal{A}$ is constructible on all of M.*

Remark 7.22. Corollary 7.21 tells us that in the setting of pointless factorization algebras constructibility does not depend on marked inclusions and marked Weiss covers. Explicitly, since the marked points by definition are a subset of the 0-dimensional strata we get that it is sufficient to check $\mathcal{A}$-locality for (ordinary) inclusions of opens that contains no marked points to establish constructibility of a pointless factorization algebra.

7.2. Localizing at isotopy equivalences. In this section we wish to extend the localization-result of [**KSW24**, §5.2] to our marked setting. That is, we want to understand what we get from localizing a certain poset of marked opens at all of the subordinate isotopy equivalences. This was first worked out in the unmarked setting for the poset of opens being all disks in [**AFT17a**], and then generalized to the stratified setting as well as using smaller poset of opens in [**KSW24**, Theorem 5.10]. We stress that because the nature of marked inclusions is different from that of ordinary inclusions the main result of this section, Theorem 7.25, differs from the statement of [**KSW24**, Theorem 5.10].

We first define what we mean by isotopy equivalences subordinate to some poset of marked opens.

Definition 7.23. Let $\mathfrak{B}$ be a poset of marked opens of X. Denote by $\mathcal{J}_{\mathfrak{B}} \subset \mathfrak{B}$ the wide subposet of those marked inclusions $U \hookrightarrow^{\mathrm{m}} V$ of opens in $\mathfrak{B}$ which are of the form

$$\bigsqcup_j U^j \hookrightarrow^{\mathrm{m}} \bigsqcup_j V^j ,$$

where for each j the opens U^j and V^j are abstractly isomorphic conical marked disks.

Note that since the marked points of X is some chosen subset of the 0-dimensional strata it follows that the isotopy equivalences are the same in both the marked and unmarked setting.

Observation 7.24. By definition, a pointless factorization algebra $\mathcal{A}$ on $\mathfrak{B}$ is constructible exactly if the underlying functor $\mathcal{A}\colon \mathfrak{B} \to \mathcal{C}$ sends all maps in $\mathcal{J}_{\mathfrak{B}}$ to equivalences in $\mathcal{C}$. In other words, $\mathcal{A}$ is constructible if it factors through the localization $\mathfrak{B} \to \mathfrak{B}[\mathcal{J}_{\mathfrak{B}}{}^{-1}]$.

In the unmarked setting and for $\mathfrak{B} = \mathrm{disk}(X)$, this localization was computed in [**AFT17a**] to be the slice ∞-category $\mathcal{D}\mathrm{isks}_{/X}$, where $\mathcal{D}\mathrm{isks} \subset \mathcal{M}\mathrm{fld}$ is the full subcategory spanned by the smooth conical manifolds which are finite disjoint unions of conical disks (c.f. Notation 6.7). See also [**KSW24**, Remark 5.11] for a list of related results.

Then, in the stratified setting it was shown in [**KSW24**, Theorem 5.10] that working with a smaller poset of opens $\mathfrak{B}$ than all disks yields the same result. That is, for $\mathfrak{B} \subseteq \mathrm{disk}(X)$ a decomposable multiplicative disk-basis of some smooth conical manifold X, the induced map

$$\mathfrak{B}[\mathcal{J}_{\mathfrak{B}}^{-1}] \xrightarrow{\simeq} \mathcal{D}\mathrm{isks}_{/X} \tag{7.9}$$

is an equivalence of ∞-categories. The reason for generalizing the statement to allow for a sufficiently nice poset of opens is that it can then be leveraged in the context of left Kan extending (pre)factorization algebras. For this one needs that the poset of opens is closed under intersections, i.e. a presieve, which $\mathrm{disk}(X)$ is not.

Recall from Notation 6.16 that $\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}$ is the ∞-category of (finite) disjoint unions of conical marked disks and embeddings which are bijective on marked points. The statement we prove in the marked setting is the following.

THEOREM 7.25. *Let X be a smooth conical maximally marked manifold and let $\mathfrak{B} \subseteq \mathrm{disk}(X)_{\mathrm{mrk}}$ be a decomposable multiplicative disk-basis of X. Let $V \in \mathrm{open}\,(X)_{\mathrm{mrk}}$ be any object. Then the induced map*

$$\big(\mathfrak{B} \cap V_{\downarrow_{\mathrm{m}}}\big)\,[(\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}})^{-1}] \xrightarrow{\simeq} \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/V} \tag{7.10}$$

is an equivalence of ∞-categories.

Remark 7.26. We want the same applications as in [**KSW24**], and the above modified statement is accounting for the indexing category when (operadic) left Kan extending being different when working with marked inclusions. Explicitly, the left Kan extension of a copresheaf $\mathcal{A}$ on $\mathfrak{B}$ to some $\mathfrak{U} \supseteq \mathfrak{B}$ is pointwise computed by $\mathrm{colim}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}} \mathcal{A}$ for each $V \in \mathfrak{U}$ (while in the ordinary setting one replaces $V_{\downarrow_{\mathrm{m}}}$ with $V_{\downarrow}$).

In the unmarked setting the statement of Theorem 7.25 is a consequence of the equivalence (7.9) from [**KSW24**, Theorem 5.10]: for any $V \in \mathrm{open}(X)$ and $\mathfrak{B} \subset \mathrm{open}(X)$ some decomposable multiplicative disk-basis of X it follows that $\mathfrak{B} \cap V_{\downarrow}$ is a decomposable multiplicative disk-basis of V. Thus, it follows that $\mathfrak{B} \cap V_{\downarrow}[(\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow}})^{-1}] \xrightarrow{\simeq} \mathcal{D}\mathrm{isks}_{/V}$.

Before we are ready to prove this statement we set up some notation and preliminary results mostly using the theory developed in Section 6.

Assumption 7.27. For the remainder of this section we assume that X is always *maximally marked* (Definition 4.22).

Notation 7.28.

(1) Every object $U \in \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}$ can be written as an external disjoint union
$$U = \coprod_{i \in I} k_i \times U_i$$
of pairwise non-isomorphic marked conical disks U_i, each appearing k_i times. We suppress the marked points from the notation.

(2) We can decompose every object $U \in \mathrm{disk}(X)_{\mathrm{mrk}}$ as an internal disjoint union
$$U = \dot{\bigcup_{i \in I}} \dot{\bigcup_{j=1}^{k_i}} U_i^j$$
where the U_i^j are marked conical disks. If two marked conical disks have the same lower index "i" they are abstractly isomorphic. For an example of this, see Figure 6.

(3) In either case we write $[U]$ for the isomorphism type of such a disjoint union of marked conical disks.

(4) For each connected marked disk D, recall from Definition 6.30 that $X_{[D^{\mathrm{mrk}},D]} \subset X$ denotes the subspace of those points whose local neighbourhoods have type D. Each $X_{[D^{\mathrm{mrk}},D]}$ is an (unstratified) manifold that is locally closed in X. Moreover, we have a decomposition $X = \dot{\cup}_{[D]} X_{[D^{\mathrm{mrk}},D]}$ since local neighbourhood types are unique (Remark 6.28).

(5) Given an isomorphism type $[U]$, we write $\mathcal{J}_{\mathfrak{B}}^{[U]} \subseteq \mathcal{J}_{\mathfrak{B}}$ for the full poset consisting only of those disjoint unions of disks which are of that type.

Remark 7.29. Since X is always assumed to be maximally marked here we abbreviate $X_{[D^{\mathrm{mrk}},D]}$ by $X_{[D]}$. This is unambiguous for the following reason: If D contains 0-dimensional strata we know that only the marked version will appear in X. Hence we have $X_{[\emptyset,D]} = \emptyset$ and $X_{[*,D]} = X_{[D]}$, where the latter is the ordinary unmarked version recalled in Definition 6.29.

Let D be a connected marked conical disk, and $X_{[D]}$ be the corresponding subspace of points with local neighbourhood of type D. For a pointless factorization algebra $\mathcal{A}$ on X that is constructible, the size of the conical disks (say of type D) are irrelevant and one can isotope to get arbitrary small such conical disks. Moreover, it turns out that it is actually sufficient to remember the origin of the conical disk, and how it is allowed to be isotoped along the corresponding subspace $X_{[D]}$. This is the heuristics behind the equivalence
$$\left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/V}\right)^{\simeq} \simeq \coprod_{[\widetilde{U}]} \prod_{i \in I} \mathrm{Conf}_{k_i}(V_{[\widetilde{U}_i]})$$

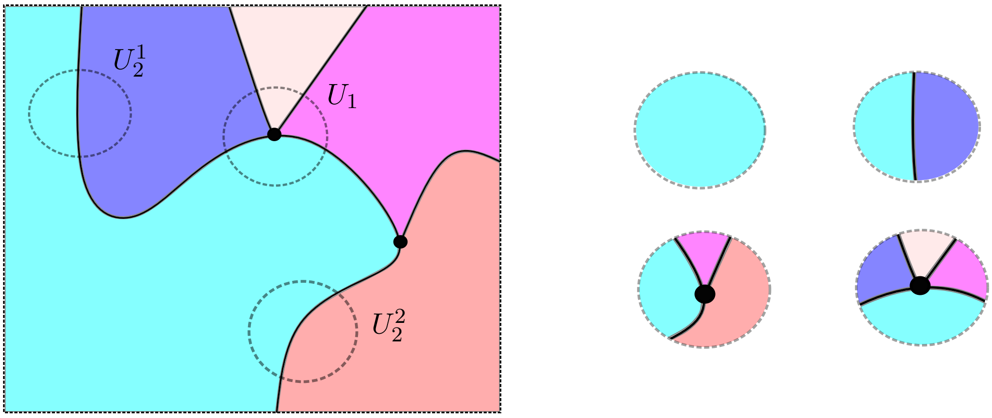


FIGURE 6. The leftmost figure is $X = \mathbb{R}^2$, where we consider both 0-dimensional strata to be marked. It also shows an object $U \in \operatorname{disk}(X)_{\mathrm{mrk}}$ and its decomposition $U = U_1 \cup U_2^1 \cup U_2^2$. The space X decomposes into four different pieces $X_{[D]}$, where D ranges over the types indicated on the right. Both basics D with 0-dimensional strata, i.e. on the bottom row, are marked.

established in Lemma 6.39. The coproduct runs over all isomorphism classes $[\widetilde{U}] = [\coprod k_i \times \widetilde{U}_i]$ with "the same marked points as V" as in Definition 6.37. Moreover, recall that $\operatorname{Conf}_{k_i}(V_{[\widetilde{U}_i]})$ refers to the unordered configuration space (Notation 6.34).

In a similar spirit to the above results we also want to express (the classifying space) of the left hand expression of Equation (7.10) in terms of configuration spaces. To that end we have the following lemma, which is a marked version of [**KSW24**, Lemma 5.15].

Lemma 7.30. *Fix a conically smooth maximally marked manifold X and let $\mathfrak{B} \subseteq \operatorname{disk}(X)_{\mathrm{mrk}}$ be a decomposable multiplcative disk-basis of X. Let $V \in \operatorname{open}(X)_{\mathrm{mrk}}$ and fix an isomorphism type $[\widetilde{U}] = [\coprod_{i\in I} k_i \times \widetilde{U}_i]$ in $\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}$ with the same marked points as V. Then we have a canonical equivalence of ∞-groupoids*

$$\left|\mathcal{J}^{[\widetilde{U}]}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}\right| \simeq \prod_{i\in I} \operatorname{Conf}_{k_i}(V_{[\widetilde{U}_i]}) \ .$$

PROOF. Let $B = \bigcup_{i,j} B_i^j \in \mathcal{J}^{[\widetilde{U}]}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}$, i.e. it is an object in $\mathfrak{B}$ of the type $[\widetilde{U}]$, which has a marked inclusion into V. Such an object exists because the isomorphism type $[\widetilde{U}]$ has the same marked points as V. Using the object B we define an open subset of $\prod_{i\in I} \operatorname{Conf}_{k_i}(V_{[\widetilde{U}_i]})$:

$$\operatorname{Conf}_{=}(B) \coloneqq \left\{ \left(S_i \in \operatorname{Conf}_{k_i}(V_{[B_i]})\right)_{i\in I} \,\middle|\, \forall i', i \in I,\, \forall 1 \le j \le k_i \colon \#(S_{i'} \cap B_i^j) = \delta_{i'i} \right\} \ .$$

In words, this consists of tuples of configurations $(S_i)_{i\in I}$ of points of local neighbourhood type $[\widetilde{U}_i]$ which induces a bijection $\pi_0(S_i) \xrightarrow{\cong} \pi_0(B_i)$. By definition of the (unordered) configuration space the points S_i are all pairwise disjoint. This assignment assembles into a functor

$$\operatorname{Conf}_{=}(-) \colon \mathcal{J}^{[\widetilde{U}]}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}} \longrightarrow \operatorname{open}\left(\prod_{i\in I} \operatorname{Conf}_{k_i}(V_{[\widetilde{U}_i]})\right); B \mapsto \operatorname{Conf}_{=}(B) \ .$$

We now want to apply Lurie's Seifert-van Kampen theorem ([**Lur17**, Theorem A.3.1]) to the functor $\mathrm{Conf}_{=}(-)$. First we need to establish that the condition of the theorem holds, i.e. for each $S := (S_i = \{s_i^1, \dots, s_i^{k_i}\})_i \in \prod_{i\in I} \mathrm{Conf}_{k_i}(V_{[\widetilde{U}_i]})$ we need to prove that the full subposet

$$\mathrm{K}_S := \{B \in \mathcal{J}^{[\widetilde{U}]}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}} \mid S \in \mathrm{Conf}_{=}(B)\} \subseteq \mathcal{J}^{[\widetilde{U}]}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}$$

is weakly contractible.

Claim: The poset K_S is cofiltered and hence weakly contractible.

- We first show that K_S is non-empty. Choose some open subset $W \subset V \subset X$ containing S. Note that by default W then contains all the marked points of V; they are in S to begin with because of the constraint on the isomorphism type $[\widetilde{U}]$. Using that $\mathfrak{B}$ is a disk-basis (of X) we can, for each $s_i^j \in S$, find an $B_i^j \in \mathfrak{B}$ with $s_i^j \in B_i^j$. Since S is a set of pairwise disjoint points and X is Hausdorff we can further assume that the B_i^j are all pairwise disjoint. Moreover, since $\mathfrak{B}$ is decomposable we can assume each B_i^j to be connected and B_i^j is of type $[\widetilde{U}_i]$ since that is the local type around s_i^j. From multiplicativity of $\mathfrak{B}$ it follows that the disjoint union $B := \sqcup_{i,j} B_i^j \hookrightarrow^{\mathrm{m}} W$ lies in K_S making the poset non-empty.
- We now show that given two objects $B, B' \in \mathrm{K}_S$ there exists a third object $W \in \mathrm{K}_S$ with maps (in K_S) into both. By applying the previous argument to $W = B \cap B'$ we obtain some $B'' \in \mathrm{K}_S$ such that $B'' \hookrightarrow^{\mathrm{m}} B$ and $B'' \hookrightarrow^{\mathrm{m}} B'$. These morphisms exists in $\mathfrak{B}$, but since $\mathrm{K}_S \subset \mathfrak{B}$ is *not* full, we need to show that these marked inclusions are in K_S. By construction, the (marked) inclusion $B'' \hookrightarrow^{\mathrm{m}} B$ induce bijections $\bigsqcup_i \pi_0(S_i) \xrightarrow{\cong} \pi_0(B'') \xrightarrow{\cong} \pi_0(B)$, and for each i, j the corresponding connected components $B_i''^j$ and B_i^j have a common point s_i^j in their deepest strata. Thus, by Theorem 6.25 it implies that the connected components are abstractly isomorphic. Hence, the marked inclusion $B'' \hookrightarrow^{\mathrm{m}} B$ lies in $\mathcal{J}_{\mathfrak{B}}$ and thus also in K_S, which is a full subposet thereof. The exact same argument applies to $B'' \hookrightarrow^{\mathrm{m}} B'$.

Applying the Seifert-van Kampen theorem gives the equivalence

$$\left|\mathcal{J}^{[\widetilde{U}]}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}\right| = \operatorname*{colim}_{B\in \mathcal{J}^{[\widetilde{U}]}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}} \{\star\} \xleftarrow{\simeq} \operatorname*{colim}_{B\in \mathcal{J}^{[\widetilde{U}]}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}} \mathrm{Conf}_{=}(B) \xrightarrow{\simeq} \prod_{i\in I} \mathrm{Conf}_{k_i}(V_{[\widetilde{U}_i]}) \;, \tag{7.11}$$

where the leftmost equivalence follows from $\mathrm{Conf}_{=}(B)$ being homeomorphic to $\prod_{i,j} B_i^j$ and hence contractible. ☐

Remark 7.31. For any isomorphism type $[U]$ which does not have the same marked points as V, the category $\mathcal{J}^{[U]}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}$ will be empty, which is why it is crucial that we exclude such isomorphism types. The reader should think of this as being an incarnation of how the enriched hom $\mathcal{M}\mathrm{fld}^{\mathrm{mrk,bij}}(U,V)_{|\Delta}$ is empty whenever U does not have the same marked points as V, c.f. (the proof of) Lemma 6.14 and Remark 6.15.

We now turn towards the proof of Theorem 7.25 using the same approach as in [**KSW24**, Theorem 5.10].

Proof of Theorem 7.25. By Theorem 6.25 we know that $\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}$ is exactly the collection of arrows in $\mathfrak{B}$ which are sent to equivalences in $\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/V}$. Hence, the statement follows if we can apply Lemma A.35 to the functor

$$F\colon D := \mathfrak{B} \cap V_{\downarrow_{\mathrm{m}}} \longrightarrow \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/V} =: E$$

with $W := \mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}$.

We need to check the two assumptions of the lemma:

(1) The first condition is that F should induce an equivalence $\left|\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}\right| \to \left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/V}\right)^{\simeq}$ of ∞-groupoids. For this, consider the commuting diagram

$$\begin{tikzcd}
 & \mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}} \arrow[dl, dashed] \arrow[dr] & \\
\operatorname*{colim}_{B\in\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}} \left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/B}\right)^{\simeq}_{\mathrm{term}} \arrow[r, hook] \arrow[d, dashed, "\simeq"] & \operatorname*{colim}_{B\in\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}} \left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/B}\right)^{\simeq} \arrow[r] \arrow[d, "\simeq"] & \left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/V}\right)^{\simeq} \arrow[d, "\simeq"] \\
\operatorname*{colim}_{B\in\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}} \mathrm{Conf}_{=}(B) \arrow[r, hook] \arrow[rr, bend right, "\simeq"'] & \operatorname*{colim}_{B\in\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}} \coprod_{[\widetilde{U}]}\prod_{i\in I} \mathrm{Conf}_{k_i}(B_{[\widetilde{U}_i]}) \arrow[r] & \coprod_{[\widetilde{U}]}\prod_{i\in I} \mathrm{Conf}_{k_i}(V_{[\widetilde{U}_i]})
\end{tikzcd}$$

where:

- $(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/B})_{\mathrm{term}}$ is the full subcategory of the slice consisting only of those objects where the structure map $D \hookrightarrow B$ is an equivalence; i.e. the subgroupoid of terminal objects.
- The right diagonal map is explicitly given as $B \mapsto (B \hookrightarrow V)$, so it tautologically factors as the dashed left diagonal map $B \mapsto (B, B \overset{\mathrm{id}}{\hookrightarrow} B)$.
- The lower left horizontal map is given for each B by the inclusion $\mathrm{Conf}_{=}(B) \hookrightarrow \prod_{i\in I} \mathrm{Conf}_{k_i}(B)$ into the summand corresponding to the type $[\widetilde{U}] = [B]$. Since we start with an open $B \in \mathfrak{B} \cap V_{\downarrow_{\mathrm{m}}}$ we automatically have that $[B]$ is an isomorphism type with the same marked points as V.
- The middle and right vertical equivalences are those of Lemma 6.39.
- The lower horizontal composite equivalence is the (rightmost) equivalence of Equation (7.11).
- The left vertical dashed map exists because if $D \hookrightarrow B$ is an equivalence then every choice of points S_i as in Remark 6.36 (and Remark 6.40) gives a tuple of configurations that lies in $\mathrm{Conf}_{=}(B)$.
- the dashed vertical map is an equivalence since both $\left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/B}\right)_{\mathrm{term}}$ and $\mathrm{Conf}_{=}(B)$ are contractible.

Now observe that since every $\left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/B}\right)^{\simeq}_{\mathrm{term}}$ is contractible we have that

$$\left|\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}\right| \simeq \operatorname*{colim}_{B\in\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}} * \simeq \operatorname*{colim}_{B\in\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}} \left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/B}\right)^{\simeq}_{\mathrm{term}} .$$

It follows that the dashed diagonal map presents the localization $\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}} \to \left|\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}\right|$ to the classifying space. We conclude that the map $\left|\mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_{\mathrm{m}}}}\right| \to \left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/V}\right)^{\simeq}$ is presented by the upper horizontal composite which, by the diagram, is a composite of equivalences.

(2) For each $B \in \mathfrak{B} \cap V_{\downarrow_\mathrm{m}}$ we have to check that the functor F induces an equivalence

$$\left| \mathcal{J}_{\mathfrak{B}\cap V_{\downarrow_\mathrm{m}}} \times_{\mathfrak{B}\cap V_{\downarrow_\mathrm{m}}} (\mathfrak{B} \cap V_{\downarrow_\mathrm{m}})_{/B} \right| \longrightarrow \left(\left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/V} \right)_{B \hookrightarrow V} \right)^{\simeq} .$$

But this map is canonically equivalent to

$$\left| \mathcal{J}_{\mathfrak{B}\cap B_{\downarrow_\mathrm{m}}} \right| \longrightarrow \left(\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/B} \right)^{\simeq}$$

which is an equivalence by the previously proven part 1 by setting $V = B$. □

Some of the consequences of the localization-result in [**KSW24**] follow immediately from the version of the localization-result established in Theorem 7.25. For example, let $\mathfrak{U}$ be any poset of marked opens and $\iota\colon \mathfrak{B} \subset \mathfrak{U}$ a decomposable multiplicative disk-basis of $\mathfrak{U}$. As in [**KSW24**, Corollary 5.17] it follows that the operadic left Kan extension $\iota_!\mathcal{A}$ of a constructible multiplicative pointless prefactorization algebra $\mathcal{A}$ on $\mathfrak{B}$ is constructible on $\mathfrak{U}$. If in the situation above we have $\mathfrak{U} = \mathrm{disk}(X)_{\mathrm{mrk}}$ we get that the restriction along ι induces an equivalence

$$\mathrm{Alg}^{\mathrm{m,cstr}}_{\mathrm{disk}(X)_{\mathrm{mrk}}} \xrightarrow{\simeq} \mathrm{Alg}^{\mathrm{m,cstr}}_{\mathfrak{B}}$$

between multiplicative and constructible prefactorization algebras on $\mathrm{disk}(X)_{\mathrm{mrk}}$ and $\mathfrak{B}$ by [**KSW24**, Corollary 5.18]. In addition the statement of [**KSW24**, Proposition 5.16] also immediately carries over.

Remark 7.32. We now comment on the part of the results in [**KSW24**, §5.2] which we are not able to translate to the marked setting and why. For example, the localization (7.9) is shown to generalize to a localization of the corresponding ∞-operads in [**KSW24**, Proposition 5.19]. As a consequence one gets an equivalence

$$\mathrm{Alg}^{\mathrm{m}}_{\mathcal{D}\mathrm{isks}_{/X}} \xrightarrow{\simeq} \mathrm{Alg}^{\mathrm{m,cstr}}_{\mathfrak{B}} ,$$

where as above $\mathfrak{B} \subseteq \mathrm{disk}(X)$ is a multiplicative decomposable disk-basis of X ([**KSW24**, Corollary 5.20]).

However, the way $\mathcal{D}\mathrm{isks}_{/X}$ is equipped with the structure of an ∞-operad goes through the symmetric monoidal structure $\coprod$ on $\mathcal{M}\mathrm{fld}$, and heavily uses that $\emptyset$ is an initial object with respect to this symmetric monoidal structure. See [**AFT17a**, Corollary 1.20] for details. Since there is no analogue of this for the marked (and bijective on marked points) setting, it is not so clear how to make $\mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/X}$ into an ∞-operad, and we do not pursue this further in this thesis.

7.3. Constructible implies marked Weiss on marked disks. It has long been folklore that constructible factorization algebras extended from disks automatically satisfy the Weiss condition. A first precise statement of this form appeared in [**AF20a**] in the unstratified setting, i.e. for locally constant factorization algebras. This was generalised in [**KSW24**] to the stratified setting and constructible factorization algebras, as well as allowing for a more general poset of opens than all disks. We follow the latter here in the quest of getting an analogous statement in the setting of constructible pointless factorization algebras.

The main result of this section is that [**KSW24**, Theorem 5.21] carries over to the marked setting:

THEOREM 7.33. *Let X be a smooth conical maximally marked manifold and let $\mathfrak{B} \subseteq \operatorname{disk}(X)_{\mathrm{mrk}}$ be a decomposable multiplicative disk-basis of X. Every left Kan extension of a constructible copresheaf on $\mathfrak{B}$ to any poset of opens $\mathfrak{U} \supseteq \mathfrak{B}$ is automatically a marked Weiss hypercosheaf.*

The following corollary is the one we actually rely on when proving that constructible pointless factorization algebras glue just like the ordinary ones.

Corollary 7.34. [**KSW24**, Corollary 5.22] *Let X be a smooth conical maximally marked manifold, $\mathfrak{B} \subseteq \operatorname{disk}(X)_{\mathrm{mrk}}$ a decomposable multiplicative disk-basis of X and $\mathfrak{U} \supseteq \mathfrak{B}$ a presieve. Then for every constructible copresheaf on $\mathfrak{B}$ the left Kan extension to $\mathfrak{U}$ is a marked Weiss cosheaf.*

PROOF. This is immediate from Theorem 7.33 because every marked Weiss hypercosheaf on a presieve is a marked Weiss cosheaf; see e.g. Remark 3.25. □

Remark 7.35. Note that the above two results implicitly asks for X to have enough good marked disks by asking for the existence of a decomposable multiplicative disk-basis $\mathfrak{B}$, as in [**KSW24**]. This is not present in the unstratified, unmarked and $\mathfrak{B} = \operatorname{disk}(X)$ version of [**AF20a**, Proposition 2.22], c.f. [**KSW24**, Remark 5.25] for some more details. However, essentially the same result (i.e. in the unstratified and unmarked setting) was obtained in [**Sán23**], also with an assumption on existence of enough good disks.

As in the previous subsection some of the corollaries of [**KSW24**] immediately carries over to the pointless setting while some do not. Explicitly, let X be a smooth conical maximally marked manifold with enough good marked disks. If $\mathfrak{U} \subseteq \operatorname{open}(X)_{\mathrm{mrk}}$ is a presieve and $\mathfrak{B} \subseteq \mathfrak{U}$ is a decomposable factorizing disk-basis of $\mathfrak{U}$ we have by [**KSW24**, Corollary 5.23] that restriction yields an equivalence

$$\mathrm{FACT}^{\mathrm{pl,cstr}}_{\mathfrak{U}} \xrightarrow{\simeq} \mathrm{FACT}^{\mathrm{pl,cstr}}_{\mathfrak{B}} = \mathrm{Alg}^{\mathrm{m,cstr}}_{\mathfrak{B}}$$

of ∞-categories. Moreover, [**KSW24**, Corollary 5.24] gives that restriction yields an equivalence

$$\mathrm{FACT}^{\mathrm{pl,cstr}}_{X} \xrightarrow{\simeq} \mathrm{Alg}^{\mathrm{m,cstr}}_{\operatorname{disk}(X)_{\mathrm{mrk}}}$$

of ∞-categories. However, we stress that [**KSW24**, Corollary 5.26] does *not* immediately translate to the setting here since we do not have an analogue of [**KSW24**, Corollary 5.20] as explained in Remark 7.32. We are now done with recording the consequences of Theorem 7.33 so it is time to turn to its proof.

PROOF OF THEOREM 7.33. Let $\mathcal{A}\colon \mathfrak{U} \to \mathcal{C}$ be a copresheaf left Kan extended from $\mathfrak{B}$, i.e. $\mathcal{A} \simeq \iota_! \mathcal{A}_{|\mathfrak{B}}$, where moreover $\mathcal{A}_{|\mathfrak{B}}$ is constructible. Since $\mathcal{A}_{|\mathfrak{B}}$ is constructible it follows that for each $U \in \mathfrak{U}$ it factors as

$$\mathcal{A}_{|\mathfrak{B} \cap U_{\downarrow_{\mathrm{m}}}} : \mathfrak{B} \cap U_{\downarrow_{\mathrm{m}}} \longrightarrow \mathfrak{Disks}^{\mathrm{mrk,bij}}{}_{/U} \longrightarrow \mathcal{C}\ , \tag{7.12}$$

where the first functor is a localization by Theorem 7.25. We need to prove that every marked Weiss hypercover $\mathfrak{W} \to^{\mathrm{m}}_{\mathrm{h}} \{M\}$, for a marked open $M \in \mathfrak{U}$, is $\mathcal{A}$-local. For this we want to apply Lemma A.33 (in particular $2 \Rightarrow 1$) to

- the full ∞-subcategory $\mathcal{D} := \mathfrak{Disks}^{\mathrm{mrk,bij}} \subset \mathfrak{Mfld}^{\mathrm{mrk,bij}}$ of smooth conical marked disks,

- and the diagram
$$F\colon \mathfrak{W}^{\triangleright} \xrightarrow{\infty\mapsto M} \mathfrak{W}\cup\{M\} \to \mathrm{RFib}(\mathcal{D}), \qquad W \mapsto \big(\mathcal{D}_{/W} \to \mathcal{D}\big)$$
of right fibrations.

We now aim to establish condition 2 of Lemma A.33. By Lemma 6.39 the map (A.4) is identified with the map of ∞-groupoids
$$\operatorname*{colim}_{W\in\mathfrak{W}} \prod_{i\in I} \mathrm{Conf}_{k_i}(W_{[U_i]}) \longrightarrow \prod_{i\in I} \mathrm{Conf}_{k_i}(M_{[U_i]}) \tag{7.13}$$
induced by the hyperprecover
$$\left\{\prod_{i\in I} \mathrm{Conf}_{k_i}(W_{[U_i]}) \mid W\in\mathfrak{W}\right\} \to^{\mathrm{h}} \prod_{i\in I} \mathrm{Conf}_{k_i}(M_{[U_i]}) \tag{7.14}$$
for each isomorphism class $U = \bigcup_{i,j} U_i^j$ (as in Notation 7.28) where $U \in \mathfrak{B}\cap M_{\downarrow_{\mathrm{m}}}$.

Since $\mathfrak{W} \to^{\mathrm{m}}_{\mathrm{h}} \{M\}$ is a marked Weiss hypercover it follows that (7.14) is a hypercover. Hence, by [**DI04**, Proposition 4.6 (c)][17] we get that (7.13) is an equivalence of spaces. Consider the commuting square of ∞-categories:
$$\begin{array}{ccc} \operatorname*{colim}_{W\in\mathfrak{W}} \mathfrak{B}\cap W_{\downarrow_{\mathrm{m}}} & \longrightarrow & \operatorname*{colim}_{W\in\mathfrak{W}} \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/W} \\ \big\downarrow & & \big\downarrow \\ \mathfrak{B}\cap M_{\downarrow_{\mathrm{m}}} & \longrightarrow & \mathcal{D}\mathrm{isks}^{\mathrm{mrk,bij}}{}_{/M} \end{array}$$
Having established condition 2 of Lemma A.33 it follows from 1 that the right vertical functor is colimit cofinal. Moreover, by setting $U \coloneqq M$ and $U \coloneqq W$ in (7.12) and passing to colimits of ∞-categories, the two horizontal maps are localizations which $\mathcal{A}$ factors through (uniquely).

Upon passing to colimits in $\mathcal{C}$ it follows that the right vertical map of the commuting diagram below is an equivalence:
$$\begin{array}{ccccc} \operatorname*{colim}_{W\in\mathfrak{W}} \mathcal{A}(W) & \xleftarrow{\simeq} & \operatorname*{colim}_{W\in\mathfrak{W}} \operatorname*{colim}_{\mathfrak{B}\cap W_{\downarrow_{\mathrm{m}}}} \mathcal{A}_{|\mathfrak{B}} & = & \operatorname*{colim}_{\operatorname{colim}_{W\in\mathfrak{W}} \mathfrak{B}\cap W_{\downarrow_{\mathrm{m}}}} \mathcal{A}_{|\mathfrak{B}} \\ \big\downarrow & & & & \big\downarrow{\simeq} \\ \mathcal{A}(M) & & \xleftarrow[\simeq]{\qquad\qquad} & & \operatorname*{colim}_{\mathfrak{B}\cap M_{\downarrow_{\mathrm{m}}}} \mathcal{A}_{|\mathfrak{B}}\,. \end{array}$$
The two horizontal maps are equivalences because $\mathcal{A}$ is a left Kan extension of its restriction to $\mathfrak{B}$. It follows that the left vertical map is an equivalence which is what we wanted to show. $\square$

[17] Note that our notion of hypercovers corresponds to their complete covers.

Part 2

The pointless higher Morita categories and dualizability

Attribution. Using the new notion of constructible *pointless* factorization algebras from Part 1 we, in this part, construct the *pointless* higher Morita categories $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$. This construction is analogous to Scheimbauer's construction of the pointed higher Morita categories $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ which uses constructible ordinary factorization algebras [**Sch14, GS18**], simply replacing the ∞-category of constructible ordinary factorization algebras $\mathrm{FACT}_X^{\mathrm{cstr}}$ with the ∞-category of constructible *pointless* factorization algebras $\mathrm{FACT}_X^{\mathrm{pl,cstr}}$. This works by the results on gluing of constructible pointless factorization algebras and symmetric monoidality established in Part 1.

Since $\mathrm{FACT}_X^{\mathrm{cstr}}$ is a symmetric monoidal subcategory of $\mathrm{FACT}_X^{\mathrm{pl,cstr}}$ we can transfer the n-dualizability-results from [**GS18**] for Scheimbauer's pointed higher Morita categories to the pointless version. However, in the pointless version $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$, we can also prove interesting $(n+1)$-dualizability results, which are not available in the pointed version $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ due to [**GS18**, Theorem 5.1]. The new contribution here was in finding a suitable modification of the definition of a constructible factorization algebra, i.e. the pointless ones from Part 1. We also provide a partial proof of a conjecture by Lurie in Section 10.2 regarding $(n+1)$-dualizability in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ for $n = 2$, which is only previously proven for $n = 1$, and $n = 2$ when $\mathcal{C} = \mathrm{Pr}$.

Assumption. Throughout this part of the thesis we fix an $\otimes$-presentable symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$ (as in Part 1).

8. Ingredients for the pointless higher Morita categories

In this section we consider the general building blocks needed to construct the pointless higher Morita categories from constructible pointless factorization algebras. We start with a brief review of the relevant marked stratified spaces in Section 8.1. Then we introduce a number of important maps between marked stratified spaces and prove that constructible pointless factorization algebras indeed push forward along these in Section 8.2. In particular, the maps introduced here are the key ingredients when defining composition and duals in the higher Morita categories.

8.1. Nice (marked) stratified spaces. We give a brief introduction to the stratified (marked) spaces relevant for the (pointless) higher Morita categories, following [**GS18**]. Note that all the stratified (marked) spaces considered here are particularly nice instances of conical (marked) manifolds, and even smooth conical (marked) manifolds, as defined in Section 4.2.

The most important types of stratifications needed in this part of the thesis comes from "affine flags", and the following definition gives the prototype of these.

Definition 8.1. The *standard affine flag* is given by

$$\{(\tfrac{1}{2},\dots,\tfrac{1}{2})\} \times (0,1)^{n-k} \subset \{(\tfrac{1}{2},\dots,\tfrac{1}{2})\} \times (0,1)^{n-k+1} \subset \cdots$$
$$\cdots \subset \{(\tfrac{1}{2})\} \times (0,1)^{n-1} \subset (0,1)^n$$

for $0 \leq k \leq n$. This flag will be the local picture of our stratifications so we call it the *standard stratified disk of index* k.

Remark 8.2. By the terminology in Section 4.2 the affine flags are nothing but certain basics $\mathbf{D}(Z,k) \coloneqq \mathbf{C}Z \times \mathbb{R}^k$, where $\mathbf{C}Z$ is the cone of Z as in Construction 4.24. In fact, for a fixed n one can see that a standard stratified disk of index k corresponds to the cone of $Z = \mathbf{S}^{n-k-1}$ stratified by the spheres of strictly smaller dimension.

Example 8.3. When $n = 1$ the standard stratified disk of index 1 corresponds to the line $\mathbf{D}(\emptyset, 1) = \mathbb{R}^1 \cong (0,1)$, while that of index 0 gives the line $(0,1)$ with a point at $\{\frac{1}{2}\}$. The latter is equivalent to $\mathbf{D}(Z,0)$ for $Z = \mathbf{S}^0$.

When $n = 2$ we have three standard stratified disks as depicted in Figure 7. Read from left to right these are $\mathbf{D}(\emptyset, 2)$, $\mathbf{D}(Z,1)$ with $Z = \mathbf{S}^0$ and $\mathbf{D}(W,0)$ where $W = \mathbf{S}^1$ stratified by $\mathbf{S}^0$, i.e. a point at the north and the south pole.

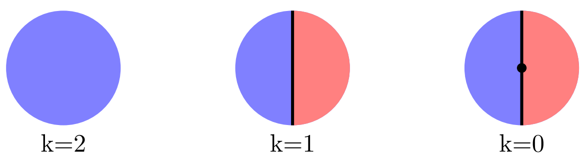


FIGURE 7. The three standard stratified disks for $n = 2$.

The colours in the above figure, as well as in subsequent illustrations, are used to easier distinguish the different connected components. We now restrict ourselves to only smooth conical manifolds (c.f. Definition 4.32) which are of a particularly easy form.

Definition 8.4. We say that a smooth conical manifold $X = (0,1)^n$ is a *nice stratified space* if it has a stratifying poset of the form

$$P \coloneqq \left\{ \emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = (0,1)^n \right\}$$

such that for every point $x \in X_i \setminus X_{i-1}$, there is an open neighbourhood of x that is abstractly isomorphic to a standard stratified disk of index i. Moreover, we assume that each X_i has a finite number of connected components and is a smooth closed i-dimensional submanifold of $(0,1)^n$. In this situation we also say that X is equipped with a *nice stratification*, and we call $X_i \setminus X_{i-1}$ for the *i-dimensional strata* (of X).

Note that the strata $X_i \setminus X_{i-1}$ coincides with the $[U]$-strata of X from Definition 6.29 for $[U]$ the standard stratified disk of index i.

Example 8.5. Giving a nice stratification for $n = 1$ amounts to giving a finite number of points in $(0,1)$. For example, a nice stratification consisting of two points s_1, s_2 can look as in Figure 8 below.

FIGURE 8. Example of a nice stratification of $(0,1)$.

Example 8.6. For $n = 2$ such a stratification amounts to the unit square with an embedded 1-manifold X_1 together with a finite collection of points X_0. The left stratified space of Figure 9 is an example of a nice stratified space. However, the right stratified space is *not* an example. It fails because two of the stratified points do *not* have neighbourhoods which are standard stratified disks. However, it is a smooth conical manifold, offering a simple illustration of the fact that working with nice stratified spaces is very restrictive.

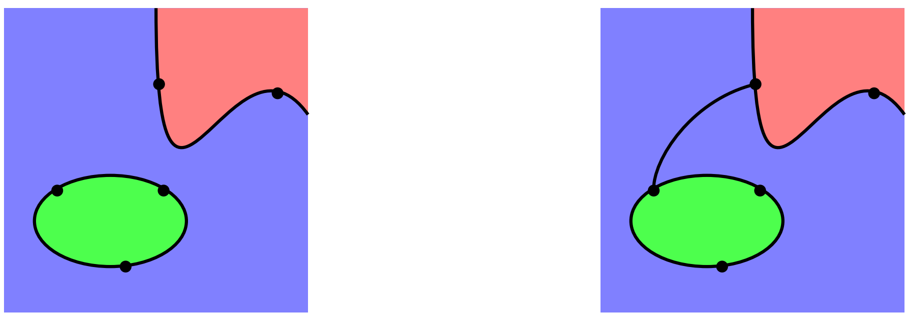


FIGURE 9. The leftmost stratified space is an example of a nice stratified space, while the rightmost is *not*.

Since we want to work with pointless factorization algebras we also need marked versions of the above. Giving a marking on X corresponds to giving a stratified open embedding $X^{\mathrm{mrk}} \to X$, where X^{mrk} is the finite collection of marked points. See e.g. Definition 4.39 and the surrounding discussion.

Definition 8.7. A *marked nice stratified space* X is a nice stratified space X which is marked by a (finite) collection of marked points X^{mrk}. Analogously, we say that a standard stratified disk is *marked* if it comes with a (possibly trivial) choice of marked points.

Recall that the only standard stratified disks that can be non-trivially marked are the ones of index $k = 0$, i.e. corresponding to pure cones $\mathbf{D}(Z, 0) = \mathbf{C}Z$, because they are the only ones with non-trivial 0-dimensional strata.

Definition 8.8. Let X be a (marked) nice stratified space. We say that an open disk D in X is a *basic (marked) stratified disk of index* k if D is abstractly isomorphic to a standard (marked) stratified disk of index k.

Remark 8.9. Every (marked) nice stratified space is in particular a conical marked manifold. Hence, it follows that every (marked) nice stratified space admits a (marked) Weiss cover by basic (marked) stratified disks as explained in Remark 4.31.

Assumption 8.10. From now on we will always assume that we are in the situation of Definition 8.4 (Definition 8.7) when simply saying "(marked) stratified space" and "(marked) stratification".

8.2. Maps between (marked) stratified spaces and pushforwards. We now turn to the relevant maps between stratified (marked) spaces. For example, we will consider "adequately stratified" maps, "collapse-and-rescale" maps and refinements. These are either maps that interact nicely with the stratification and the basic disks, or is of a particularly simple form. Here we prove that constructible pointless factorization algebras push forward along all of these maps.

We start with "adequately stratified" maps. For this we first need the following definition.

Definition 8.11. Let $f\colon X \to Y$ be a map of marked stratified spaces, i.e. a map of stratified spaces such that $f(X^{\mathrm{mrk}}) \subseteq Y^{\mathrm{mrk}}$. We say that f is a *locally trivial fibration* between marked stratified spaces if, for each point $y \in Y$, there exists an open (stratified) neighbourhood U of y together with an abstract isomorphism of marked stratified opens $f^{-1}(U) \cong U \times F$, where F is some marked stratified space and $U \times F$ is the marked product stratified space as in Example 4.21.

Note that f restricted to each $X_i \setminus X_{i-1}$ is a locally trivial fibration in the usual sense. We need some extra conditions on these types of maps to ensure that they behave well with respect to basic marked stratified disks. The following definition (modulo marked points) is due to [**Gin15**, §6.1]. [18]

Definition 8.12. Let $f\colon X \to Y$ be a locally trivial fibration between stratified marked spaces. We say that f is *adequately stratified* if Y has an open cover $\{V_i\}_{i\in I}$ by trivializing basic marked stratified disks such that

1. $f^{-1}(V_i) \overset{\psi}{\cong} V_i \times F$ has a cover by basic marked stratified disks (in X) of the form $\{\psi^{-1}(V_i \times D_j)\}_{j\in J}$,
2. for marked inclusions $T \hookrightarrow^{\mathrm{m}} U$, where T and U are basic marked stratified disks in V with the same index, we have a marked inclusion $\psi^{-1}(T \times D_j) \hookrightarrow^{\mathrm{m}} \psi^{-1}(U \times D_j)$ of basic marked stratified disks (of X) of the same index for all $j \in J$.

[18] Note that what Ginot calls "good neighborhoods", c.f. [**Gin15**, Definition 21], exactly boils down to being abstractly isomorphic to the same standard stratified disk.

Example 8.13. Some examples of adequately stratified maps used in this thesis are:

- locally trivial stratified marked fibrations, e.g. the map ∇ in Section 5.2
- fold maps which appear in Construction 10.5.
- (local) diffeomorphisms of stratified marked spaces.
- for $f\colon X \to Y$ any adequately stratified map it follows that $f \times \mathrm{id}\colon X \times (0,1) \to Y \times (0,1)$ (where $(0,1)$ is trivially stratified) is also adequately stratified.

The conditions placed on a locally trivial fibration to be adequately stratified are tailored to get the following lemma, which is a modification of [**Gin15**, Proposition 26].

Lemma 8.14. *Let $f\colon X \to Y$ be an adequately stratified map between marked stratified spaces. Pushforward along f induces a functor*

$$f_\star\colon \mathrm{F\small{ACT}}_X^{\mathrm{pl,cstr}} \longrightarrow \mathrm{F\small{ACT}}_Y^{\mathrm{pl,cstr}} .$$

Proof. We know from Lemma 4.54 that the pushforward preserves pointless factorization algebras, so we only need to check that $f_\star\mathcal{A}$ is constructible. Additionally, by Theorem 7.20 it is sufficient to check that $f_\star\mathcal{A}$ is constructible on some open cover of Y. We choose to work with the open cover $\{V_i\}_{i\in I}$ from Definition 8.12. Consider some marked inclusion $T \hookrightarrow^{\mathrm{m}} U$ of two abstractly isomorphic stratified disks both subordinate to some V_i. Recall that $f^{-1}(V_i) \cong V_i \times F$. Pick some marked Weiss cover $\mathcal{W}$ (in X) of F using the basic marked stratified disks D_j. Since $\mathcal{A}$ is constructible on X we know that each marked inclusion $\psi^{-1}(T \times D_j) \hookrightarrow^{\mathrm{m}} \psi^{-1}(U \times D_j)$ is sent to an equivalence in $\mathcal{C}$. It follows that the marked refinement $T \times \mathcal{W} \to^{\mathrm{m}} U \times \mathcal{W}$ is $\mathcal{A}$-local. Using that $\mathcal{A}$ is a marked Weiss cosheaf we get the desired equivalence:

$$f_\star\mathcal{A}(T) = \mathcal{A}(T \times F) \simeq \operatorname*{colim}_{T\times\mathcal{W}} \mathcal{A} \xrightarrow{\simeq} \operatorname*{colim}_{U\times\mathcal{W}} \mathcal{A} \simeq \mathcal{A}(U \times F) = f_\star\mathcal{A}(U) . \qquad \square$$

We now turn to the so-called "collapse-and-rescale"-maps. These are pivotal when defining composition in the higher Morita category.

Definition 8.15. [**Sch14**, Definition 3.1.1] Let $0 \le b \le a \le 1$ such that $(b,a) \neq (0,1)$. Let $\rho_a^b : [0,1] \to [0,1]$ be the piecewise linear map defined as follows: If $a = b$ let $\rho_a^b = \mathrm{id}_{[0,1]}$ and if $b < a$, let

$$\rho_a^b(x) = \begin{cases} \frac{x}{1-(a-b)}, & x \le b, \\ \frac{b}{1-(a-b)}, & b \le x \le a, \\ \frac{x-(a-b)}{a-(a-b)}, & a \le x, \end{cases}$$

By slight abuse of notation we also denote the restriction $\rho_a^b\colon (0,1) \to (0,1)$ by the same map and call this the *collapse-and-rescale* map. Moreover, if either a or b are marked points we require the corresponding image under ρ_a^b to be marked.

In words, this map "collapses" everything between b and a to the point $c \coloneqq \frac{b}{1-(a-b)}$ and rescales the compliment back to $(0,1)$. See Figure 10 for a depiction of the different cases.

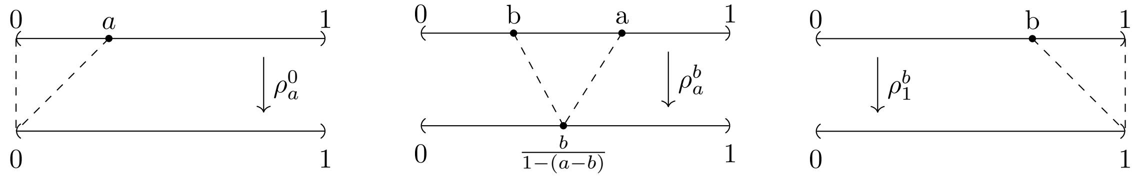


FIGURE 10. The collapse-and-rescale map illustrated for the three different cases; $b = 0$, $0 < b < a < 1$ and $a = 1$ when read from left to right.

This map is *not* adequately stratified in the case $0 < b < a < 1$ for the simple reason that the preimage of an interval containing $c = \frac{b}{1-(a-b)}$ contains both b and a and is hence *not* a basic stratified disk. Hence, we have to provide a separate argument for pushing forward along these maps.

Lemma 8.16. *Pushing forward along the collapse-and-rescale maps* $\rho \coloneqq \rho_a^b \colon (0,1) \to (0,1)$ *induces a functor of constructible pointless factorization algebras*

$$\rho_\star \colon \mathrm{FACT}^{\mathrm{pl,cstr}}_{(0,1)} \longrightarrow \mathrm{FACT}^{\mathrm{pl,cstr}}_{(0,1)} .$$

PROOF. We know from Lemma 4.54 that the pushforward preserves pointless factorization algebras, so we only need to check that $f_\star\mathcal{A}$ is constructible. Moreover, by Corollary 7.21 we only need to check constructibility away from the marked 0-dimensional strata at $c = \frac{b}{1-(a-b)}$. In this case we have that the preimage of any inclusion $U \hookrightarrow V$, where U and V are a connected interval not containing the marked point c, is an inclusion of the same type. By constructibility of $\mathcal{A}$ it immediately follows that $f_\star\mathcal{A}(U) \xrightarrow{\simeq} f_\star\mathcal{A}(V)$. $\square$

The above lemma can also be proven by building a convenient marked Weiss cover as in the proof of [**KSW24**, Lemma 5.5] (which is in fact used to get Corollary 7.21). This kind of approach is what we generalize when considering the collapse-and-rescale map together with copies of the identity map.

Definition 8.17. Let $\rho = \rho_{a^{k+1}}^{b^{k+1}}$ for some fixed $0 < b^{k+1} < a^{k+1} < 1$. Define

$$\rho^{(k+1)} \coloneqq \mathrm{id}^k \times \rho \times \mathrm{id}^l \colon (0,1)^{k+l+1} \longrightarrow (0,1)^{k+l+1} .$$

Here, the source $(0,1)^{k+l+1}$ is stratified by the flag

$$\emptyset \subseteq \left\{(a^1,\dots,b^{k+1}),(a^1,\dots,a^{k+1})\right\} \times (0,1)^l \subseteq \{(a^1,\dots,a^k)\} \times (0,1)^{l+1} \subseteq \cdots \subseteq (0,1)^{k+l+1}$$

where $0 < a^j < 1$, $j \in \{1,\dots,k\}$, while the target $(0,1)^{k+l+1}$ is stratified by

$$\emptyset \subseteq \{a^1,\dots,a^k,c^{k+1}\} \times (0,1)^l \subseteq \{a^1,\dots,a^k\} \times (0,1)^{l+1} \subseteq \cdots \subseteq (0,1)^{k+l+1}$$

where $c^{k+1} = \frac{b^{k+1}}{1-(a^{k+1}-b^{k+1})}$.

In words, this simply corresponds to the stratification having two l-dimensional hyperplanes in the x_{k+1}-direction which the copy of ρ collapses and rescales to obtain one l-dimensional hyperplane in the x_{k+1}-direction.

Example 8.18. We give an example of the map $\rho^{(k+1)}$ from Definition 8.17. If we set $k = 1$ and $l = 0$ we get a piece-wise linear map as illustrated in Figure 11, where $0 < a^1 < 1$ as well as $0 < b^2 < a^2 < 1$ are given in the figure. The dashed lines are *not* part of the stratification, and are only there to illustrate that the 0-dimensional strata are given by the intersection of the vertical line at $x_1 = a^1$ and the horizontal lines at $x_2 = b^2$, respectively $x_2 = a^2$.

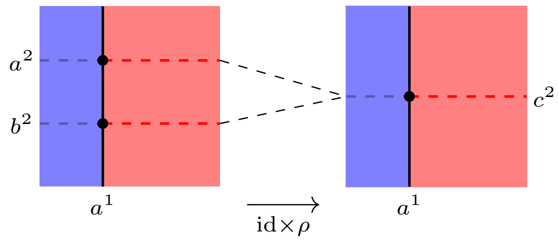


FIGURE 11. An example of $\rho^{(k+1)}$ for $k = 1$ and $l = 0$.

Lemma 8.19. *Let $\rho = \rho_{a^k}^{b^k}$ be a collapse-and-rescale map. Pushing forward along $\rho^{(k+1)} := \mathrm{id}^k \times \rho \times \mathrm{id}^l$ from Definition 8.17 induces a functor of constructible pointless factorization algebras*

$$(\rho^{(k+1)})_\star := (\mathrm{id}^k \times \rho \times \mathrm{id}^l)_\star \colon \mathrm{FACT}^{\mathrm{pl,cstr}}_{(0,1)^{k+l+1}} \longrightarrow \mathrm{FACT}^{\mathrm{pl,cstr}}_{(0,1)^{k+l+1}} . \tag{8.1}$$

PROOF. As before, we only need to check constructibility of the pushforward. Let $\mathcal{A} \in \mathrm{FACT}^{\mathrm{pl,cstr}}_{(0,1)^{k+l+1}}$, i.e. a constructible pointless factorization algebra in the left hand side of (8.1). The only non-trivial check is if we have a marked inclusion $U \hookrightarrow^{\mathrm{m}} V$ of two basic disks both non-trivially intersecting the hyperplane $c^{k+1} \times (0,1)^l$, so we assume this is the case. Let $\widetilde{U} := (\rho^{(k+1)})^{-1}(U)$ and $\widetilde{V} := (\rho^{(k+1)})^{-1}(V)$. Consider the marked Weiss cover of $\widetilde{V}$

$$\mathcal{W}_{\widetilde{V}} := \left\{ \widetilde{V} \setminus \{e \times (0,1)^l\} \mid e \in (b^{k+1}, a^{k+1}) \right\} .$$

This restricts to a marked Weiss cover of $\widetilde{U}$ by setting $\mathcal{W}_{\widetilde{U}} := \{W \cap \widetilde{U} \mid W \in \mathcal{W}_{\widetilde{V}}\}$. Let W_1 and W_2 denote the two connected components of any $W \in \mathcal{W}_{\widetilde{V}}$. Then we have

$$\mathcal{A}(W \cap U) \simeq \mathcal{A}(W_1 \cap U) \otimes \mathcal{A}(W_2 \cap U) \xrightarrow{\simeq} \mathcal{A}(W_1) \otimes \mathcal{A}(W_2) \simeq \mathcal{A}(W)$$

by multiplicativity and constructibility of $\mathcal{A}$. That is, the inclusion $W \cap U \to W$ is $\mathcal{A}$-local for every $W \in \mathcal{W}_{\widetilde{V}}$. Hence, the top horizontal map of the following commutative square is $\mathcal{A}$-local

$$\begin{tikzcd} \operatorname*{colim}_{\mathcal{W}_{\widetilde{U}}} \mathcal{A} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \operatorname*{colim}_{\mathcal{W}_{\widetilde{V}}} \mathcal{A} \arrow[d, "\simeq"] \\ \mathcal{A}(\widetilde{U}) \arrow[r] & \mathcal{A}(\widetilde{V}) \end{tikzcd}$$

The two horizontal arrows are $\mathcal{A}$-local because $\mathcal{A}$ is a marked Weiss cosheaf. It follows that the bottom horizontal map is also $\mathcal{A}$-local which is what we needed to show. □

We now consider refinements of marked stratifications. We will use this to "add points" to the stratification later on. This is useful when producing morphisms from objects of the pointless higher Morita categories.

Definition 8.20. Let $f\colon (X \to P) \to (Y \to Q)$ be a (conically smooth) map between two stratified marked spaces. Denote by X_p the substratified space corresponding to $p \in P$. We say that f is a *refinement* if it is a homeomorphism of the underlying topological spaces and the restriction $f_{|}\colon X_{|_p} \to Y_{|_{f(p)}}$ is an isomorphism onto its image for each $p \in P$.

Lemma 8.21. *Let $f\colon X \to Y$ be a refinement. Pushforward along f induces a functor*

$$f_\star\colon \mathrm{F{\small ACT}}_X^{\mathrm{pl,cstr}} \longrightarrow \mathrm{F{\small ACT}}_Y^{\mathrm{pl,cstr}}\ .$$

PROOF. As before we only need to check constructibility. Since the preimage of a (possibly marked) stratified basic disk is again a (marked) stratified disk, just possibly of a higher index, constructibility of $f_\star\mathcal{A}$ follows immediately from that of $\mathcal{A}$. □

Example 8.22. The simplest example relevant to us is adding a (marked) point to the stratification. An example is given by the refinement f illustrated in Figure 12 below. Let $\mathcal{A}$ be a locally constant factorization algebra on $(0,1)$ with values in VECT, i.e. characterised by an associative unital algebra $\mathbf{A}$ as explained in Example 4.49. Then the pushforward $f_\star\mathcal{A}$ evaluated at an interval containing the added point is simply $\mathbf{A}$ as an $\mathbf{A}$-$\mathbf{A}$-bimodule, or put differently it is the identity 1-morphism of $\mathbf{A}$ in the Morita 2-category.

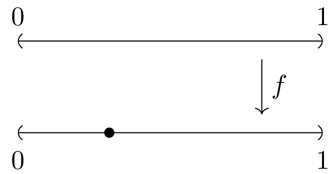


FIGURE 12. A simple example of a refinement f going from $(0,1)$ with trivial stratification to $(0,1)$ stratified by one point.

Recall from Definition 4.17 the definition of a stratified open embedding. With this we give a variant of [**Gin15**, Proposition 25]:

Proposition 8.23. *Let $\iota\colon X \hookrightarrow Y$ be a stratified open embedding of stratified spaces which sends marked points to marked points. Moreover, assume that for any marked inclusion $U \hookrightarrow^{\mathrm{m}} V$ of two basic marked stratified disks of the same index in Y we have that either $\iota^{-1}(U)$ and $\iota^{-1}(V)$ are both the empty set or basic marked stratified disks of the same index in X. Then pushing forward along ι induces a functor*

$$\iota_\star\colon \mathrm{F{\small ACT}}_X^{\mathrm{pl,cstr}} \to \mathrm{F{\small ACT}}_Y^{\mathrm{pl,cstr}}.$$

PROOF. Once again we only need to address the constructibility. Let $\mathcal{A} \in \mathrm{F{\small ACT}}_X^{\mathrm{pl,cstr}}$, and let $U \hookrightarrow^{\mathrm{m}} V$ be a marked inclusion of two basic marked stratified disks of the same index in Y. If $\iota^{-1}(U) = \emptyset = \iota^{-1}(V)$ we have that $\iota_\star\mathcal{A}(U) \simeq \mathbb{1}_{\mathcal{C}} \simeq \iota_\star\mathcal{A}(V)$ is an equivalence because the trivial factorization algebra (Example 2.22) is constructible. If instead $\iota^{-1}(U) \hookrightarrow^{\mathrm{m}} \iota^{-1}(V)$ are both basic marked stratified disks of the same index in X it follows that $\iota_\star\mathcal{A}(U) \to \iota_\star\mathcal{A}(V)$ is an equivalence because $\mathcal{A}$ is constructible. □

Example 8.24. Let $\iota\colon X \to Y$ be a stratified open embedding (which sends marked points to marked points). Moreover, let ι be such that $\iota(X)$ is the union of strata in Y, i.e. $\iota(X) =$

$\cup_{k\leq i} X_k$ for some $i \leq n$. Then ι satisfies the conditions of Proposition 8.23. [19] An explicit example of a map of this form for $n = 2$ and $k = 1$ is given in Figure 13.

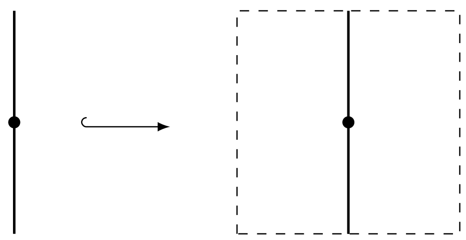


FIGURE 13. An example of a map satisfying the conditions of Proposition 8.23. We outline $(0, 1)^2$ with dashed lines to illustrate that the value of the push forward is trivial away from the line.

[19]It seems like Ginot in [**Gin15**, Proposition 25] proves that one can push forward constructible factorization algebras along these kinds of maps. However, the same argument works verbatim for more general maps which is why the condition on ι in Proposition 8.23 is phrased differently here.

9. The pointless higher Morita categories

In this section we define the (even higher) pointless higher Morita categories built from constructible pointless factorization algebras. We start by giving the formal construction for $n = 1$ and outline the general case in Section 9.1. This follows the construction of the pointed higher Morita categories of Scheimbauer from [**Sch14, GS18**]. Then we explain that the construction of even higher Morita categories from [**JFS17**] also applies to the pointless version of this thesis in Section 9.2. Lastly, in Section 9.3 we provide a symmetric monoidal functor which witnesses that the pointed higher Morita categories embeds into the pointless version. We also conjecture how the pointless higher Morita categories of this thesis relates to the (also pointless) higher Morita categories constructed by Haugseng.

Assumption 9.1. From now on all marked stratified spaces X will be maximally marked (c.f. Definition 4.22). Explicitly, that means that all of the 0-dimensional strata appearing is marked or equivalently that $X^{\mathrm{mrk}} = X_0$.

9.1. Formal construction. The *pointed* higher Morita categories were first constructed by Scheimbauer in [**Sch14**] using constructible factorization algebras. This construction was then reworked for $n = 1$ in [**GS18**], while the generalization to $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ will appear in [**Sch**] (together with a comparison to the original construction). Here we explain the construction for $n = 1$ adapted to the *pointless* setting, closely following [**GS18**, Appendix A]. Explicitly, we replace constructible factorization algebras by their *pointless* version to obtain the *pointless* higher Morita categories.

In more detail, the pointless Morita $(\infty, 2)$-category $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ is obtained by constructing a category object in $(\infty, 1)$-categories, i.e. a functor $\Delta^{\mathrm{op}} \to \mathrm{CAT}_\infty$ which satisfies the Segal condition. This is done in a few steps:

(1) First the relevant stratifications are packaged into a category Strat (Section 9.1.1).
(2) Then we explain that constructible pointless factorization algebras assemble into a functor out of Strat (Section 9.1.2).
(3) To obtain an actual category object in CAT_∞ from the above one needs to left Kan extend along a suitable functor $\pi\colon \mathrm{Strat} \to \Delta^{\mathrm{op}}$ and prove the Segal condition. From this one can then extract an $(\infty, 2)$-category $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ (Section 9.1.3).

The $(\infty, 2)$-category $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ from above is equipped with a symmetric monoidal structure as explained in Section 9.1.4. In Section 9.1.5 we extract an informal description of $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ from the formal construction. Lastly, we explain how the above procedure generalizes to define $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ in Section 9.1.6, but refrain from writing out full details as this will appear in [**Sch**].

Remark 9.2. We want to highlight that the construction of the *pointed* higher Morita categories using constructible factorization algebras requires the fact that constructible factorization algebras glue, which was long claimed in the literature but only proven much later in [**KSW24**, Theorem 6.1]. Moreover, since we have established that constructible *pointless* factorization algebras also glue in Theorem 5.3 we are able to leverage the construction of Scheimbauer's pointed higher Morita categories to obtain the pointless higher Morita categories.

9.1.1. The category of stratifications. We first package the stratifications relevant for $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ into a suitable category. It will be convenient to include the endpoints $\{0, 1\}$ of the interval $(0, 1)$ we work with.

Definition 9.3. [**GS18**, Definition A.1] Let Strat be the category whose objects are finite subsets $\{0,1\} \subset S \subset [0,1]$. Morphisms from S to S' are continuous maps $f\colon [0,1] \to [0,1]$ that are order-preserving, endpoint-preserving and stratum-preserving, i.e. $f(S) \subset S'$. We often denote an object S in Strat by $s_0 = 0 < s_1 < \cdots < s_k < 1$, using the linear order inherited from $[0,1]$.

Definition 9.4. Let $S \in$ Strat be an arbitrary object. The *associated marked stratified space* $\overline{\mathrm{A}}_{\mathrm{m}}(X)$ is defined by

$$\overline{\mathrm{A}}_{\mathrm{m}}(S) \coloneqq \big(S \subset [0,1]\big) \ ,$$

together with the maximal marking.

Explicitly, using the maximal marking above means that all of S corresponds to marked points. In some sense the associated marked stratified space gives us a second way of thinking about objects and morphisms of Strat which is helpful for example in the following observation.

Observation 9.5. Morphisms in Strat are generated by morphisms $f\colon [0,1] \to [0,1]$ of one of the following four elementary types.

(R) *rescaling:* If S has elements $0 < s_1 < \ldots s_l < 1$ and S' has elements $0 < s'_1 < \cdots < s'_l < 1$, then there is a morphism $S \to S'$ for any homeomorphism $f\colon [0,1] \to [0,1]$ such that $f(s_j) = s'_j$, $f(0) = 0$ and $f(1) = 1$. We think of such a map as a rescaling of the connected components of the associated marked stratified spaces, i.e. $f\colon \overline{\mathrm{A}}_{\mathrm{m}}(S) \to \overline{\mathrm{A}}_{\mathrm{m}}(S')$.

(A) *adding points or refinement:* Let S' have objects $0 < s_1 < \cdots < s_l < 1$ and fix some $1 \le j \le l$. Then the identity $\mathrm{id}\colon [0,1] \to [0,1]$ is a morphism $S' \setminus \{s_j\} \to S'$. This corresponds to adding the point s_j to the stratification, i.e. $\mathrm{id}\colon \overline{\mathrm{A}}_{\mathrm{m}}(S' \setminus \{s_j\}) \to \overline{\mathrm{A}}_{\mathrm{m}}(S')$. See Figure 12 for a simple illustration of this (after restricting to $(0,1)$).

(IC) *inner collapse:* Let S have elements $0 < s_1 < \cdots < s_l < 1$ and let S' have elements $0 < s'_1 < \cdots < s_{l-1} < 1$. Fix some $1 \le j < l$. Let $f_j\colon [0,1] \to [0,1]$ be the endpoint-preserving map such that

$$f_j(s_i) = \begin{cases} s'_i, & \text{if } i \le j, \\ s'_{i-1}, & \text{if } i > j, \end{cases}$$

and which is linear between the points. This gives a morphism $f_j\colon S \to S'$ which corresponds to a map of (marked) stratified spaces $\overline{\mathrm{A}}_{\mathrm{m}}(S) \to \overline{\mathrm{A}}_{\mathrm{m}}(S')$ which collapses the closed interval $[s_j, s_{j+1}]$ to a point. See the middle picture of Figure 10 for an illustration (after restricting to $(0,1)$) coming from a collapse-and-rescale map.

(OC) *outer collapse:* Let S have elements $0 < s_1 < \cdots < s_l < 1$ and S' have elements $0 < s'_1 < \cdots < s'_{l-1} < 1$. Consider the endpoint-preserving maps $f_l, f_r\colon [0,1] \to [0,1]$ where

$$f_l(s_i) = \begin{cases} 0 & \text{if } i = 1, \\ s'_{i-1} & \text{if } i > 1, \end{cases}$$

and

$$f_r(s_i) = \begin{cases} s'_i & \text{if } i < l, \\ 1 & \text{if } i = l, \end{cases}$$

and which is linear between the points. These give morphisms $f_l, f_r\colon S \to S'$. On the associated (marked) stratified spaces $\overline{\mathrm{A}}_{\mathrm{m}}(S) \to \overline{\mathrm{A}}_{\mathrm{m}}(S')$ this corresponds to collapsing the interval $[0, s_1]$ respectively $[s_l, 1]$ to a point. We think of this as "cutting off" the left and right ends, respectively. For an example (after restricting to $(0,1)$) coming from the collapse-and-rescale maps see the left and right illustrations of Figure 10.

With this one can check claims about Strat by checking the claim for each generator, which is convenient.

The category Strat can be related to the opposite simplex category Δ^{op}. This will be useful when constructing a category object later on.

Lemma 9.6. [**GS18**, Lemma A.2] *The functor* $\pi\colon \mathrm{Strat} \to \Delta^{\mathrm{op}}$*, which on objects sends* S *to* $\pi_0\big((0,1) \setminus S\big)$*, is a localization.*

Remark 9.7. We spell out what the functor π sends the generating morphisms of Strat to. The rescalings (R) are simply sent to identities, while adding a point (A) gives a degeneracy map. Finally, both the inner and outer collapse maps (IC) (OC) induces face maps.

9.1.2. Treating factorization algebras functorially. We now want to work with constructible pointless factorization algebras on $(0,1)$ with stratification determined by objects of Strat in a functorial way. First of all this requires us to go from stratifications of $[0,1]$ to stratifications of $(0,1)$, i.e. we get rid of the end-points.

Definition 9.8. Let $S = \{0 < s_1 < \cdots < s_l < 1\} \in \mathrm{Strat}$. We define the *associated marked stratified space* (without endpoints) to be

$$\mathrm{A_m}(S) \coloneqq \big(\{s_1, \ldots, s_l\} \subset (0,1)\big)\ ,$$

once again with the maximal marking.

We now turn to the morphisms of Strat, and explain how we can push forward constructible pointless factorization algebras along a restricted part of the maps of Strat.

Definition 9.9. [**GS18**, Definition A.4] Let $f\colon S \to S'$ be a morphism in Strat. Set $s_{\min} \coloneqq \max f^{-1}(0)$ and $s_{\max} \coloneqq \min f^{-1}(1)$. We define the *restricted pushforward* to be the functor

$$\begin{aligned} f_\sharp\colon \mathrm{F{\scriptsize ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(S)} &\longrightarrow \mathrm{F{\scriptsize ACT}}^{\mathrm{pless}}_{\mathrm{A_m}(S')} \\ \mathcal{F} &\longmapsto f_\sharp \mathcal{F} \coloneqq \Big(f_{|(s_{\min}, s_{\max})}\Big)_\star \mathcal{F}_{|(s_{\min}, s_{\max})}\ . \end{aligned} \tag{9.1}$$

That is, it is defined to be the pushforward along $f_{|(s_{\min}, s_{\max})}$.

Note that all morphisms $f\colon S \to S'$ in Strat are continuous and send marked points to marked points (because $f(S) \subset S'$). Thus it follows from Lemma 4.54 that the restricted pushforward preserves pointless factorization algebras making $f_\sharp$ well-defined. We even want to argue that the restricted pushforward lands in constructible pointless factorization algebras, but first we look at what this assignment is on the generators of Strat.

Example 9.10. We simplify Definition 9.9 for the four different generators of Strat and explain how to obtain the restricted pushforward of a general morphism from this.

(1) If f is of type (R), (A) or (IC) we have $f_\sharp \mathcal{F} = {f_{|(0,1)}}_\star \mathcal{F}$.

(2) For $f = f_l$ we have $s_{\min} = s_1$, $s_{\max} = 1$ and for $f = f_r$ we have $s_{\min} = 0$, $s_{\max} = s_l$. The restricted pushforwards are then given by

$$(f_l)_\sharp \mathcal{F} = {f_{|(s_1,1)}}_\star \mathcal{F}_{|(s_1,1)} \quad \text{and} \quad (f_r)_\sharp \mathcal{F} = {f_{|(0,s_l)}}_\star \mathcal{F}_{|(0,s_l)}\ .$$

(3) A general morphism f of Strat can be written as a composition $g_1 \circ \cdots \circ g_k$, where all the g_i's are of the form (R), (A), (IC) and (OC). We claim that $f_\sharp \mathcal{F} = g_{1\sharp} \circ \cdots \circ g_{k\sharp}\mathcal{F}$. This follows from how the pushforward along a composition is a concatenation of pushforwards and

$$(g_1 \circ g_2)_{|\left(\max(g_1\circ g_2)^{-1}(0),\min(g_1\circ g_2^{-1}(1)\right)} = g_{1|\left(\max(g_1^{-1}(0)),\min(g_1^{-1}(1)\right)} \circ g_{2|\left(\max(g_2^{-1}(0)),\min(g_2^{-1}(1)\right)} \,.$$

As mentioned, we want the restricted pushforward from (9.1) to even land in $\mathrm{F{\scriptstyle ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(S')}$. To that end we have the result below, which is a pointless analogue of [**GS18**, Proposition A.6].

Proposition 9.11. *For a constructible pointless factorization algebra $\mathcal{F}$, the restricted pushforward $f_\sharp\mathcal{F}$ along any morphism f in* Strat *is again constructible.*

Proof. Upon restricting to $(0,1)$ we see that a rescaling map (R) corresponds to a particularly easy adequately stratified map, so constructibility of the restricted pushforward follows from Lemma 8.14. The case of adding a point (A) is covered by Lemma 8.21. The remaining maps, i.e. the collapse maps (IC) and (OC) both follow from the argument in Lemma 8.16. $\square$

As a consequence of Proposition 9.11 we get the pointless analogue of [**GS18**, Proposition A.8].

Corollary 9.12. *Assigning the ∞-category of constructible pointless factorization algebras on a marked stratified space yields a functor*

$$\mathrm{Strat} \xrightarrow{\mathrm{F{\scriptstyle ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(-)}} \mathrm{C{\scriptstyle AT}}_\infty \,.$$

To prove that constructible pointless factorization algebras push forward along (inner and outer) collapse maps one can also use the proof of [**GS18**, Lemma A.7]. This is because the Weiss covers used there are in fact *marked* Weiss covers (for our choice of maximal marking).

9.1.3. The pointless $(\infty,2)$-Morita category $\mathrm{A{\scriptstyle LG}}_1^{\mathrm{pl}}(\mathcal{C})$. We now have all the pieces to give the formal construction of $\mathrm{A{\scriptstyle LG}}_1^{\mathrm{pl}}(\mathcal{C})$.

First we perform a left Kan extension to get a functor out of Δ^{op} instead of Strat.

Definition 9.13. [**GS18**, Definition A.10] Let $\mathrm{Alg}_1^{\mathrm{pl}} = \mathrm{Alg}_1^{\mathrm{pl}}(\mathcal{C})$ be the left Kan extension of $\mathrm{F{\scriptstyle ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(-)}$ along the functor π from Lemma 9.6. Explicitly,

$$\begin{tikzcd} \mathrm{Strat} \arrow[r, "\mathrm{F{\scriptstyle ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(-)}"] \arrow[d, "\pi"'] & \mathrm{C{\scriptstyle AT}}_\infty \\ \Delta^{\mathrm{op}} \arrow[ur, dashed, "\mathrm{Alg}_1^{\mathrm{pl}}"'] & \end{tikzcd} .$$

Recall from Definition B.1 the definition of a (1-uple) category object (in $\mathrm{C{\scriptstyle AT}}_\infty$). For this we need $\mathrm{Alg}_1^{\mathrm{pl}}$ to satisfy the Segal condition, which it indeed does. Explicitly, we get this as a corollary of Theorem 5.3, i.e. that constructible pointless factorization algebras glue, together with the proof of [**GS18**, Theorem A.11] which is the analogous statement for the pointed version using constructible ordinary factorization algebras.

Corollary 9.14. *The simplicial object in* CAT_∞ *given by* $\mathrm{Alg}_1^{\mathrm{pl}} = \mathrm{Alg}_1^{\mathrm{pl}}(\mathcal{C})$ *satisfies the Segal condition. Thus,* $\mathrm{Alg}_1^{\mathrm{pl}}$ *is a category object in* CAT_∞*.*

We stress that the original construction of the pointed higher Morita categories from [**Sch14**], as well as the reformulation of [**GS18, Sch**], relies on the fact that constructible factorization algebras glue which was only established later in [**KSW24**, Theorem 6.1].

In total, we have that $\mathrm{Alg}_1^{\mathrm{pl}}(\mathcal{C})$ is a category object in CAT_∞. This gives rise to a double ∞-category as in Observation B.5.

Remark 9.15. Informally, the double ∞-category obtained from $\mathrm{Alg}_1^{\mathrm{pl}}(\mathcal{C})$ has

- objects are (families of) constructible factorization algebras on $(0,1)$ with the trivial stratification, [20]
- vertical 1-morphisms are given by maps of such (families of) constructible factorization algebras,
- horizontal 1-morphisms are (families of) constructible pointless factorization algebras on $(0,1)$ stratified by a marked point, and
- the corresponding commutative squares are filled with maps of (families of) pointless constructible factorization algebras.

Meanwhile, the $(\infty,2)$-category we wish to obtain should not have any non-trivial vertical morphisms. To extract such an $(\infty,2)$-category we use the right adjoint of Proposition B.7, which in this case is made more explicit in Construction B.6. In words, it simply forgets about the non-trivial vertical 1-morphisms. This gives a 2-fold Segal space, which upon completing yields an honest $(\infty,2)$-category. We refer the reader to Appendix B.1 for more details on this.

Definition 9.16. The *pointless Morita* $(\infty,2)$*-category* $\mathrm{ALG}_1^{\mathrm{pl}} = \mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ is the $(\infty,2)$-category underlying the double ∞-category $\mathrm{Alg}_1^{\mathrm{pl}}$ as in Construction B.8.

Remark 9.17. In the pointless version it is necessary to complete the new Segal space produced from the double ∞-category. This is in contrast to the pointed higher Morita categories of Scheimbauer which are automatically complete by a variant of the argument in [**Sch14**, Proposition 3.2.34]. The argument relies on working with pointed bimodules (and point-preserving bimodule homomorphisms), and hence does not translate to the pointless setting. Observe that the fact that we have to complete is consistent with the (pointless) higher Morita categories constructed by Haugseng, which also requires the completion-step; c.f. [**Hau17**, Definition 4.40].

9.1.4. Symmetric monoidal structure on $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$. We briefly explain how the symmetric monoidal structure of the $(\infty,2)$-category $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ is constructed. This construction is identical to the construction of the symmetric monoidal structure on $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ as written in [**GS18**, Appendix A.4].

Recall that a symmetric monoidal $(\infty,2)$-category can be represented by a monoid object in $(\infty,2)$-categories. This can in turn come from a monoid in category objects in CAT_∞, which is what is constructed in the pointless version of [**GS18**, Definition A.14] below.

[20]See also Section 9.1.5 on how one obtains a family of constructible (pointless) factorization algebras as objects (1-morphisms).

Definition 9.18. Let $\mathrm{Alg}_1^{\mathrm{pl},\otimes}$ be the left Kan extension of $\mathrm{F{\scriptsize ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(-)^{\amalg -}}$ along $\mathrm{id} \times \pi$, i.e.

$$\begin{tikzcd} \mathrm{F{\scriptsize IN}}_* \times \mathrm{Strat} \arrow[r, "\mathrm{F{\scriptsize ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(-)^{\amalg -}}"] \arrow[d, "\mathrm{id}\times\pi"'] & \mathrm{C{\scriptsize AT}}_\infty \\ \mathrm{F{\scriptsize IN}}_* \times \Delta^{\mathrm{op}} \arrow[ur, dashed, "\mathrm{Alg}_1^{\mathrm{pl},\otimes}"'] & \end{tikzcd} .$$

With the above definition one directly gets the below as a corollary of the proof of [**GS18**, Theorem A.15].

THEOREM 9.19. *The functor* $\mathrm{Alg}_1^{\mathrm{pl},\otimes}$ *satisfies the Segal condition in both* $\mathrm{F{\scriptsize IN}}_*$ *and* Δ^{op} *and thus determines a symmetric monoidal* $(\infty, 2)$*-category.*

Establishing the Segal condition in Δ^{op} is a straightforward variant of the proof of [**GS18**, Theorem A.11], while the Segal condition in $\mathrm{F{\scriptsize IN}}_*$ is analogous to the argument from Proposition 5.11 equipping $\mathrm{F{\scriptsize ACT}}^{\mathrm{pl,cstr}}_X$ with a symmetric monoidal structure. As before, the crucial ingredient in both cases is that constructible pointless factorization algebras glue.

9.1.5. Extracting an informal description for $n = 1$. The pointless higher Morita categories have an informal description analogous to that of the pointed version of Scheimbauer. Here we briefly explain how to recover this informal description in the setting of $\mathrm{A{\scriptsize LG}}_1^{\mathrm{pl}}(\mathcal{C})$, as well as highlight the difference between the pointed and pointless version in this informal language. More details on the informal description can be found in [**GS18**, §2.2 & 2.3].

Fix an object $[k] \in \Delta^{\mathrm{op}}$. Then $(\mathrm{Alg}_1^{\mathrm{pl}})_k$, defined as a left Kan extension, assigns a colimit indexed by $\mathrm{Strat}_{/[k]}$. Explicitly,

$$(\mathrm{Alg}_1^{\mathrm{pl}})_k = \operatorname*{colim}_{S \in \mathrm{Strat}_{/[k]}} \mathrm{F{\scriptsize ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(S)} .$$

The objects of $\mathrm{Strat}_{/[k]}$ are all stratifications $S \subset [0,1]$ with $k+2$ elements, and morphisms are simply rescalings.

In words, an object of $(\mathrm{Alg}_1^{\mathrm{pl}})_k$ is essentially a family of constructible pointless factorization algebras $\mathcal{F}_S$, for each such stratification S, such that for any rescaling map $f\colon S \to S'$ between any two such stratifications we have an induced equivalence $f_\sharp \mathcal{F}_S \simeq \mathcal{F}_{S'}$. In the informal description one instead specify just a representative of this collection; the rest of the family can be recovered by pushing forward along rescaling maps (which indeed induces equivalences).

Explicitly, just specifying a representative we have that

(0) an *object* or 0-*morphism* of the pointless higher Morita category $\mathrm{A{\scriptsize LG}}_1^{\mathrm{pl}}(\mathcal{C})$ corresponds to an element of $(\mathrm{Alg}_1^{\mathrm{pl}})_0$, i.e. a locally constant factorization algebra on $(0,1)$.

(1) a *1-morphism* is given by an element of $(\mathrm{Alg}_1^{\mathrm{pl}})_1$, i.e. a constructible pointless factorization algebra on $(0,1)$ stratified by one point which is also marked.

Notation 9.20. We often say that an object, respectively 1-morphism of $\mathrm{A{\scriptsize LG}}_1^{\mathrm{pl}}(\mathcal{C})$ is *encoded* by a constructible pointless factorization algebra $\mathcal{F}$. In this situation $\mathcal{F}$ is understood to be defined on $(0,1)$, respectively $(0,1)$ stratified by a marked point.

We now explain how to extract the source and target objects of a 1-morphism, as well as how 1-morphisms are composed. For this recall Remark 9.7 which explains what morphisms of Δ^{op} the generating morphisms of Strat are sent to (under π).

- *Source and target:* Let $\mathcal{F}$ encode a 1-morphism of $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$. The source, denoted $s(\mathcal{F})$, is extracted by pushing forward $\mathcal{F}$ along the outer collapse map f_r from (OC). Similarly, the target, $t(\mathcal{F})$, is obtained by pushing forward $\mathcal{F}$ along the outer collapse map f_l instead. At the level of stratified marked spaces this corresponds to reading the figures from left to right.
- *Composition:* Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two constructible pointless factorization algebras on $(0,1)$ with a marked stratified point. Moreover, assume that $t(\mathcal{F}_1) \simeq s(\mathcal{F}_2)$ (making the 1-morphisms composable). We first glue $\mathcal{F}_1$ and $\mathcal{F}_2$ along their common target respectively source to obtain, after a rescaling, a constructible pointless factorization algebra $\mathcal{F}$ on $(0,1)$ stratified by 2 marked points, i.e. an object of $(\mathrm{Alg}_1^{\mathrm{pl}})_2$. To obtain an honest 1-morphism from this one has to push forward along the inner collapse map from (IC).

Note also that the degeneracy maps are given by adding points to the stratification (A) which simply corresponds to adding an identity 1-morphism. We also extract the following useful geometric picture for the (symmetric) monoidal product:

- *Monoidal product:* Let $\mathcal{F}$ and $\mathcal{G}$ encode two objects of $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$. First glue $\mathcal{F}$ and $\mathcal{G}$ to define a constructible pointless factorization algebra on two disjoint copies of $(0,1)$, and visualize the two copies as being "stacked" above one another. To obtain an actual object of $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ from this one pushes forward along the fold map $\nabla\colon (0,1) \amalg (0,1) \to (0,1)$, which one can think of as "fusing" the two disjoint copies of $(0,1)$.

The same visual picture also applies for 1-morphisms after possibly rescaling to make sure that the marked stratified point on both copies of $(0,1)$ agree.

Remark 9.21. The difference between the pointed and pointless version of the higher Morita categories for $n = 1$ arises at the level of 1-morphisms of $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ since that is where the 0-dimensional strata enters the conversation. Similarly, because 2-morphisms are morphisms of constructible factorization algebras, respectively of constructible pointless factorization algebras the difference propagates also to higher morphisms.

9.1.6. The general construction. We give some details on how the formal construction of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ works for $n > 1$. For the full details we refer to the forthcoming paper [**Sch**].

To construct $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ the idea is to work with the n-fold product of Strat, instead of just one copy. The first thing to understand is how to extract an associated (maximally marked) stratified space of the n-cube $[0,1]^n$ from an object $\bar{S} \in \mathrm{Strat}^n$. Before we outline this for general n, we give two examples for $n = 2$ to give the reader some intuition.

Example 9.22. Let $n = 2$ and consider the following three objects $S_1 = \{0, s_1^1, s_2^1, 1\}$, $S_2 = \{0, 1\}$ and $S_2' = \{0, s_1^2, s_2^2, s_3^2, 1\}$ of Strat. From the pair $(S_1, S_2) \in \mathrm{Strat}^2$ we extract a stratification of $[0,1]^2$ whose restriction to $(0,1)^2$ is illustrated in the leftmost square of Figure 14. Similarly, the pair $(S_1, S_2') \in \mathrm{Strat}^2$ instead gives rise to the rightmost stratified space of Figure 14 after restricting to $(0,1)^2$. Here all of the six stratified points are set to be marked points. The dashed horizontal lines are only for visual aid and *not* part of the stratification.

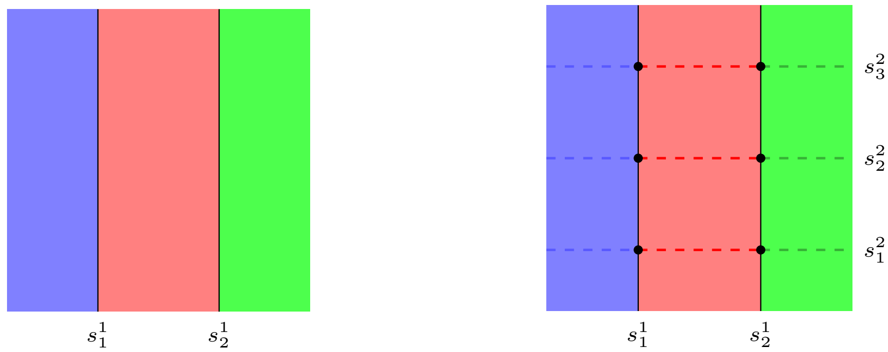


FIGURE 14. Examples of marked stratified spaces extracted from an object of Strat^2.

Now let $\bar{S} = (S_1, \dots, S_n) \in \mathrm{Strat}^n$ be an arbitrary object. Informally, the *associated maximally marked stratified space* $\mathrm{A_m}(\bar{S})$ (defined on $(0,1)^n$) has

- $(n-1)$-dimensional hyperplanes at $x_1 = \{s_i^1\}$, for each $s_i^1 \in S_1 \setminus \{0,1\}$.
- For each element $s_j^2 \in S_2 \setminus \{0,1\}$ the intersection of the above $(n-1)$-dimensional hyperplane with the $(n-1)$-dimensional hyperplane at $x_2 = \{s_j^2\}$ determines an $(n-2)$-dimensional strata. Observe that the latter $(n-1)$-dimensional hyperplane is *not* part of the stratification, only the intersection is.
- This iterates all the way to S_n, which determines the 0-dimensional strata (and hence also the marked points).

Observation 9.23. Let $\bar{S} = (S_1, \dots, S_n) \in \mathrm{Strat}^n$ be an object such that $S_k = \{0,1\}$, for some $1 \le k \le n$. From the general procedure outlined above we observe that the associated (maximally marked) stratified space has *no* non-trivial strata of dimension $(n-k)$ or lower.

Example 9.24. In the leftmost square of Figure 14 we have $S_2 = \{0,1\}$ and hence there is no $2-2=0$-dimensional strata appearing. Similarly, choosing $(S_1, S_2) \in \mathrm{Strat}^2$ such that $S_1 = \{0,1\}$ gives the trivial stratification after restricting to $(0,1)^2$ regardless of what S_2 is.

Just like for $n = 1$, the constructible pointless factorization algebras push forward along the maps of Strat^n. Explicitly, recall the generating morphisms of Strat from Observation 9.5. The fact that we can push forward constructible pointless factorization algebras along rescalings (R) or maps that add points (A) (i.e. refinements) does not depend on working with $(0,1)$ and directly generalizes to the setting of Strat^n. However, in the setting of Strat^n we need versions of the inner and outer collapse maps (IC) and (OC) that collapses and rescales only the stratification in one direction at a time, while acting by identity in the remaining ones. Explicitly, we need to be able to push forward constructible pointless factorization algebras along $\mathrm{id}^k \times \rho \times \mathrm{id}^l \colon (0,1)^n \to (0,1)^n$ for $n = k+l+1$, where $(0,1)^n$ comes with a stratification extracted from Strat^n generalizing the situation for $n = 1$. This is exactly the content of

Lemma 8.19. Hence, we get the top horizontal functor in the diagram

$$\begin{tikzcd} \mathrm{Strat}^n \arrow[r, "\textsc{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(-)}"] \arrow[d, "\pi^n"'] & \textsc{Cat}_\infty \\ \left(\Delta^{\mathrm{op}}\right)^n \arrow[ur, dashed, "\mathrm{Alg}^{\mathrm{pl}}_n"'] & \end{tikzcd} . \tag{9.2}$$

We define $\mathrm{Alg}^{\mathrm{pl}}_n$ to be the left Kan extension of $\textsc{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(-)}$ along π^n, i.e. the above dashed arrow.

The above defines an n-uple simplicial object in $\textsc{Cat}_\infty$. To actually get an $(\infty, n+1)$-category from this one first needs to show that $\mathrm{Alg}^{\mathrm{pl}}_n$ is an *n-uple category object* in $\textsc{Cat}_\infty$ (Definition B.1). That is, for each $1 \leq i \leq n$ and every $k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n \geq 0$ one needs to show that

$$\mathrm{Alg}^{\mathrm{pl}}_n(\mathcal{C})_{k_1,\ldots,k_{i-1},\bullet,k_{i+1},\ldots,k_n}$$

satisfies the Segal condition. This will again follow by the fact that constructible pointless factorization algebras glue by a (more complicated) version of Corollary 9.14.

Moreover, $\mathrm{Alg}^{\mathrm{pl}}_n(\mathcal{C})$ even determines an *n-fold Segal object* in $\textsc{Cat}_\infty$ (Definition B.3). Explicitly, for each $1 \leq i \leq n-1$ and for every $k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n \geq 0$ one needs to show that the degeneracy map

$$\mathrm{Alg}^{\mathrm{pl}}_n(\mathcal{C})_{k_1,\ldots k_{i-1},0,\ldots,0} \longrightarrow \mathrm{Alg}^{\mathrm{pl}}_n(\mathcal{C})_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n} \tag{9.3}$$

is a weak equivalence. Let $\vec{k} = (k_1, \ldots, k_{i-1}, 0, \ldots, 0)$ and $\vec{k}' = (k_1, \ldots, k_{i-1}, 0, k_{i+1}, \ldots, k_n)$ as above.

Remark 9.25. Giving an object in the left hand side of (9.3) corresponds to giving a family of pointless factorization algebras which are constructible with respect to the maximally marked stratified space associated to some object $\bar{S} \in \mathrm{Strat}^n{}_{/\vec{k}}$. Similarly, an object in the right hand side is determined by some object $\bar{S}' \in \mathrm{Strat}^n{}_{/\vec{k}'}$. For notational simplicity we argue by choosing a representative of this collection. The rest of the family is recovered by pushing forward along rescaling maps. See Section 9.1.5 for more details on this for $n = 1$.

Observation 9.23 tells us that since $k_i = 0$ the associated maximally marked space of $\bar{S}$ and $\bar{S}'$ has no strata of dimension $(n - i)$ or lower. [21] Recall from Remark 9.7 that the degeneracy maps are induced by adding points to the stratification. However, adding points to go from $\bar{S}$ to $\bar{S}'$ gives the same associated stratified space since we are only adding points in the entries $i+1, \ldots, n$. Thus, by pushing forward along a rescaling map we get an equivalence between the constructible pointless factorization algebras corresponding to $\bar{S}$ and $\bar{S}'$. This can in turn be promoted to an equivalence between the corresponding families of constructible factorization algebras which are all related by rescaling. It follows that the map (9.3) is indeed a weak equivalence.

In summary, we get that the functor $\mathrm{Alg}^{\mathrm{pl}}_n(\mathcal{C})$ from (9.2) determines an n-fold Segal object of $\textsc{Cat}_\infty$. As in the setting of $n = 1$ we use Construction B.8 to extract its underlying $(\infty, n+1)$-category. We tentatively summarize this in the following definition, but stress that this still relies on the full details for the above constructions which will appear in [**Sch**].

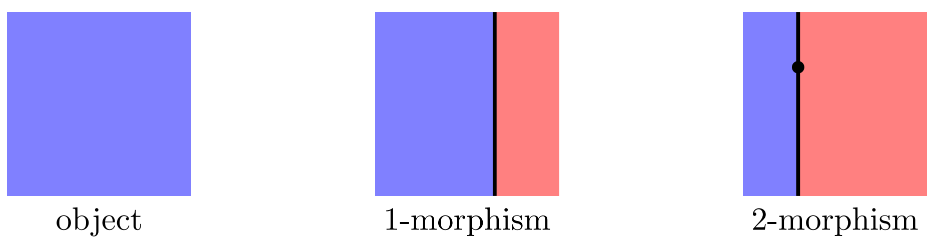
[21] In particular, since $i \leq n-1$ it follows that at most (i.e. when $i = n-1$) we can have 1-dimensional strata, but there will be no 0-dimensional strata appearing and thus also no marked points. Hence, the argument is the same as in the pointed setting.

Definition 9.26. Let $\mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C})$ denote the n-fold Segal object in CAT_∞ sketched above. We define the *pointless higher Morita category* $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ to be the underlying $(\infty, n+1)$-category obtained from Construction B.8.

Remark 9.27. We want to stress that the difference between the pointed and pointless higher Morita categories constructed from constructible ordinary respectively pointless factorization algebras arises at the level of n-morphisms (and as a consequence also the higher morphisms). The stratified spaces encoding n-morphisms are the only ones having 0-dimensional strata and hence marked points, so this is the only place constructible pointless factorization algebras differ from the ordinary constructible factorization algebras. Since e.g. $(n+1)$-morphisms are maps of n-morphisms, i.e. structure-preserving, these are also different as a consequence of the structure they need to preserve being different.

Lastly, there is nothing about the construction of the symmetric monoidal structure from Theorem 9.19 that hinges on working with $n = 1$. By substituting Strat with Strat^n one can adapt the construction to also equip $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ with a symmetric monoidal structure.

Remark 9.28. For later convenience we briefly extend the informal description of the pointless higher Morita categories from Section 9.1.5 to $n = 2$. Objects of $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$ are encoded by locally constant factorization algebras while 1-morphisms are encoded by constructible factorization algebras on $(0,1)^2$ stratified by a line at some fixed $x_1 = a_1$, for $0 < a_1 < 1$. The 2-morphisms are encoded by constructible *pointless* factorization algebras on $(0,1)^2$ stratified by a line at $x_1 = a_1$ and a marked point in its interior given by the intersection of $x_1 = a_1$ and $x_2 = a_2$. That is, at the level of stratified (marked) spaces we have:



Convention 9.29. Recall that the 1-morphisms in $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ are read from left to right, as explained in Section 9.1.5. Thus, we also read 1-morphisms of $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$ from left to right. Figure 15 illustrated how to extract the source and target 1-morphisms of a 2-morphism of $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$ at the level of marked stratified spaces. As we can see, the convention is to read 2-morphisms from bottom to top.

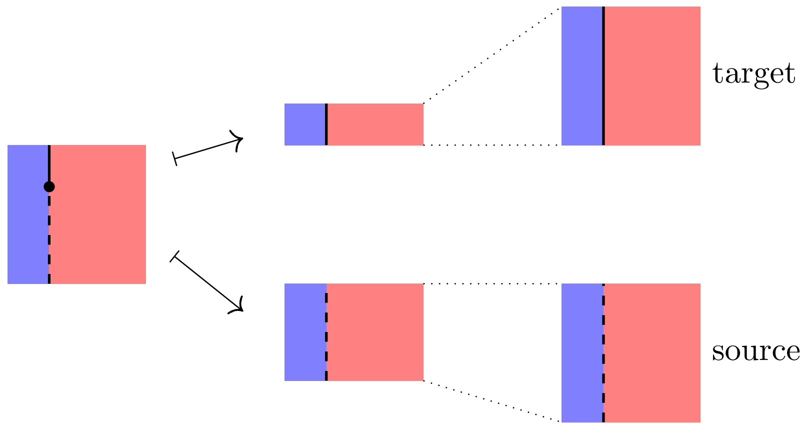


FIGURE 15. An illustration of how one extracts the source and target 1-morphisms of a 2-morphism in $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$.

9.2. The even higher pointless Morita categories. In [**JFS17**], Johnson-Freyd and Scheimbauer constructs the even higher Morita categories. That is, they allow for the target category $\mathcal{C}$ to be a symmetric monoidal (∞, m)-category, for $m \geq 1$. Using the higher morphisms of $\mathcal{C}$ one then obtain versions of the higher Morita categories which are $(\infty, n+m)$-categories in contrast to "only" $(\infty, n+1)$-categories. We give a short summary of their constructions, and explain how this also applies to the pointless higher Morita categories constructed in this thesis.

From an (∞, m)-category $\mathcal{C}$ one can construct an $(m-1)$-uple simplicial diagram of $(\infty, 1)$-categories as follows:

$$\begin{aligned} \mathcal{C}^{\square}_{\vec{\bullet}} \colon (\Delta^{\mathrm{op}})^{m-1} &\longrightarrow \mathrm{CAT}_\infty \\ \vec{l} &\mapsto \mathcal{C}^{\square}_{\vec{l}} = \tau_{(\infty,1)}[\Theta^{\vec{l}}, \mathcal{C}] \ . \end{aligned} \tag{9.4}$$

Here, $\Theta^{\vec{l}}$ is a strict higher category appearing as an object in Joyal's category Θ_m, viewed as an (∞, m)-category, while $\tau_{(\infty,1)}[\Theta^{\vec{l}}, \mathcal{C}]$ is the (underlying) $(\infty, 1)$-category of functors into $\mathcal{C}$. For example, for $\vec{l} = (1, \dots, 1, 0 \dots, 0)$ where the first l entries are ones, one can think of the $(\infty, 1)$-category $\tau_{(\infty,1)}[\Theta^{\vec{l}}, \mathcal{C}]$ as extracting the $(\infty, 1)$-category of l-morphisms. We refer the reader to [**JFS17**, §4 & 5] for more details on this. Moreover, by [**JFS17**, Theorem 5.11] the $(m-1)$-uple simplicial diagram (9.4) is actually a complete $(m-1)$-fold Segal object in CAT_∞.

Remark 9.30. Compared to the notation in [**JFS17**] the above diagram $\mathcal{C}^{\square}$ is denoted by $\mathcal{C}^{\boxed{\text{strong}}}$. They also construct two additional alternatives for $\mathcal{C}^{\square}$ which corresponds to lax and oplax versions of bimodules. See for example [**JFS17**, Definition 5.14] for the three variants of the diagram from Equation (9.4). We never need these variants in this thesis, but note that Theorem 9.34 applies to all three versions.

The two steps for constructing the even higher (pointless) Morita categories are:

(1) first apply the construction of the pointless higher Morita category not just to a fixed $(\infty, 1)$-category but rather to the diagram (9.4) (subject to some conditions) to obtain an $(n+m-1)$-uple simplicial object in CAT_∞.

(2) secondly one needs to ensure that this $(n+m-1)$-uple simplicial object satisfies the Segal condition so that it actually determines an $(n+m-1)$-fold Segal object in CAT_∞, or in other words an $(\infty, n+m)$-category.

For the first step to work we need to put some conditions on the diagram obtained from $\mathcal{C}$. To understand what these should be we highlight the following direct consequence of the construction of the pointless higher Morita categories.

Observation 9.31. Let $\mathcal{C}$ be $\otimes$-presentable, and let $\mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C})_{\vec{\bullet}}$ denote the n-fold Segal object in CAT_∞ constructed in Corollary 9.14 for $n=1$, and outlined in Section 9.1.6 for general n. The assignment

$$\mathcal{C} \mapsto \mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C})_{\vec{\bullet}}$$

is functorial for $\otimes$-cocontinuous functors.

Hence, the reasonable condition to put on the diagram $\mathcal{C}^\square$ is the following:

Definition 9.32. Let $\mathcal{C}$ be a symmetric monoidal (∞, m)-category. The diagram $\mathcal{C}^\square$ is called $\otimes$-*presentable* if it is an $(m-1)$-uple simplicial diagram of $\otimes$-presentable categories and $\otimes$-cocontinuous functors.

Remark 9.33. Observe that for the version of the higher Morita category of Scheimbauer one typically asks for the diagram $\mathcal{C}^\square$ to be $\otimes$-sifted cocomplete. However, the proof that constructible factorization algebras glue from [**KSW24**] requires $\mathcal{C}$ to be $\otimes$-presentable. In the current version of [**KSW24**] we did not check whether one can use e.g. [**AFT17a**, Corollary 2.28], which says that $\mathcal{D}\mathrm{isks}_{/X}$ is sifted, to show that $\otimes$-sifted cocompleteness is sufficient. The version of Haugseng requires $\mathcal{C}^\square$ to be $\otimes$-GR-cocomplete. See [**JFS17**, Definition 8.3] for the exact definitions. Since we do not have a good understanding of the colimits needed for the pointless higher Morita categories constructed from pointless factorization algebras we ask for $\otimes$-presentable here.

As explained in [**JFS17**, Remark 8.4] it follows from their results, [**Lur09a**, Lemma 5.4.5.5] and Observation 9.31 (which is the analogue of [**JFS17**, Theorem 8.2]) that the assignment

$$\begin{aligned} \mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C}^\square_{\vec{\bullet}})_{\vec{\bullet}} \colon (\Delta^{\mathrm{op}})^n \times (\Delta^{\mathrm{op}})^{m-1} &\longrightarrow \mathrm{CAT}_\infty \\ (\vec{k}, \vec{l}) &\mapsto \mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C}^\square_{\vec{l}})_{\vec{k}} \end{aligned} \tag{9.5}$$

determines an $(n+m-1)$-uple simplicial object in CAT_∞. This concludes step 1 from above.

We now turn towards step 2. For this $(n+m-1)$-uple simplicial object in CAT_∞ to give rise to an $(\infty, n+m)$-category we need to ensure that it is an n-fold Segal object internal to (complete) $(m-1)$-fold Segal objects in CAT_∞. For the higher Morita categories of Scheimbauer and Haugseng this is proven in [**JFS17**, Theorem 8.5]. We will now explain that the same arguments carries over to the pointless higher Morita categories constructed in this thesis.

THEOREM 9.34. *Let $\mathcal{C}$ be a symmetric monoidal (∞, m)-category such that $\mathcal{C}^\square$ is $\otimes$-presentable. Then $\mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C}^\square_{\vec{\bullet}})_{\vec{\bullet}}$ from* (9.5) *is an n-fold Segal object internal to complete $(m-1)$-fold Segal objects in CAT_∞. Hence, it has an underlying $(\infty, n+m)$-category, denoted $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$, which we call the "even higher pointless Morita category".*

Proof. For fixed $\vec{l}$ we know that $\mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C}^{\square}_{\vec{l}})_{\vec{\bullet}}$ is a n-fold Segal object in Cat_∞. Hence, it remains to prove that $\mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C}^{\square}_{\vec{\bullet}})_{\vec{k}}$ is a complete $(m-1)$-fold Segal object in Cat_∞ for fixed $\vec{k}$. We know that $\mathcal{C}^{\square}_{\vec{\bullet}}$ is a complete $(m-1)$-fold Segal object in Cat_∞ from [**JFS17**, Remark 8.4]. Hence, the Segal condition follows if we know that the functor $\mathrm{Alg}_n^{\mathrm{pl}}(-)_{\vec{k}}$ preserves fibered products. We show this in Proposition 9.37, which is an adaptation of [**JFS17**, Proposition 8.17]. □

Before giving the missing proposition, we record the following useful observation.

Observation 9.35. Fix some tuple $\bar{S} \in \mathrm{Strat}^n{}_{/\vec{k}}$. Recall that $\mathrm{A_m}(\bar{S})$ denotes the associated (maximally marked) stratified space defined for $n = 1$ in Definition 9.8 and sketched in Section 9.1.6 for general n. Then, the (fully faithful) inclusion of the fiber

$$\mathrm{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(\bar{S})}(\mathcal{C}) \hookrightarrow \mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C})_{\vec{k}} \tag{9.6}$$

is an equivalence. The essential surjectivity follows from how all of the stratifications appearing on the right hand side can be rescaled (in a compatible way) to the fixed stratification $\mathrm{A_m}(\bar{S})$, just like for ordinary factorization algebras.

For the proof of the following proposition we also introduce the following notation:

Notation 9.36. Let $\mathrm{Cat}_\infty^{\mathrm{pr}}$ denote the ∞-category of $\otimes$-presentable ∞-categories and $\otimes$-cocontinuous functors, while $\mathrm{Cat}_\infty^{\mathrm{sm}}$ denotes the ∞-category of symmetric monoidal ∞-categories and symmetric monoidal functors.

We now give the analogue of [**JFS17**, Proposition 8.17], as well as a more detailed version of the proof from [**JFS17**] adapted to the pointless setting.

Proposition 9.37. *For fixed $\vec{k} \in (\Delta^{\mathrm{op}})^n$, the functor $\mathrm{Alg}_n^{\mathrm{pl}}(-)_{\vec{k}}$ from $\mathrm{Cat}_\infty^{\mathrm{pr}}$ to Cat_∞ preserves fibered products.*

Proof. The equivalence (9.6) is natural in $\mathcal{C}$, so the proposition follows if we can show that $\mathrm{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(\bar{S})}(-)$ preserves fibered products. We will show this for any stratified maximally marked space X here.

It will be convenient to first argue at the level of algebras, for which we need the following: By [**Lur09a**, Proposition 5.5.3.13] and the fact that fibered products of symmetric monoidal ∞-categories are computed pointwise we get that the fibered product (computed in $\mathrm{Cat}_\infty^{\mathrm{sm}}$) of $\otimes$-presentable ∞-categories along $\otimes$-cocontinuous functors is again a $\otimes$-presentable ∞-category. Moreover, the projection maps are $\otimes$-cocontinuous.

Recall that $\mathrm{Fact}^{\mathrm{pl,cstr}}_X(-)$ is a subfunctor of $\mathrm{Alg}_{\mathrm{open}(X)_{\mathrm{mrk}}}(-)$, where the latter refers to algebras over the ∞-operad $\mathrm{open}(X)^{\otimes}_{\mathrm{mrk}}$. The functor $\mathrm{Alg}_{\mathrm{open}(X)_{\mathrm{mrk}}}(-)\colon \mathrm{Cat}_\infty^{\mathrm{sm}} \to \mathrm{Cat}_\infty$ preserves limits. First of all, the functor $\mathrm{Alg}_{\mathrm{open}(X)_{\mathrm{mrk}}}(-)$ factors as $\mathrm{Cat}_\infty^{\mathrm{sm}} \xrightarrow{\mathrm{Und}} \mathrm{Opd}_\infty \to \mathrm{Cat}_\infty$, where the first functor $\mathrm{Und}\colon \mathrm{Cat}_\infty^{\mathrm{sm}} \to \mathrm{Opd}_\infty$ sends a symmetric monoidal ∞-category to its underlying ∞-operad. By [**Lur17**, Section 2.2.4] the functor Und has a left adjoint, called the symmetric monoidal envelope, and thus by virtue of being a right adjoint it preserves limits. The second functor $\mathrm{Opd}_\infty \to \mathrm{Cat}_\infty$ is given by a mapping space, and hence also preserves limits. Explicitly, if we start with a fibered product $\mathcal{D}^{\otimes} \times_{\mathcal{E}^{\otimes}} \mathcal{F}^{\otimes}$ we obtain the pullback square:

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{open}(X)_{\mathrm{mrk}}}(\mathcal{D} \times_{\mathcal{E}} \mathcal{F}) \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathrm{open}(X)_{\mathrm{mrk}}}(\mathcal{D}) \arrow[d] \\ \mathrm{Alg}_{\mathrm{open}(X)_{\mathrm{mrk}}}(\mathcal{F}) \arrow[r] & \mathrm{Alg}_{\mathrm{open}(X)_{\mathrm{mrk}}}(\mathcal{E}) \end{tikzcd} \tag{9.7}$$

It remains to show that this pullback square restricts to constructible pointless factorization algebras. We do this by arguing that the pullback restricts (separately) to constructible pointless prefactorization algebras, multiplicative pointless prefactorization algebras and to marked Weiss cosheaves.

- For constructibility, note that we have an equivalence in the fibered product $\mathcal{D}^{\otimes} \times_{\mathcal{E}^{\otimes}} \mathcal{F}^{\otimes}$ if and only if its images in each component is an equivalence. Hence, the square (9.7) restricts to constructible pointless prefactorization algebras.
- For multiplicativity we use that a morphism in the fibered product is cocartesian if and only if its images in each component is cocartesian. Hence, the square also restricts to a pullback of the corresponding ∞-categories of multiplicative pointless prefactorization algebras.
- For the marked Weiss cosheaf conditions we first note that by [**Lur09a**, Lemma 5.4.5.5] a diagram (of a specific shape) in the fiber product is a colimit diagram if and only if the projections are. Applying this for any diagram coming from a marked Weiss cover gives that an algebra in the fiber product is a marked Weiss cosheaf if and only if each projection is a marked Weiss cosheaf.

Hence, we in total get that the square (9.7) indeed restricts to constructible pointless factorization algebras, so we are done. □

Having established that the theory of even higher Morita categories also applies to our pointless version we give an interesting example of this from the literature.

Example 9.38. The main example of an even higher Morita category here is $\mathrm{ALG}_2^{\mathrm{pl}}(\mathrm{Pr})$ constructed by Brochier, Jordan and Snyder in [**BJS21**]. Strictly speaking, they employ Haugseng's model for their construction, which we expect to be equivalent to our pointless version. However, they also explain how to understand their structures in terms of structure maps of constructible factorization algebras, c.f. [**BJS21**, Figure 1-4]. In fact, they even draw intuition from the results of [**GS18**] for part of their dualizability-results.

Recall from [**BJS21**, Definition 2.7] that Pr is the 2-category of locally presentable categories, cocontinuous functors and natural transformations. For notational simplicity we will abbreviate and simply say e.g. 'category' for an object of Pr below. The symmetric monoidal structure is given by the Deligne-Kelly tensor products; c.f. [**BJS21**, Definition 2.8]. Moreover, the diagram $\mathrm{Pr}^{\square}$ is $\otimes$-presentable by e.g. the same arguments outlined in [**JFS17**, Example 8.11]. Informally, the 4-category $\mathrm{ALG}_2^{\mathrm{pl}}(\mathrm{Pr})$ is given as follows:

- objects are braided tensor categories,
- 1-morphisms are tensor categories with central structures, [22]
- 2-morphisms are centered bimodule categories,
- 3-morphisms are bimodule functors of such, and

[22]this is a different way of packaging the data of a bimodule category of braided tensor categories. See [**BJS21**, Definition-Proposition 3.2] for more details, as well as [**BJS21**, Figure 2] which connects this to constructible factorization algebras.

- 4-morphisms are natural transformations of such.

The composition of 1- and 2-morphisms is defined using the balanced Deligne-Kelly tensor product. In [**BJS21**] they show explicitly that this indeed is well-defined, i.e. that the composition they define indeed is a 1- respectively 2-morphism. If one instead works with our pointless version one would get that the composition is given by the balanced Deligne-Kelly tensor product as a consequence of the marked Weiss cosheaf condition as explained in Example 5.6. Within this 4-category they obtain sufficient conditions for 3- and 4-dualizability, we recall this in more detail in Remark 10.27 later.

Recall that our main motivation for constructing the pointless version of the higher Morita category is to examine n and $(n+1)$-dualizability therein. However, with the even higher version there is some potential ambiguity. A priori the two actions below could give different $(\infty, n+1)$-categories

- first truncate $\mathcal{C}$ to get an $(\infty,1)$-category before applying $\mathrm{ALG}_n^{\mathrm{pl}}(-)$, or
- first apply $\mathrm{ALG}_n^{\mathrm{pl}}(-)$ and then truncate the resulting $(\infty, n+m)$-category to an $(\infty, n+1)$-category.

However, from the construction of the even higher pointless Morita category one can see that the above two actions are equivalent. That is, we have the following result.

Proposition 9.39. [**GS18**, Proposition A.20] *The inclusion $\tau_{(\infty,1)}\mathcal{C} \hookrightarrow \mathcal{C}$ induces an equivalence of $(\infty, n+1)$-categories*

$$\tau_{(\infty,n+1)}\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C}) \simeq \mathrm{ALG}_n^{\mathrm{pl}}(\tau_{(\infty,1)}\mathcal{C}) \simeq \mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C}^{\square}_{0,\dots,0}) \ .$$

Remark 9.40. Because of the above proposition we choose *not* to introduce special notation for the even higher Morita categories here. Whenever one is interested in up to $(n+1)$-dualizability in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ it is sufficient to consider $\mathrm{ALG}_n^{\mathrm{pl}}(\tau_{(\infty,1)}\mathcal{C})$. Similarly, we also do not spell out the symmetric monoidal structure of the even higher Morita categories here since we do not need it, and simply refer the interested reader to [**JFS17**] and [**Sch**]

9.3. Comparing higher Morita categories. With the formal definition of the pointless higher Morita categories we can now properly compare it to the pointed version by Scheimbauer. We also conjecture an equivalence between the pointless higher Morita categories constructed in this thesis and the (also pointless) higher Morita categories of Haugseng from [**Hau17**].

Recall from Lemma 4.56 that restriction along the inclusion $\iota_X \colon \mathrm{open}(X)_{\mathrm{mrk}} \to \mathrm{open}(X)$ induces a functor $\iota_X^* \colon \mathrm{FACT}_X^{\mathrm{cstr}} \to \mathrm{FACT}_X^{\mathrm{pl,cstr}}$ of constructible (pointless) factorization algebras. Moreover, the functor ι_X^* is shown to be symmetric monoidal in Proposition 5.12. This is the key ingredient for proving the following theorem.

THEOREM 9.41. *Let $\mathcal{C}$ be a symmetric monoidal $\otimes$-presentable ∞-category. We have a symmetric monoidal functor of $(\infty, n+1)$-categories*

$$\iota^* \colon \mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C}) \longrightarrow \mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C}) \ , \tag{9.8}$$

where the former refers to the higher Morita category of Scheimbauer.

PROOF. For simplicity, let $n=1$. The argument for general n is entirely analogous. Recall from Definition 9.18 how the symmetric monoidal structure on $\mathrm{Alg}_1^{\mathrm{pl}}(\mathcal{C})$ is constructed. By replacing constructible pointless algebras by their ordinary version the same definition also gives the symmetric monoidal structure on $\mathrm{Alg}_1^{\mathrm{ptd}}(\mathcal{C})$.

We claim that restriction along $\iota := \iota_{A(-)^{\amalg -}}$ induces a natural transformation

$$\iota^* := \iota^*_{A(-)^{\amalg -}} : \mathrm{Fact}^{\mathrm{cstr}}_{\mathrm{A}(-)^{\amalg -}} \Rightarrow \mathrm{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(-)^{\amalg -}} \ . \tag{9.9}$$

First fix some $I \in \mathrm{Fin}_*$ and let $f\colon S \to S'$ be any morphism of Strat. Consider the square

$$\begin{tikzcd} \mathrm{Fact}^{\mathrm{cstr}}_{\mathrm{A}(S)^{\amalg I}} \arrow[r, "\amalg_{i\in I} f_\sharp"] \arrow[d, "\iota^*_{A(S)^{\amalg I}}"'] & \mathrm{Fact}^{\mathrm{cstr}}_{\mathrm{A}(S')^{\amalg I}} \arrow[d, "\iota^*_{A(S')^{\amalg I}}"] \\ \mathrm{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(S)^{\amalg I}} \arrow[r, "\amalg_{i\in I} f_\sharp"'] & \mathrm{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(S')^{\amalg I}} \end{tikzcd} \tag{9.10}$$

where $f_\sharp$ is the restricted pushforward from Definition 9.9. Here, $\mathrm{A_m}(S)$ is the associated maximally marked space from Definition 9.8, while $\mathrm{A}(S)$ is the corresponding trivially marked version. By Proposition 9.11 the restricted push forward preserves constructible (pointless) factorization algebras. Thus Lemma 4.59 and Lemma 4.55 tells us that the square (9.10) commutes. Now fix $S \in$ Strat and let $f\colon I_+ \to J_+$ be any map in Fin_*. Then the square analogous to (9.10) commutes by (the proof of) Proposition 5.12. In summary we have established that ι^* indeed induces a natural transformation as in (9.9). This in turn induces a natural transformation

$$\iota^*\colon \mathrm{Alg}_1^{\mathrm{ptd},\otimes}(\mathcal{C}) \Rightarrow \mathrm{Alg}_1^{\mathrm{pl},\otimes}(\mathcal{C}) \ .$$

This gives rise to a symmetric monoidal functor between the two underlying symmetric monoidal $(\infty, 2)$-categories. For general n the proof works exactly the same way by simply replacing $\pi\colon \mathrm{Strat} \to \Delta^{\mathrm{op}}$ with its n-fold product $\pi^n\colon \mathrm{Strat}^n \to (\Delta^{\mathrm{op}})^n$. $\square$

Recall that the map $\iota_X\colon \mathrm{open}(X)_{\mathrm{mrk}} \to \mathrm{open}(X)$ is *not* fully faithful when X has non-trivial marking, so the above constructed functor between the pointed and pointless higher Morita categories is also *not* fully faithful in general. However, it is only at the level of stratifications giving rise to n-morphisms that this happens. In fact, $\iota^*\colon \mathrm{Fact}^{\mathrm{cstr}}_{\mathrm{A}(\bar{S})} \to \mathrm{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(\bar{S})}$ is an equality for all configurations $\bar{S} = (S_1, \dots, S_n) \in \mathrm{Strat}^n$ where $S_k = \{0, 1\}$ for at least one $k \leq n$. Hence, we get the following as a consequence of Theorem 9.41.

Corollary 9.42. *Let $\mathcal{C}$ be a symmetric monoidal $\otimes$-presentable ∞-category. The functor* (9.8) *induces an equivalence on the truncations to $(\infty, n-1)$-categories. Explicitly,*

$$\tau_{(\infty,n-1)}\iota^*\colon \tau_{(\infty,n-1)}\mathrm{Alg}_n^{\mathrm{ptd}}(\mathcal{C}) \xrightarrow{\simeq} \tau_{(\infty,n-1)}\mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C}) \ .$$

It has long been conjectured that the higher Morita categories constructed by Haugseng and Scheimbauer should be related. See for example [**Hau17**, Conjecture 1.9]. It is hence natural to expect our pointless version constructed here to be equivalent to Haugseng's version, which we denote by $\mathrm{Alg}_n^{\mathrm{H}}(\mathcal{C})$ here.

Conjecture 9.43. *Let $\mathcal{C}$ be a symmetric monoidal $\otimes$-presentable ∞-category. The symmetric monoidal $(\infty, n+1)$-categories $\mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C})$ and $\mathrm{Alg}_n^{\mathrm{H}}(\mathcal{C})$ are equivalent.*

Remark 9.44. Note that this kind of comparison seems to be difficult to make precise. There is a known equivalence at the level of objects, but an extension of this to the entire Morita theory (i.e. iterative bimodules) is not yet present in the literature. One approach is to exhibit bimodules of associative algebras as constructible factorization algebras on the line stratified with a point. This is well-understood for constructible factorization algebras on $[0, \infty)$ and right modules, see e.g. [**KSW24**, Example 5.27], and has a rather straightforward extension to the setting with bimodules. With such an identification on can then bootstrap this to bimodules of bimodules (etc) using a version of Dunn's additivity theorem for constructible (pointless) factorization algebras; such a result is work in progress by Anja Švraka.

10. Dualizability in pointless higher Morita categories

We start by explaining how the results from [**GS18**] proving that $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ is fully n-dualizable directly translates to give full n-dualizability of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ in Section 10.1. For future convenience we also explain how to extract the dualizability data witnessing that an object of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ is n-dualizable from the constructions of [**GS18**]. In Section 10.2 we consider $(n+1)$-dualizability in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ by giving a conjecture of Lurie and partially proving it for $n = 2$. For a brief introduction to dualizability in higher categories we refer the reader to Appendix B.2.

10.1. Full n-dualizability of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$.

In this section we explain how the full n-dualizability results of the pointed higher Morita categories $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ from [**GS18**] carry over to our pointless version. Then we sketch how one extracts the full n-dualizability data of a fixed object of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$, first for $n = 2$ and then for general n, from the explicit constructions of [**GS18**].

Corollary 10.1. *Let $\mathcal{C}$ be a symmetric monoidal (∞, m)-category and let $\mathcal{C}^{\square}$ be $\otimes$-presentable (see Definition 9.32). The symmetric monoidal $(\infty, n+m)$-category $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ is fully n-dualizable, i.e.*

(1) every object of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ has a dual; and
(2) if $1 \leq k < n$, any k-morphism in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ has both a left and a right adjoint.

PROOF. Direct consequence of [**GS18**, Theorem 4.1], which is the same statement for $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$, and Theorem 9.41 together with Proposition B.25. $\square$

Remark 10.2. There are three versions of the even higher Morita categories constructed in [**JFS17**], see also Remark 9.30. Since full n-dualizability is detected in the (∞, n)-truncation of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$, which agrees for all three versions of the even higher Morita categories, Corollary 10.1 applies to all three versions.

Remark 10.3. Recall that the pointed higher Morita category is automatically complete, while the pointless version is not, and hence we need to complete the latter. However, this does not influence the results regarding duals and adjoints. In fact, the mapping Segal spaces in the completion is the completion of the mapping Segal spaces ([**Hau17**, Lemma 5.50]), so one does not need to complete to check if an n-fold Segal space has adjoints for k-morphisms. See also [**Hau18**, Remark 11.8].

It will be helpful for results in Section 10.2 to extract the dualizability-data of an object of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ from the proof of [**GS18**, Theorem 4.1]. Before considering the general case, we first make this explicit for $n = 2$. It will be sufficient to understand the left adjoints here, and we refer the reader to [**GS18**] for the analogous construction of right adjoints.

We first explain how to get dual objects in $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$ in the following construction which is a combination of [**GS18**, Definition 3.5 & Construction 3.6].

Construction 10.4. Let $\mathcal{R}$ denote an arbitrary object of $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$ encoded by the constructible factorization algebra $\mathcal{F}$. Let $r\colon (0,1) \to (0,1)$ denote the *reflection map* given by $x_1 \mapsto 1 - x_1$. From this the *reversing map* is defined to be $\mathrm{rev} := r \times \mathrm{id}$. The dual of $\mathcal{R}$ is denoted $\mathcal{R}^{\mathrm{rev}}$ and is encoded by the constructible factorization algebra $\mathrm{rev}_{\star}\mathcal{F}$, i.e. the pushforward of $\mathcal{F}$ along the reversing map.

The reversing map is adequately stratified, so we can indeed push forward along it. The push forward is illustrated in Figure 16, where we have added the page framing as well as its image under the reversing map. The framing is *not* part of the morphism, however, it is a useful tool for book-keeping.

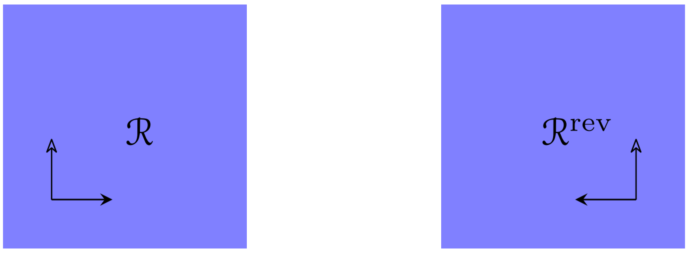


FIGURE 16. An arbitrary object of $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$ and its dual.

The corresponding evaluation 1-morphism is constructed in three steps in [**GS18**, Construction 3.19], which we briefly recall here.

Construction 10.5.

(1) First consider $\mathcal{R}$ as an identity 1-morphism. This is done by adding a line at e.g. $x_1 = \frac{1}{2}$ to the trivial stratification of $(0,1)^2$) and observing that $\mathcal{R}$ is clearly constructible with respect to this stratification.
(2) The evaluation map comes from bending the right side of this square *up* and to the *left* in the ambient 3-dimensional space, resulting in the leftmost surface of Figure 17.
(3) To obtain an actual 1-morphism of $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$ one then has to push forward the resulting constructible factorization algebra along a projection map to $(0,1)^2$ which forgets about the vertical direction. This projection map is adequately stratified exactly when we also equip the target $(0,1)^2$ with a line at $x_1 = \frac{1}{2}$. [23]

The last two steps combine to give the *fold map*, denoted f, as in [**GS18**, Definition 3.18 & Construction 3.19]. Pushing forward along it gives the evaluation 1-morphism

$$\mathrm{ev}_{\mathcal{R}} = \mathcal{R}^{\circlearrowleft} \colon \mathcal{R}^{\mathrm{rev}} \otimes \mathcal{R} \to \mathbb{1} \ .$$

Similarly, the coevaluation 1-morphism is instead given by folding the left side of the square from Construction 10.5 1 *down* and to the *right*. See [**GS18**, Construction 3.21] for details. This is illustrated in the rightmost surface of Figure 17 and (after pushing forward along the projection to $(0,1)^2$) results in a 1-morphism

$$\mathrm{coev}_{\mathcal{R}} = \mathcal{R}^{\circlearrowright} \colon \mathbb{1} \to \mathcal{R} \otimes \mathcal{R}^{\mathrm{rev}} \ .$$

---

[23] If the stratification we started with was not at $x_1 = \frac{1}{2}$ one instead has to modify the steps here by adding suitable rescalings. This only makes the construction more notationally involved, so we choose to work with the standard stratification here.

FIGURE 17. Evaluation and coevaluation 1-morphisms before projecting.

Remark 10.6. We make the choice of illustrating the above procedure before projecting down to $(0,1)^2$ because the corresponding figures are the ones that will be helpful when examining $(n+1)$-dualizability in Section 10.2. For the same reason we also illustrate the corresponding unit and counit 2-morphisms witnessing that the evaluation and coevaluation has left adjoints below before projecting down to $(0,1)^2$.

In [**GS18**] they construct left adjoints for any 1-morphism by a bending procedure similar to the above but in the first and second multiplicative directions. Let $\mathbf{A}\colon \mathcal{R} \to \mathcal{S}$ be an arbitrary 1-morphism encoded by a constructible factorization algebra $\mathcal{F}$ on $(0,1)^2$ stratified by a line at $x_1 = \frac{1}{2}$. The corresponding counit 2-morphism witnessing that $\mathbf{A}$ has a left adjoint is that of [**GS18**, Construction 3.16] whose steps are outlined below.

Construction 10.7.

(1) One first bends the top of the stratified square $(0,1)^2$ to the *right* and *down*. Explicitly, this can be done e.g. by the diffeomorphism ϕ of [**GS18**, Example 3.15]. This introduces a rotation, namely 180 degrees clockwise rotation before a 180 degrees counterclockwise rotation back, in the lower right corner of the square.

(2) Lastly, one uses a product of two suitable collapse and rescale maps, e.g. $\tau = \rho_a^b \times \rho_c^d$ as chosen in [**GS18**, Example 3.15], to obtain an honest 2-morphism. We prove that one indeed can push forward constructible (pointless) factorization algebras along τ in Lemma 10.11 below.

Applying the above steps to the 1-morphism $\mathrm{coev}_{\mathcal{R}} = \mathcal{R}^{\varsigma}$ we get that the counit 2-morphism witnessing that $\mathrm{coev}_{\mathcal{R}}$ has a left adjoint corresponds to a constructible factorization algebra on the surface of Figure 18 (before projecting to the plane and collapsing-and-rescaling). Using the notation of [**GS18**, Construction 3.16] the counit 2-morphism is

$$\varepsilon_{\mathrm{coev}_{\mathcal{R}}} = \left(\mathcal{R}^{\varsigma}\right)^{\cap} \colon \mathrm{coev}_{\mathcal{R}} \circ_{\mathcal{R}\otimes\mathcal{R}^{\mathrm{rev}}} \mathrm{coev}_{\mathcal{R}}^{\mathrm{L}} \Rightarrow \mathrm{id}_{\mathbb{1}} \ ,$$

where $\mathrm{coev}_{\mathcal{R}}^{\mathrm{L}}$ denotes the left adjoint. [24]

[24] Just looking at the pictures here one might think that $\mathrm{coev}_{\mathcal{R}}^{\mathrm{L}}$ is equivalent to $\mathrm{ev}_{\mathcal{R}}$. However, this is not the case because of the non-trivial rotation of the framing in the former.

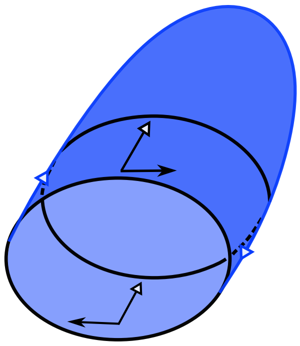


FIGURE 18. The counit 2-morphism $\varepsilon_{\mathrm{coev}_{\mathcal{R}}}$.

The unit witnessing that $\mathrm{coev}_{\mathcal{R}}$ has a left adjoint is produced in [**GS18**, Construction 3.17]. In words this construction takes the square from Construction 10.7 1 and bends the bottom of the square to the *left* and *up* instead, e.g. by the inverse diffeomorphism ϕ^{-1}. This gives a constructible factorization algebra on the surface of Figure 19 below. After projecting to the plane and collapsing-and-rescaling one gets the 2-morphism

$$\eta_{\mathrm{coev}_{\mathcal{R}}} = (\mathcal{R}^{\zeta})^{\cup} \colon \mathrm{id}_{\mathcal{R}\otimes\mathcal{R}^{\mathrm{rev}}} \Rightarrow \mathrm{coev}_{\mathcal{R}}^{\mathrm{L}} \circ_{\mathbb{1}} \mathrm{coev}_{\mathcal{R}} \ .$$

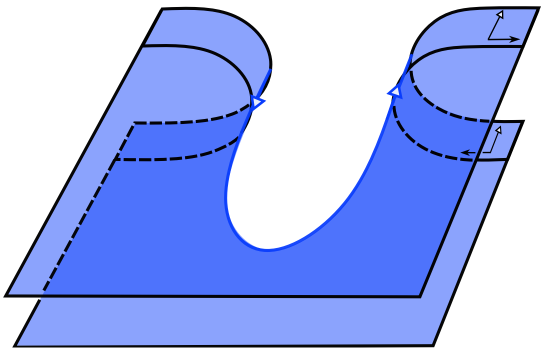


FIGURE 19. The unit 2-morphism $\eta_{\mathrm{coev}_{\mathcal{R}}}$.

The unit and counit 2-morphisms witnessing that $\mathrm{ev}_{\mathcal{R}}$ has a left adjoint are produced in the exact same way. At the level of stratified surfaces before pushing forward along the projection and collapse-and-rescale maps we have that the counit 2-morphism $\varepsilon_{\mathrm{ev}_{\mathcal{R}}}$ is given by a constructible factorization algebra on the leftmost surface of Figure 20. Similarly, the unit 2-morphism $\eta_{\mathrm{ev}_{\mathcal{R}}}$ is given by a constructible factorization algebra on the rightmost surface of Figure 20.

Remark 10.8. Observe that for the folding procedure to obtain the evaluation and coevaluation 1-morphisms one is folding (or bending) in the 1st multiplicative direction and the ambient space, which one can think of as the 0th multiplicative direction. Similarly, to obtain the unit and counit 2-morphisms one bends in the 2nd multiplicative direction and the 1st multiplicative direction, which here takes the role of the ambient space from above.

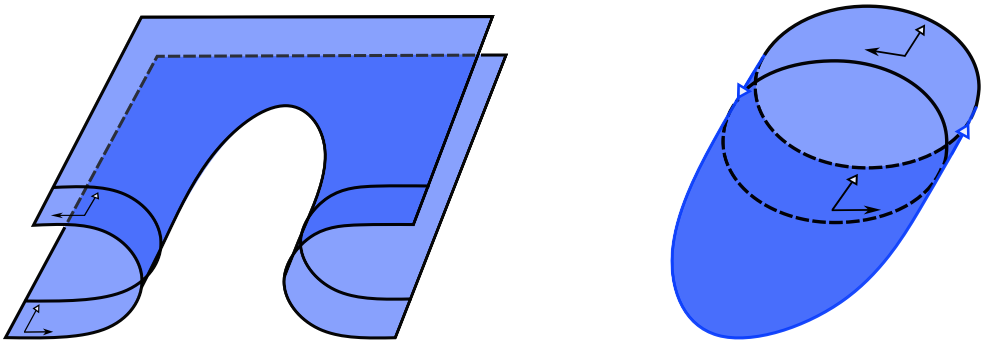


FIGURE 20. The counit and unit 2-morphisms $\varepsilon_{\mathrm{ev}_{\mathcal{R}}}$ respectively $\eta_{\mathrm{ev}_{\mathcal{R}}}$.

For general n one obtains the dualizability-data by leveraging the bending procedures for $n = 2$ together with suitable copies of identities. We outline how to obtain the k-morphisms, for $k \leq n$, appearing in the n-dualizability data of an object of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$.

(1) The evaluation 1-morphism of an object encoded by a constructible factorization algebra $\mathcal{F}$ is obtained by the steps in Construction 10.5 applied to the first coordinate. Explicitly, one adds an $(n-1)$-dimensional hyperplane to the stratification, e.g. at $x_1 = \frac{1}{2}$, then push forward along the fold map and $(n-1)$ copies of the identity map, i.e. $f \times \mathrm{id}^{n-1}$. This is again adequately stratified, c.f. Example 8.13. The coevaluation 1-morphism is constructed the same way just using the second version of the fold map.

(2) To obtain the counit 2-morphisms of that data one use the bending procedure from Construction 10.7 times $(n-2)$-copies of the identity map. Similarly, the unit 2-morphism is constructed the same way modulo using e.g. the inverse diffeomorphism.

(k) In general, to obtain the counit k-morphism one has to push forward the corresponding constructible (pointless) factorization algebra along $\mathrm{id}^{k-2} \times (\tau \circ \phi) \times \mathrm{id}^{n-k}$, where we borrowed the notation from Construction 10.7. The unit k-morphism is obtained similarly by using e.g. the inverse diffeomorphism. See Lemma 10.11 for a proof that pushing forward along these maps indeed induces functors of constructible (pointless) factorization algebras.

For complete details on the above outlined procedure we refer the reader to [**GS18**].

Remark 10.9. From the constructions of [**GS18**] it is clear why one can never get $(n+1)$-dualizability in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ by similar arguments. Put simply, we are "out of directions to bend". This is also evident in how the nature of an n-morphism of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ is very different from that of an $(n+1)$-morphism. The former is a constructible (pointless) factorization algebra for a particular stratification while the latter instead is a morphism of such constructible (pointless) factorization algebras.

Remark 10.10. Gwilliam and Scheimbauer also prove an interesting result regarding $(n+1)$-dualizable objects of $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$. Namely, they prove that any $(n+1)$-dualizable object of $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ is equivalent to the unit locally constant factorization algebra on $(0,1)^n$ [**GS18**, Theorem 5.1]. In other words, there are no interesting $(n+1)$-dualizable objects in $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$, and the reason for this is the pointings introduced at the level of n-morphisms. This is the main motivation for constructing the pointless higher Morita category as explained in detail in Section 1.2.

We now give the postponed proof that we can indeed push forward constructible (pointless) factorization algebras along maps of the form

$$(\tau\circ\phi)^{(k)} := \mathrm{id}^{k-2}\times(\tau\circ\phi)\times\mathrm{id}^{n-k}\colon (0,1)^n \longrightarrow (0,1)^n, \qquad \text{for } 2\le k\le n\ , \tag{10.1}$$

where ϕ is the diffeomorphism from [**GS18**, Example 3.15] and $\tau = \rho_a^b\times\rho_c^d$ for a suitable choice of $0<b<a<1$ and $0<d<c<1$ as in [**GS18**, Example 3.15]. [25] Such a suitable choice, as well as the maps ϕ and τ, are illustrated for $n=2$ in Figure 21. The source copy of $(0,1)^n$ in (10.1) is stratified such that it encodes a $(k-1)$-morphism of $\mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C})$. Explicitly, the stratification is given by the flag

$$\emptyset \subseteq \{(a^1,\dots,a^{k-1})\}\times(0,1)\subseteq\{(a^1,\dots,a^{k-2})\}\times(0,1)^2\subseteq\cdots\subseteq(0,1)^n\ .$$

The target copy of $(0,1)^n$ is then seen to be encoding a k-morphism of $\mathrm{Alg}_n^{\mathrm{pl}}(\mathcal{C})$.

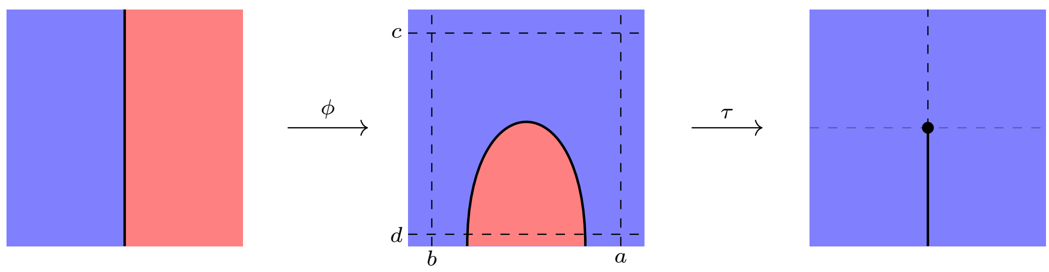


FIGURE 21. An illustration of the composite map $\tau\circ\phi$ for $n=2$ producing a 2-morphism in $\mathrm{Alg}_2^{\mathrm{pl}}(\mathcal{C})$ from a 1-morphism.

Lemma 10.11. *Consider $(\tau\circ\phi)^{(k)}$ from (10.1) where the stratifications are as explained above. Pushing forward along $(\tau\circ\phi)^{(k)}$ induces a functor of constructible (pointless) factorization algebras*

$$((\tau\circ\phi)^{(k)})_\star\colon \mathrm{Fact}^{\mathrm{pl,cstr}}_{(0,1)^n}\longrightarrow\mathrm{Fact}^{\mathrm{pl,cstr}}_{(0,1)^n}\ .$$

The same is true when replacing ϕ with ϕ^{-1}.

PROOF. We only spell out the details for ϕ here, the version with ϕ^{-1} is a direct modification. The map ϕ is adequately stratified, so by Lemma 8.14 we know that pushing forward along it induces a functor of constructible (pointless) factorization algebras. Hence, we only need to address $\tau^{(k)} := \mathrm{id}^{k-2}\times\tau\times\mathrm{id}^{n-k}\colon(0,1)^n\to(0,1)^n$. As usual, we have from Lemma 4.54 that the pushforward preserves (pointless) factorization algebras. Thus, we only need to check constructibility.

Let $\mathcal{A}$ be a constructible factorization algebra in the source of $\tau^{(k)}$, i.e. a constructible factorization algebra on $(0,1)^n$ with stratification induced from already having pushed forward along $\phi^{(k)} := \mathrm{id}^{k-2}\times\phi\times\mathrm{id}^{n-k}$. The only types of inclusions of basic disks whose preimages are not inclusions of basic disks are those that intersect the (non-trivial) $(n-k+1)$-dimensional strata of the target. Let $U\overset{\mathrm{m}}{\hookrightarrow}V$ be an inclusion of basic disks of this type. The preimages

[25] In some more detail: one first restrict the stratified space to the x_{k-1} and x_k-coordinates and then chooses a,b,c,d as in [**GS18**, Example 3.15].

$\widetilde{U} := (\tau^{(k)})^{-1}(U)$ and $\widetilde{V} := (\tau^{(k)})^{-1}(V)$ then intersect the $(n-k+1)$-dimensional strata of the source twice. Such an inclusion, and the corresponding preimages, is illustrated for $n=2$ in Figure 22. Let e_1 and e_2 be the maximum, respectively minimum x_{k-1}-coordinates of the two intersections of $\widetilde{V}$ with the $(n-k+1)$-dimensional strata. This is also illustrated in Figure 22 for $n=2$. With this we define the following Weiss cover of $\widetilde{V}$

$$\mathcal{W}_{\widetilde{V}} := \{\widetilde{V} \setminus (0,1)^{k-2} \times f \times (0,1)^{n-k+1} \mid f \in (e_1, e_2)\} \ . \tag{10.2}$$

The restriction $\mathcal{W}_{\widetilde{U}} := \{W \cap \widetilde{U} \mid W \in \mathcal{W}\}$ is a Weiss cover of $\widetilde{U}$. By multiplicativity and constructibility of $\mathcal{A}$ we have that $W \cap \widetilde{U} \hookrightarrow W$ is $\mathcal{A}$-local for each $W \in \mathcal{W}_{\widetilde{V}}$. We conclude, as in the proof of Lemma 8.19, that $\widetilde{U} \hookrightarrow \widetilde{V}$ is $\mathcal{A}$-local which in turn proves the lemma. □

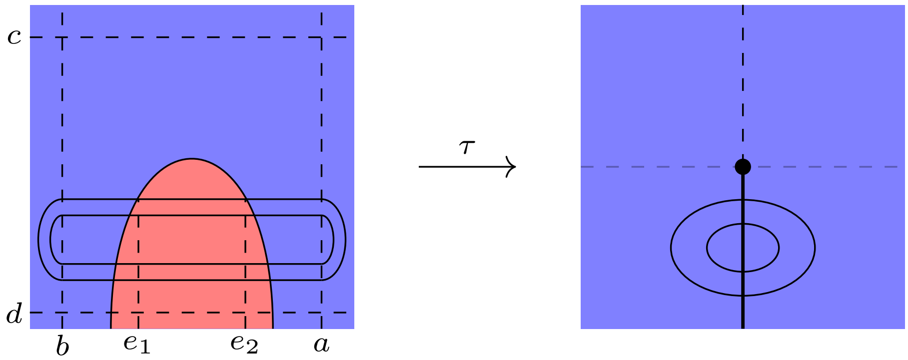


FIGURE 22. An illustration of an inclusion whose preimages are not basic disks and the choice of e_1 and e_2 used to build the Weiss cover in (10.2).

10.2. Towards higher dualizability in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$**.** We now turn towards expectations and results regarding $(n+1)$-dualizability in the context of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$. In particular, we partially prove a result conjectured by Lurie for $n=2$. Some of the statements in this section are phrased using factorization homology. We refer the reader unfamiliar with this notion to Section 11 for the definition and some basic results.[26]

We will freely use the equivalence between locally constant factorization algebras on $\mathbb{R}^n$ and E_n-algebras in this section and only distinguish them with the following notation.

Notation 10.12. Throughout this section we will denote factorization algebras by mathscr-letters like $\mathscr{R}, \mathscr{S}$ while the corresponding E_n-algebras are denoted by mathcal-letters like $\mathcal{R}, \mathcal{S}$.

We will also use the following assumption throughout.

Assumption 10.13. Let $\mathbf{A}\colon \mathscr{R} \to \mathscr{S}$ be a 1-morphism in $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$. We assume that this is equivalent to an $(\mathcal{R}, \mathcal{S})$-bimodule (in $\mathcal{C}$) which we also denote $\mathbf{A}$.

[26] The discussion in Section 11 is tailored to the oriented setting. Thus, to get the right notions for this section one needs to replace 'oriented manifold' by 'framed manifold' and 'framed E_d-algebra' by 'E_d algebra'. See also Remark 11.4.

In the setting of constructible ordinary factorization algebras one should get the above for pointed modules by a more complicated version of the argument in [**KSW24**, Example 5.27], see also [**KSW24**, Remark 5.28]. However, some of the intermediate results used there are not yet established in the pointless setting as explained in Remark 7.32. We now give some notation and setup needed to state Lurie's conjecture.

Definition 10.14. Let $\mathcal{R}$ be an object of $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$. A left (right) $\mathcal{R}$-module $\mathbf{A}$ is called *dualizable* if it has a left (right) adjoint as an $(\mathcal{R}, \mathbb{1}_{\mathcal{C}})$-bimodule ($(\mathbb{1}_{\mathcal{C}}, \mathcal{R})$-bimodule) in $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$.

Proposition 10.15. [**Lur17**, Proposition 4.6.2.13] *An $(\mathcal{R}, \mathcal{S})$-bimodule $\mathbf{A}$ has a left (right) adjoint if and only if $\mathbf{A}$ is dualizable as a left $\mathcal{R}$-module (right $\mathcal{S}$-module).*

As an illustration of how useful the above proposition can be we consider the following result which first appeared in [**Lur09b**]. Recall the reverse object $\mathcal{R}^{\mathrm{rev}}$ from Construction 10.4. This corresponds to the opposite multiplicative structure at the level of E_1-algebras.

THEOREM 10.16. *An object $\mathcal{R}$ of $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ is 2-dualizable if, and only if, it is dualizable as an object of $\mathcal{C}$ and as an $\mathcal{R} \otimes \mathcal{R}^{\mathrm{rev}}$-module.*

PROOF. Every object is 1-dualizable with dual given by $\mathcal{R}^{\mathrm{rev}}$ as explained in Construction 10.4. The evaluation map is given by $\mathcal{R}$ as an $(\mathcal{R}^{\mathrm{rev}} \otimes \mathcal{R}, \mathbb{1}_{\mathcal{C}})$-bimodule, while the coevaluation is $\mathcal{R}$ as a $(\mathbb{1}_{\mathcal{C}}, \mathcal{R} \otimes \mathcal{R}^{\mathrm{rev}})$-bimodule. By Theorem B.24 it is sufficient to provide e.g. left adjoints of the evaluation and coevaluation maps, which by Proposition 10.15 exactly corresponds to $\mathcal{R}$ being dualizable as an $\mathcal{R}^{\mathrm{rev}} \otimes \mathcal{R}$-module and as a $\mathbb{1}_{\mathcal{C}}$-module. The latter condition simply means that $\mathcal{R}$ is dualizable as an object of $\mathcal{C}$. $\square$

This provides a proof of Conjecture-Theorem 10.17 below regarding $(n+1)$-dualizability in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ when $n = 1$. To give the general statement we first need some setup. Let $\mathcal{R}$ be an object of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$, or equivalently, let $\mathcal{R}$ be an E_n-algebra. Let ε be a small non-zero number. Consider a k-disk with the trivial (page) framing. Then $S^{k-1} \times \mathbb{R} \cong S^{k-1} \times (0, \varepsilon)$ bounds the k-disk and also inherits its framing. The framing of $S^{k-1} \times \mathbb{R}^{n-k+1}$ is the product framing with the trivial framing on $\mathbb{R}^{n-k}$. At the level of factorization homology we then get an induced action

$$\int_{S^{k-1} \times \mathbb{R}^{n-k+1}} \mathcal{R} \otimes \int_{\mathbb{R}^n} \mathcal{R} \xrightarrow{\triangleright} \int_{\mathbb{R}^n} \mathcal{R} \cong \mathcal{R} \tag{10.3}$$

for each $k \in \{0, 1, \dots, n\}$. Here we use the convention that $S^{-1} = \emptyset$. Two explicit examples of this action is given in Example 10.20 and Example 10.21.

The below was first conjectured in [**Lur09b**, Remark 4.1.27], where it is claimed to follow from the proof of the cobordism hypothesis.

Conjecture-Theorem 10.17. *Let $\mathcal{C}$ be a symmetric monoidal $\otimes$-presentable ∞-category. Let $\mathcal{R}$ denote an object of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ and let $\mathcal{R}$ be the corresponding E_n-algebra. The object $\mathcal{R}$ is $(n+1)$-dualizable in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ if, and only if, it is dualizable over the factorization homologies*

$$\int_{S^{k-1} \times \mathbb{R}^{n-k+1}} \mathcal{R} \tag{10.4}$$

for $k = 0, 1, \dots, n$.

This was proven by Lurie for $n = 1$ as recalled in Theorem 10.16. For $n = 2$ and $\mathcal{C} = \mathrm{Pr}$ the theorem is proven in [**BJS21**]. See Remark 10.27 for some more details on the latter. We will give a (partial) proof of Conjecture-Theorem 10.17 for $n = 2$ assuming that Conjecture 10.26 holds.

Remark 10.18. The condition of Conjecture-Theorem 10.17 for $k = 0$ boils down to asking for $\mathcal{R}$ to be dualizable as an object of $\mathcal{C}$. Hence, to understand $(n+1)$-dualizability in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ one inherently needs to understand dualizability in the underlying category $\mathcal{C}$.

Remark 10.19. We highlight the following observation regarding the minimal number of conditions present in Conjecture-Theorem 10.17. From Theorem B.24 we know that it is enough to provide a partial set of full $(n+1)$-dualizability data to prove that an object $\mathcal{R} \in \mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ is $(n+1)$-dualizable. This boils down to providing e.g. left adjoints (and corresponding unit and counit $(n+1)$-morphisms) to the 2^n n-morphisms appearing in the partial set of full n-dualizability data of the object. This is in stark contrast to the condition in (10.4) which only requires left adjoints (and corresponding unit and counit $(n+1)$-morphisms) to $(n+1)$ n-morphisms.

Before we address the proof of Conjecture-Theorem 10.17 for $n = 2$ we need some preliminary results. However, first of all we give some intuition and explanation regarding the factorization homologies showing up in (10.4) as well as the action from (10.3).

Example 10.20. Let $n = 1 = k$. The framing of $S^0 \times \mathbb{R}$ is the one inherited from the page framing of the 1-disk, i.e. the interval $(-1, 1)$ together with the page framing. One convenient way to picture this is to imagine bending the 1-disk in the ambient space, i.e. in the x_2-direction of $\mathbb{R}^2$ say. The 1-disk after this bending is illustrated in Figure 23 together with $S^0 \times \mathbb{R}$ (corresponding to the two intervals) with its induced framing. The action (10.3) is induced by the inclusion $(S^0 \times \mathbb{R}) \amalg \mathbf{D}^1 \to \mathbf{D}^1$, or in words by "stacking" the two smaller intervals onto the 1-disk.

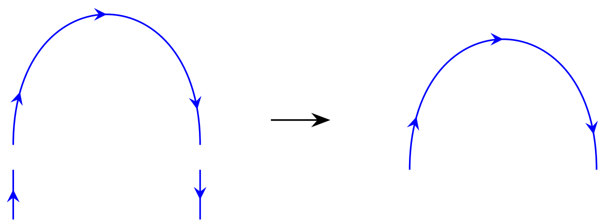


FIGURE 23. The inclusion inducing the action (10.3) for $n = 1 = k$.

Example 10.21. Now let $n = 2 = k$. Then $S^1 \times \mathbb{R}$ comes with the framing inherited from the page framing of the 2-disk. This is illustrated in Figure 24 below, which also shows the inclusion giving rise to the action (10.3) in this case. If one instead considers $n = 2$ and $k = 1$ we get a thickened version of the situation illustrated in Figure 23, i.e. the figure times the identity on a copy of $(0, 1)$ going into the page.

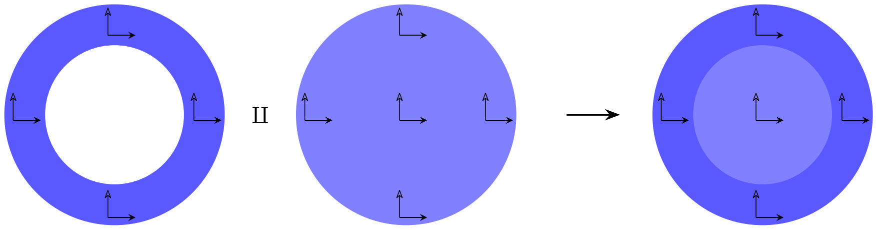


FIGURE 24. The inclusion inducing the action (10.3) for $n = 2 = k$.

We now start giving some preliminary constructions and results needed to prove Conjecture-Theorem 10.17 for $n = 2$.

Definition 10.22. Let $n \in \mathbb{N}$ and let $(0,1)^n = \mathrm{A_m}(\bar{S})$ for $\bar{S} \in \mathrm{S{\scriptsize TRAT}}^n$ where each $S_i = \{0, s_i, 1\}$, $i \in \{0, 1, \dots, n-1\}$, while S_n has $k+2$ many points. Meanwhile, $(0,1) = \mathrm{A_m}(S_n)$ is the stratified (marked) spaces associated with S_n. Define

$$\mathrm{pr}_n \colon (0,1)^n \longrightarrow (0,1), \qquad (x_1, \dots, x_n) \mapsto x_n \ .$$

In words, pr_n is the projection onto the last coordinate.

With the chosen (marked) stratifications above we have that pr_n is adequately stratified as in Definition 8.12. Thus, by Lemma 8.14 pushing forward along pr_n gives a functor

$$(\mathrm{pr}_n)_\star \colon \mathrm{F{\scriptsize ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(\bar{S})} \longrightarrow \mathrm{F{\scriptsize ACT}}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(S_n)} \ . \tag{10.5}$$

Moreover, $(\mathrm{pr}_n)_\star$ is even a symmetric monoidal functor as it is clearly compatible with the symmetric monoidal structure on $\mathrm{F{\scriptsize ACT}}^{\mathrm{pl,cstr}}_{(0,1)^n}$, respectively $\mathrm{F{\scriptsize ACT}}^{\mathrm{pl,cstr}}_{(0,1)}$ from Proposition 5.11.

Observe that a constructible pointless factorization algebra $\mathcal{F}$ on the left hand side of (10.5) encodes an n-morphism of $\mathrm{A{\scriptsize LG}}^{\mathrm{pl}}_n(\mathcal{C})$, while its image $(\mathrm{pr}_n)_\star\mathcal{F}$ encodes a 1-morphism of $\mathrm{A{\scriptsize LG}}^{\mathrm{pl}}_1(\mathcal{C})$. Thus, $(\mathrm{pr}_n)_\star$ induces a functor at the level of the pointless higher Morita categories which we prove in Lemma 10.24. For this we will need the following notation.

Notation 10.23. Let $X_{\vec{\bullet}}$ be an n-fold simplicial object in $\mathcal{D}$. We abbreviate i many copies of 1 in $\vec{\bullet}$ by (i) as in e.g. [**JFS17**]. Explicitly, $X_{\vec{\bullet},1,1,\dots,1,\vec{\bullet}} = X_{\vec{\bullet},(i),\vec{\bullet}}$, where the former had i many copies of 1.

Lemma 10.24. *Fix $\mathcal{R}, \mathcal{S}$ two $(n-2)$-morphisms of $\mathrm{A{\scriptsize LG}}^{\mathrm{pl}}_n(\mathcal{C})$, for $n \geq 2$, and let $\mathrm{Hom}_{\mathrm{A{\scriptsize LG}}^{\mathrm{pl}}_n}(\mathcal{R}, \mathcal{S})$ denote the corresponding $(\infty, 2)$-category of morphisms. The functor* (10.5) *induces a functor of $(\infty, 2)$-categories*

$$(\mathrm{pr}_n)_\star \colon \mathrm{Hom}_{\mathrm{A{\scriptsize LG}}^{\mathrm{pl}}_n}(\mathcal{R}, \mathcal{S}) \longrightarrow \mathrm{A{\scriptsize LG}}^{\mathrm{pl}}_1(\mathcal{C}) \tag{10.6}$$

which moreover is symmetric monoidal.

PROOF. Let $(\mathrm{Alg}^{\mathrm{pl}}_n)_{\vec{\bullet}}$ be the n-fold Segal object in $\mathrm{C{\scriptsize AT}}_\infty$ from Section 9.1.6 determining the pointless higher Morita category $\mathrm{A{\scriptsize LG}}^{\mathrm{pl}}_n(\mathcal{C})$. Then the left hand side of (10.6) is given by the following Segal object in $\mathrm{C{\scriptsize AT}}_\infty$

$$\mathrm{Hom}_{\mathrm{A{\scriptsize LG}}^{\mathrm{pl}}_n}(\mathcal{R}, \mathcal{S})_\bullet = \{\mathcal{R}\} \times_{(\mathrm{Alg}^{\mathrm{pl}}_n)_{(n-2),0,\bullet}} (\mathrm{Alg}^{\mathrm{pl}}_n)_{(n-1),\bullet} \times_{(\mathrm{Alg}^{\mathrm{pl}}_n)_{(n-2),0,\bullet}} \{\mathcal{S}\} \ . \tag{10.7}$$

From (10.7) we can forget about the fixed target and source to only get $(\mathrm{Alg}^{\mathrm{pl}}_n)_{(n-1),\bullet}$. Let $\bar{S} \in \mathrm{S{\scriptsize TRAT}}^{n-1}$ be an $(n-1)$-tuple of stratifications which all are of the form $S_i = \{0, s_i, 1\}$ for some $0 < s_i < 1$, i.e. such that $\pi^{n-1}(\bar{S}) = [1]^{n-1}$. Thus, $(\mathrm{Alg}_n^{\mathrm{pl}})_{(n-1),\bullet}$ is a left Kan extension of $\textsc{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(\bar{S},-)}\colon \textsc{Strat} \to \textsc{Cat}_\infty$, and it suffices to show that we get an induced natural transformation

$$(\mathrm{pr}_n)_\star\colon \textsc{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(\bar{S},-)} \Rightarrow \textsc{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(-)}\ .$$

This follows from the following diagram commuting from any two $S_n, S_n' \in \textsc{Strat}$ and any generating morphism f of $\textsc{Strat}$

$$\begin{tikzcd}
\textsc{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(\bar{S},S_n)} \arrow[r, "(\mathrm{id}^{n-1}_{(0,1)}\times f)_\star"] \arrow[d, "(\mathrm{pr}_n)_\star"'] & \textsc{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(\bar{S},S_n')} \arrow[d, "(\mathrm{pr}_n)_\star"] \\
\textsc{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(S_n)} \arrow[r, "f_\star"'] & \textsc{Fact}^{\mathrm{pl,cstr}}_{\mathrm{A_m}(S_n')}
\end{tikzcd}\ .$$

Thus, we indeed get a natural transformation between $(\mathrm{Alg}_n^{\mathrm{pl}})_{(n-1),\bullet}$, respectively $(\mathrm{Alg}_1^{\mathrm{pl}})_\bullet$. The adaptation to include the symmetric monoidal structure is straightforward. □

With the above construction we can now give the following corollary which states that one can transfer adjunctability-data from $\textsc{Alg}_n^{\mathrm{pl}}(\mathcal{C})$ to $\textsc{Alg}_1^{\mathrm{pl}}(\mathcal{C})$.

Corollary 10.25. *If an n-morphism $\mathbf{M}\colon \mathbf{A} \to \mathbf{B}$ of $\textsc{Alg}_n^{\mathrm{pl}}(\mathcal{C})$ has a left (right) adjoint, then the image of $\mathbf{M}$ in $\textsc{Alg}_1^{\mathrm{pl}}(\mathcal{C})$ also has a left (right) adjoint.*

Proof. Follows from the functor (10.6) being symmetric monoidal and Proposition B.25. □

The converse implication to that of Corollary 10.25 is far from trivial. That is, let $\mathbf{M}$ again be an n-morphism of $\textsc{Alg}_n^{\mathrm{pl}}(\mathcal{C})$, and assume you know that $(\mathrm{pr}_n)_\star\mathbf{M}$ has a left adjoint 1-morphism in $\textsc{Alg}_1^{\mathrm{pl}}(\mathcal{C})$. Does this give rise to an adjunction in $\textsc{Alg}_n^{\mathrm{pl}}(\mathcal{C})$? We conjecture that this indeed is the case.

Conjecture 10.26. *Let $\mathbf{M}$ be an n-morphism in $\textsc{Alg}_n^{\mathrm{pl}}(\mathcal{C})$. If $(\mathrm{pr}_n)_\star\mathbf{M}$ has a left (right) adjoint in $\textsc{Alg}_1^{\mathrm{pl}}(\mathcal{C})$, then $\mathbf{M}$ has a left (right) adjoint n-morphism in $\textsc{Alg}_n^{\mathrm{pl}}(\mathcal{C})$.*

The difficulty likes in "promoting" the adjoint 1-morphism in $\textsc{Alg}_1^{\mathrm{pl}}(\mathcal{C})$ to an n-morphism of $\textsc{Alg}_n^{\mathrm{pl}}(\mathcal{C})$. This requires producing a constructible pointless factorization algebra on $(0,1)^n$ (stratified by the full flag), where part of the data comes from a constructible pointless factorization algebra on $0,1)$ (stratified by a marked point). Explicitly, for $n = 2$ one has (after gluing) constructible pointless factorization algebras on the stratified spaces of Figure 25 from which one wants to produce a constructible pointless factorization algebra encoding a 2-morphism in $\textsc{Alg}_2^{\mathrm{pl}}(\mathcal{C})$.

FIGURE 25. The pieces from which one needs to define a 2-morphism of $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$ to prove Conjecture 10.26 for $n = 2$.

It seems like a marked version of the localization-result of ∞-operads as in [**KSW24**, Proposition 5.19 & Corollary 5.20] could be helpful for this as it would allow one to move between posets of marked opens of $(0, 1)$ and $(0, 1)^n$ as long as their localization agrees. Unfortunately, we do not (yet) have an analogue of this in the setting of marked opens and constructible pointless factorization algebras as explained in Remark 7.32.

Remark 10.27. Recall the description of $\mathrm{ALG}_2^{\mathrm{pl}}(\mathrm{Pr})$ of [**BJS21**] recalled in Example 9.38. In that setting Conjecture 10.26 is proven in [**BJS21**, Proposition 5.17 & Lemma 5.19]. Given a 2-morphism, i.e. in their notation an $\mathcal{A}$-centered $(\mathcal{C}, \mathcal{D})$-bimodule $\mathcal{M}$, one first forgets about the $\mathcal{A}$-central structure to get $\mathcal{M}$ as a 1-morphism of $\mathrm{ALG}_1^{\mathrm{pl}}(\mathrm{Pr})$. Then, if $\mathcal{M}$ has a left adjoint 1-morphism of $\mathrm{ALG}_1^{\mathrm{pl}}(\mathrm{Pr})$, i.e. a $(\mathcal{D}, \mathcal{C})$-bimodule ${}^{\vee}\mathcal{M}$, they equip the left adjoint with a canonical $\mathcal{A}$-central structure. That is, they exactly promote the left adjoint 1-morphism of $\mathrm{ALG}_1^{\mathrm{pl}}(\mathrm{Pr})$ to a 2-morphism of $\mathrm{ALG}_2^{\mathrm{pl}}(\mathrm{Pr})$. Moreover, they also show that the unit and counit 2-morphisms of $\mathrm{ALG}_1^{\mathrm{pl}}(\mathrm{Pr})$ extends or promotes to unit and counit 3-morphisms of $\mathrm{ALG}_2^{\mathrm{pl}}(\mathrm{Pr})$ in [**BJS21**, Lemma 5.19].

In addition, Brochier, Jordan, Safronov and Snyder claim that the argument specific to $\mathcal{C} = \mathrm{Pr}$ also applies for arbitrary target categories in [**BJSS21**] without further justification, which is not straightforward. See in particular [**BJSS21**, Proposition 2.20 & Theorem 2.22]. In private communication with the authors they suggested that their approach might still go through by a version of the proof of Proposition 10.15 but a revised version is yet to appear.

When the original n-morphism is of a particularly easy form we can give an affirmative answer to Conjecture 10.26. Part of the n-morphisms appearing in the dualizability-data of an object is of this form.

Observation 10.28. Let $\mathbf{M}$ denote an n-morphism of $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ encoded by a constructible pointless factorization algebras $\mathcal{F}$. That is, $\mathcal{F}$ is defined on $(0, 1)^n$ stratified by the full flag. Assume that $\mathcal{F}$ evaluated at a standard disk is only non-trivial if the disk intersects the 1- or 0-dimensional strata, i.e. the strata encoding the source and target $(n-1)$-morphisms of $\mathbf{M}$ as well as $\mathbf{M}$ itself. Then a left (right) adjoint of $(\mathrm{pr}_n)_\star\mathbf{M}$ in $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ immediately determines a left (right) adjoint of $\mathbf{M}$ in $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$ by Proposition 8.23.

The above situation exactly corresponds to the setting where there is no non-trivial information forgotten by pushing forward along pr_n. We now finally give the proof of Conjecture-Theorem 10.17 for $n = 2$.

PROOF OF CONJECTURE-THEOREM 10.17 FOR $n = 2$ ASSUMING CONJECTURE 10.26. We first consider the forward implication. That is, assume that $\mathcal{R}$ is 3-dualizable in $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$. In particular, all the 2-morphisms $\varepsilon_{\mathrm{ev}}, \eta_{\mathrm{ev}}, \varepsilon_{\mathrm{coev}}, \eta_{\mathrm{coev}}$ appearing in the 2-dualizability data of $\mathcal{R}$ has left adjoints. By Corollary 10.25 it follows that the images of the 2-morphisms under the functor (10.6) appearing in the 2-dualizability data has left adjoints as 1-morphisms of $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$. We denote the images by

$$\overline{\varepsilon}_{\mathrm{ev}}, \overline{\eta}_{\mathrm{ev}}, \overline{\varepsilon}_{\mathrm{coev}}, \overline{\eta}_{\mathrm{coev}} \ . \tag{10.8}$$

Recall from Convention 9.29 and in particular Figure 15 how one extracts the source and target 1-morphisms from a 2-morphism. By directly reading off from the figure Figure 18 we have that (after pushing forward along pr_n) the source object of $\overline{\varepsilon}_{\mathrm{coev}}$ is equivalent to the $k = 2$-condition of (10.4). Similarly, from Figure 19 and Figure 20 we get that the source object of $\overline{\eta}_{\mathrm{coev}}$ as well as that of $\overline{\varepsilon}_{\mathrm{ev}}$ corresponds to the $k = 0$-condition. Meanwhile the source object of $\overline{\eta}_{\mathrm{ev}}$ is the trivial object and corresponds to the $k = 0$ condition. By Proposition 10.15 it follows that the existence of left adjoints of the 1-morphisms in (10.8) is equivalent to them being dualizable as left modules of their source object which we have seen to be given exactly by that of (10.4).

For the converse implication we will use Theorem B.24 which tells us that to prove 3-dualizability of $\mathcal{R}$ it is sufficient to provide left adjoints of all 2-morphisms appearing in the 2-dualizability data. As explained above we get left adjoints of the 2-dualizability data transferred to $\mathrm{ALG}_1^{\mathrm{pl}}(\mathcal{C})$ as in (10.8) from the conditions in (10.4). The left adjoints of $\overline{\varepsilon}_{\mathrm{coev}}$ and $\overline{\eta}_{\mathrm{ev}}$ (and the corresponding unit and counit morphisms) immediately give rise to 2-morphisms (and corresponding unit and counit 3-morphisms) of $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$ by Observation 10.28. The remaining two cases are promoted to $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$ by Conjecture 10.26, thus witnessing that $\mathcal{R}$ is indeed 3-dualizable in $\mathrm{ALG}_2^{\mathrm{pl}}(\mathcal{C})$. $\square$

Part 3

Enriched skeins and factorization homology

Attribution. This part of the thesis is based on the first part of [**KKMP**], a joint article with Corina Keller, Lukas Müller and Jan Pulmann. Amongst my main contributions to that article were Section 4.3 and 7.1. Section 4.3 contains the proof that enriched skein categories satisfy excision, while Section 7.1 contains a computation of internal endomorphism algebras for surfaces with boundary and how they behave with respect to fusion of surfaces. This part contains the former, which corresponds to the proof that enriched skein categories compute factorization homology in $\mathcal{V}$-Cat, together with the necessary enriched category theoretic setup.

Assumption. In this part of the thesis we assume the enriching category $\mathcal{V}$ to be a complete and cocomplete symmetric monoidal closed category.

11. Background on factorization homology and enriched category theory

We start this section by recalling the definition of oriented factorization homology, as well as its characterizing properties in Section 11.1. Then we introduce some notation and recall the 2-category $\mathcal{V}$-Cat in Section 11.2. We also need to understand two models for the enriched relative tensor products; the truncated bar construction and the enriched Tambara relative tensor product. These are introduced in Section 11.3. Lastly, for the convenience of the reader we recall the definition of $\mathcal{V}$-enriched ribbon categories in Section 11.4. The main example of such an enriched category from [**KKMP**] is highlighted in Example 11.28.

11.1. Factorization homology.

We first briefly recall the definition of oriented factorization homology, as well as the conditions on the target category to ensure its existence. Then we explain how oriented factorization homology is characterised by three properties of which a certain gluing property called excision is the most interesting. We refer the reader to e.g. [**Lur17**] or [**AF15**] for more details.

Definition 11.1. Let $\mathcal{M}\mathrm{fld}_d^{\mathrm{or}}$ denote the topological category (i.e. ∞-category) whose

- objects are oriented d-dimensional manifolds, and
- the space of morphisms $\mathrm{Emb}^{\mathrm{or}}(X, Y)$ is the space of oriented smooth embeddings of X into Y equipped with the compact-open topology.

Let $\mathcal{D}\mathrm{isk}_d^{\mathrm{or}} \subset \mathcal{M}\mathrm{fld}_d^{\mathrm{or}}$ denote the full ∞-subcategory whose objects are disjoint unions of oriented disks. The disjoint union induces symmetric monoidal structures on these ∞-categories which we denote by $\mathcal{M}\mathrm{fld}_d^{\mathrm{or},\otimes}$ and $\mathcal{D}\mathrm{isk}_d^{\mathrm{or},\otimes}$, respectively.

We now define the algebraic input for oriented factorization homology.

Definition 11.2. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. A *framed* E_d*-algebra* $\mathcal{A}$ (in $\mathcal{C}$) is a symmetric monoidal functor

$$\mathcal{A}\colon \mathcal{D}\mathrm{isk}_d^{\mathrm{or},\otimes} \longrightarrow \mathcal{C}^{\otimes} .$$

Explicitly, $\mathcal{C}^{\otimes}$ is as in Definition A.17, and giving a (strict) symmetric monoidal functor $\mathcal{A}$ as above corresponds to giving an algebra map which preserves cocartesian morphisms.

Definition 11.3. Let $\mathcal{C}$ be a symmetric monoidal ∞-category which is $\otimes$-sifted cocomplete. *Factorization homology* $\int_{(-)}\mathcal{A}$ *with coefficients in the framed* E_d*-algebra* $\mathcal{A}$ is defined as the left Kan extension of $\mathcal{A}$ along the inclusion $\iota\colon \mathcal{D}\mathrm{isk}_d^{\mathrm{or},\otimes} \hookrightarrow \mathcal{M}\mathrm{fld}_d^{\mathrm{or},\otimes}$. Explicitly,

$$\begin{tikzcd} \mathcal{D}\mathrm{isk}_d^{\mathrm{or},\otimes} \arrow[r, "\mathcal{A}"] \arrow[d, hook, "\iota"'] & \mathcal{C}^{\otimes} \\ \mathcal{M}\mathrm{fld}_d^{\mathrm{or},\otimes} \arrow[ur, dashed, "\int_{(-)}\mathcal{A}"'] & \end{tikzcd} .$$

Remark 11.4. In this part of this thesis we only work with oriented manifolds, or even more concretely with oriented surfaces. Hence, we have given the definitions above tailored to this setting. By replacing "oriented manifolds" with "framed manifolds", and "framed E_d-algebra" by just "E_d-algebra" above one instead gets the definition of (framed) factorization homology.

Remark 11.5. The left Kan extension has a pointwise formula which is a sifted colimit. Thus, assuming $\mathcal{C}^{\otimes}$ to be sifted cocomplete ensures existence of oriented factorization homology.

For this to be a symmetric monoidal functor one additionally asks for the tensor product to commute with these sifted colimits. See [**AF15**, Proposition 3.7] for more details.

Remark 11.6 (Factorization homology is pointed)**.** Since we work with (oriented) embeddings when defining (oriented) factorization homology the value of factorization homology on any oriented manifold M is naturally *pointed* by the inclusion $\emptyset \hookrightarrow M$ of the empty manifold. Explicitly, the inclusion $\emptyset \hookrightarrow M$ induces a map

$$\int_{\emptyset} \mathcal{A} \cong 1_{\mathcal{C}} \longrightarrow \int_M \mathcal{A} \ .$$

Remark 11.7 (Pointless factorization homology)**.** In light of Part 1 of this thesis one might ask if there is a corresponding *pointless* version of factorization homology. The definition should be straightforward by replacing the ∞-categories of d-dimensional manifold and disks by marked versions, where the embeddings are required to induce bijections on the marked points as in Section 6.1 and in particular Notation 6.16. Moreover, $\mathcal{C}$ should be $\otimes$-presentable instead of just $\otimes$-sifted cocomplete. The name reflects how the pointing induced by the inclusion of the empty set would no longer be present.

This type of definition is already introduced by Ben-Zvi, Brochier and Jordan in the setting of surfaces, see Section 2.2 and in particular Definition 2.5 in [**BZBJ18b**]. They moreover state (and use) that excision holds in this setting without proof. We expect results like excision and the fact that factorization homology gives rise to factorization algebras, c.f. [**KSW24**, §6.4], to carry over to the pointless analogue. However, making this precise is beyond the scope of this thesis.

We now turn to the characterizing properties of factorization homology. Just like how ordinary homology theories are characterized by satisfying the Eilenberg–Steenrod axioms, there is an analogous list of properties characterizing (oriented) factorization homology. To formulate the gluing property, which goes under the name of excision, we first need the following definition.

Definition 11.8. A *collar-gluing* of an oriented manifold M is a decomposition $M = M_- \bigcup_{M_0} M_+$. Here, M_- and M_+ are open subsets of M while $M_0 = M_- \cap M_+$ comes with a direct product structure, i.e. a diffeomorphism $\theta\colon M_0 \xrightarrow{\cong} N \times (-1,1)$ of oriented manifolds.

We refer the reader to [**AF15**, Definition 3.13 & Remark 3.14] for more details.

Example 11.9. Figure 26 illustrates an example of a collar-gluing of $M = \Sigma_{2,0}$, i.e. the genus 2 surface. Explicitly, we have $M_- \cong \Sigma_{1,1} \cong M_+$, while $M_0 \cong \mathbf{S}^1 \times (-1,1)$.

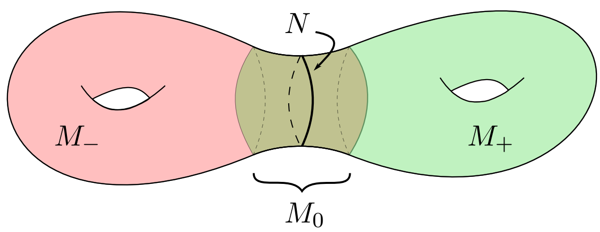


FIGURE 26. Example of a collar-gluing of the genus 2 surface.

Factorization homology for the product manifold $N \times (-1,1)$ has a natural framed E_1-algebra structure induced by embeddings of open intervals. This in turn gives rise to a framed E_1-algebra structure on M_0 under the diffeomorphism $\theta\colon M_0 \cong N \times (-1,1)$.

Construction 11.10. Fix two oriented embeddings

$$\mu_-\colon [-1,1)\sqcup(-1,1)\hookrightarrow[-1,1) \quad \text{and} \quad \mu_+\colon (-1,1)\sqcup(-1,1]\hookrightarrow(-1,1]$$

such that $\mu_-(-1)=-1$ and $\mu_+(1)=1$. Under the diffeomorphism θ these maps give embeddings

$$\mathrm{act}_-\colon M_-\sqcup M_0\longrightarrow M_- \quad \text{and} \quad \mathrm{act}_+\colon M_0\sqcup M_+\longrightarrow M_+ \ .$$

The maps act_- and act_+ induce right, respectively left $\int_{M_0}\mathcal{A}$-module structures on $\int_{M_-}\mathcal{A}$, respectively $\int_{M_+}\mathcal{A}$.

The right module structure induced by act_- for the collar-gluing from Example 11.9 is illustrated in Figure 27.

We can now properly state the gluing-result for factorization homology.

Lemma 11.11. [**AF15**, Lemma 3.18] *Let $M=M_-\bigcup_{M_0}M_+$ be a collar-gluing of oriented d-manifolds and let $\mathcal{A}$ be a framed E_d-algebra in $\mathcal{C}$, an $\otimes$-sifted cocomplete symmetric monoidal ∞-category. Then there is an equivalence of ∞-categories*

$$\int_M\mathcal{A}\simeq\int_{M_-}\mathcal{A}\underset{\int_{M_0}\mathcal{A}}{\otimes}\int_{M_+}\mathcal{A} \ ,$$

where the right hand side relative tensor product is computed by the colimit of the 2-sided bar construction

$$\ldots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \mathcal{M}_-\otimes\mathcal{N}\otimes\mathcal{N}\otimes\mathcal{M}_+ \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathcal{M}_-\otimes\mathcal{N}\otimes\mathcal{M}_+ \rightrightarrows \mathcal{M}_-\otimes\mathcal{M}_+$$

for $\mathcal{M}_-:=\int_{M_-}\mathcal{A}$, $\mathcal{M}_+:=\int_{M_+}\mathcal{A}$ and $\mathcal{N}:=\int_{M_0}\mathcal{A}$.

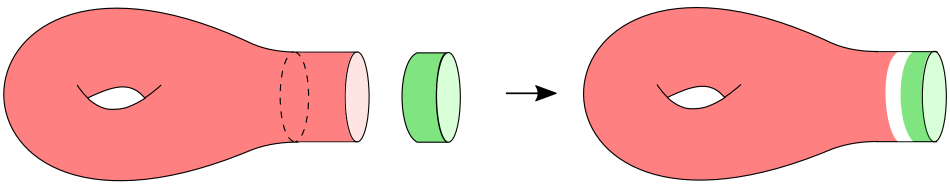


FIGURE 27. The map which induces the right $\int_{M_0}\mathcal{A}$-module structure on $\int_{M_-}\mathcal{A}$. The green collar depicts the product manifold $N\times(-1,1)$.

Excision is a very powerful computational tool, as it allows one to compute factorization homology of complicated manifolds from simpler pieces. We now give the full list of characterizing properties of oriented factorization homology, of which excision is typically the most involved to prove.

THEOREM 11.12. [**AF15, AFT17a**] *Let $\mathcal{A}$ be a framed E_2-algebra in $\mathcal{C}$. The functor $\int_{-}\mathcal{A}$ is characterized by the following properties:*

(1) If U is contractible then there is an equivalence in $\mathcal{C}$

$$\int_U\mathcal{A}\simeq\mathcal{A}.$$

(2) If $M_0\cong N\times(-1,1)$ for some 1-manifold N, then the inclusion of intervals inside a larger interval induces a canonical framed E_1-structure on $\int_{M_0}\mathcal{A}$.

(3) $\int_{-}\mathcal{A}$ satisfies excision.

11.2. The target: $\mathcal{V}$-Cat. Recall that we assume the enriching category $\mathcal{V}$ to be a complete and cocomplete symmetric monoidal closed category. We first introduce some notation before defining the 2-category $\mathcal{V}$-Cat of enriched categories, as well as explaining its natural monoidal product, following [**KKMP**, §3]. Then we recall a result from [**KKMP**] regarding when $\mathcal{V}$-Cat admits all bicolimits, ensuring that it is a suitable target for factorization homology.

Notation 11.13. Let $\mathcal{C}$ be a $\mathcal{V}$-enriched category. Its *underlying category*, denoted $\mathcal{C}_0$, is the ordinary category with the same objects as $\mathcal{C}$, and with morphisms $\mathcal{C}_0(c,c') := \mathcal{V}(1_\mathcal{V}, \mathcal{C}(c,c'))$.

If we have a morphism $f \in \mathcal{C}_0(c,c')$, *precomposition* with f is the morphism in $\mathcal{V}$

$$\mathcal{C}(c',c'') \xrightarrow{-\circ f} \mathcal{C}(c,c'')$$

given by the composition

$$\mathcal{C}(c',c'') \xrightarrow{\cong} \mathcal{C}(c',c'') \otimes^\mathcal{V} 1_\mathcal{V} \xrightarrow{\mathrm{id}\otimes^\mathcal{V} f} \mathcal{C}(c',c'') \otimes^\mathcal{V} \mathcal{C}(c,c') \xrightarrow{\circ^\mathcal{C}} \mathcal{C}(c,c'') ,$$

where in the last step we use the composition of $\mathcal{C}$. *Postcomposition* by f, i.e. $f \circ -$, is defined similarly.

Moreover, if $\mathcal{C}$ is monoidal (see Section 11.4 for the definition), we analogously define

$$\mathcal{C}(e,e') \xrightarrow{-\otimes^\mathcal{C} f} \mathcal{C}(e \otimes^\mathcal{C} c, e' \otimes^\mathcal{C} c')$$

and

$$\mathcal{C}(c' \otimes^\mathcal{C} d', c'' \otimes^\mathcal{C} d'') \xrightarrow{-\circ(f\otimes^\mathcal{C} g)} \mathcal{C}(c \otimes^\mathcal{C} d, c'' \otimes^\mathcal{C} d'') ,$$

where $g \in \mathcal{C}_0(d,d')$.

Definition 11.14. Let $(\mathcal{V}, \otimes^\mathcal{V})$ be a symmetric monoidal category. We define $\mathcal{V}$-Cat to be the following 2-category; objects are *$\mathcal{V}$-enriched categories* $\mathcal{C}, \mathcal{D}$. The 1-morphisms are *$\mathcal{V}$-enriched functors* $F \colon \mathcal{C} \to \mathcal{D}$, i.e. a function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ together with morphisms

$$F_{c,c'} \colon \mathcal{C}(c,c') \longrightarrow \mathcal{D}(F(c), F(c'))$$

in $\mathcal{V}$. The 2-morphisms are *$\mathcal{V}$-natural transformations* $\alpha \colon F \Rightarrow G$ between $F, G \colon \mathcal{C} \to \mathcal{D}$, with components $\alpha_c \colon 1_\mathcal{V} \to \mathcal{D}(F(c), G(c))$, for all $c \in \mathcal{C}$, making the following diagram commute

$$\begin{tikzcd}
\mathcal{C}(c,c') \arrow[r, "F_{c,c'}"] \arrow[d, "G_{c,c'}"] & \mathcal{D}(F(c), F(c')) \arrow[d, "\alpha_{c'}\circ -"] \\
\mathcal{D}(G(c), G(c')) \arrow[r, "-\circ\alpha_c"] & \mathcal{D}(F(c), G(c'))
\end{tikzcd} .$$

We denote the set of $\mathcal{V}$-natural transformations from F to G by $\mathcal{V}\text{-Nat}(F,G)$. For $\mathcal{C}, \mathcal{D} \in \mathcal{V}$-$\mathsf{Cat}$ we denote the category of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations between them by $\mathcal{V}\text{-Fun}(\mathcal{C}, \mathcal{D})$.

We have that a $\mathcal{V}$-functor $F \colon \mathcal{C} \to \mathcal{D}$ is an *equivalence* if and only if it is essentially surjective and fully faithful. In the enriched context, a functor is called *fully-faithful* if each morphism $F_{c,c'}$ is an isomorphism in $\mathcal{V}$ [**Kel82**, §1]. We now turn towards the tensor product on $\mathcal{V}$-Cat which uses that $\mathcal{V}$ is closed and complete.

Definition 11.15. Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathcal{V}$-categories. Define $\mathcal{C} \times \mathcal{D}$ to be the $\mathcal{V}$-category whose objects are pairs $(c, d) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$ and its morphisms are defined by

$$\mathcal{C} \times \mathcal{D}((c, d), (c', d')) := \mathcal{C}(c, c') \otimes^{\mathcal{V}} \mathcal{D}(d, d') \in \mathcal{V} .$$

The 2-category $\mathcal{V}$-Cat is closed under the above tensor product [**Kel82**, Section 2.3]. That is, there is an equivalence of categories

$$\mathrm{Hom}_{\mathcal{V}\text{-}\mathsf{Cat}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathrm{Hom}_{\mathcal{V}\text{-}\mathsf{Cat}}(\mathcal{C}, [\mathcal{D}, \mathcal{E}]) \tag{11.1}$$

which is 2-natural in the $\mathcal{V}$-categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$. Here, $[\mathcal{D}, \mathcal{E}]$ is the $\mathcal{V}$-category whose objects are $\mathcal{V}$-functors $F \colon \mathcal{D} \to \mathcal{E}$ and morphisms are defined by the objects

$$\mathcal{V}\text{-}\mathbf{Nat}(F, G) := \int_{d \in \mathcal{D}} \mathcal{E}(F(d), G(d)) \ \in \mathcal{V} ,$$

where the right hand side is a $\mathcal{V}$-enriched end.

From Equation (11.1) we see that the internal hom is right adjoint to the natural tensor product of $\mathcal{V}$-Cat. Thus, the tensor product from Definition 11.15 preserves colimits in $\mathcal{V}$-Cat. It only remains to prove that $\mathcal{V}$-Cat admits enough colimits to ensure existence of factorization homology. We refer the reader to [**KKMP**, Appendix A.3] for more details and a proof of the following proposition.

Proposition 11.16. [**KKMP**, Proposition A.4] *Let $\mathcal{V}$ be a cocomplete and symmetric monoidal closed category. Then the 2-category $\mathcal{V}$-Cat admits all bicolimits.*

Thus, we get that $\mathcal{V}$-Cat is a suitable target for factorization homology given Part 3. We also briefly give an example of an enriching category which is of particular interest in [**KKMP**] in the context of deformation quantization. See also Remark 12.16 for a situation using this enriching category.

Example 11.17. Let M be a left $\mathbb{C}[[\hbar]]$-module, where $\mathbb{C}[[\hbar]]$ denotes the formal power series ring. Consider the submodules $(\hbar^n M)_{n \in \mathbb{N}}$ and denote $M_n := M_n := M/\hbar^n M$. Then $(M_n, p_n)_{n \in \mathbb{N}}$ is an inverse system of $\mathbb{C}[[\hbar]]$-modules, where p_n denotes the canonical projections $p_n \colon M_n \to M_{n-1}$. The *$\hbar$-adic completion* of M, denoted $\widehat{M}$, is then defined as the inverse limit

$$\widehat{M} := \varprojlim_{n} M_n = \{(x_n) \in \prod_n M_n \,|\, p_n(x_n) = x_{n-1}\} .$$

If the canonical map $M \to \widehat{M}$ is an isomorphism we say that M is a *complete* $\mathbb{C}[[\hbar]]$-module. We denote the category of complete $\mathbb{C}[[\hbar]]$-modules by $\widehat{\mathbb{C}[[\hbar]]\text{-}\mathrm{Mod}}$. Its symmetric monoidal structure comes from the completed tensor product. Moreover, $\mathcal{V} = \widehat{\mathbb{C}[[\hbar]]\text{-}\mathrm{Mod}}$ is also complete and cocomplete symmetric monoidal closed, c.f. [**KKMP**, §3.5] and the references therein. Thus, it provides an example of an enriching category such that factorization homology in $\mathcal{V}$-Cat exists.

11.3. Enriched relative tensor products. Factorization homology satisfy excision, as recalled in Lemma 11.11. This is formulated using the two-sided bar construction as a model for the relative tensor product. However, it turns out to be convenient to have access to more models. We briefly introduce the three (equivalent) models present in [**KKMP**, §3.4], before giving details on the two models relevant for the proof of Theorem 13.2.

Let $\mathcal{M}$ and $\mathcal{N}$ be right, respectively left $\mathcal{V}$-enriched module categories for some monoidal $\mathcal{V}$-enriched category $\mathcal{A}$. We have three models for the enriched relative tensor product:

- Enriched Tambara relative tensor product: $\mathcal{M} \times_{\mathcal{A}} \mathcal{N}$.
  This is an enriched version of the relative tensor product from [**Tam01**], which is a categorical analogue of the relative tensor product of modules. This is used in [**Coo23**] when proving excision of skein categories in the $\mathbb{k}$-linear setting.
- Truncated two-sided bar construction: $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$.
  The two-sided bar construction is used when formulating excision for factorization homology. Since $\mathcal{V}$-Cat is a 2-category we only need to understand the truncated version here.
- Coend tensor product: $\mathcal{M} \otimes_{\mathcal{A}}^{\int} \mathcal{N}$.
  This is, to the best of our knowledge, a new model for the relative tensor product in $\mathcal{V}$-Cat defined using enriched coends.

Remark 11.18. We do not need the coend tensor product from [**KKMP**, §3.4.3] in this thesis, but want to highlight it here since this new model turns out to be very agreeable for computations. In particular, it is used to compute the internal endomorphism algebras associated to the pointing of enriched skein categories. We refer the interested reader to Section 6.3 and 7.1 of [**KKMP**] for a precise definition of the internal endomorphism algebra and some concrete computations showcasing how convenient this model is.

11.3.1. The enriched Tambara tensor product. We briefly give the construction of the enriched Tambara tensor product in terms of generators and relations following [**KKMP**, §3.4.1]. This construction uses the notion of $\mathcal{V}$-graphs, and we refer the reader unfamiliar with this notion to Appendix C.1 for more details and references. Throughout this section $\mathcal{A}$ is a monoidal $\mathcal{V}$-category and $\mathcal{M}$ and $\mathcal{N}$ are right, respectively left $\mathcal{A}$-module $\mathcal{V}$-categories.

Definition 11.19. Define the $\mathcal{V}$-graph Ω as follows: objects are given by $\mathrm{Ob}(\Gamma) = \mathrm{Ob}(\mathcal{M} \times \mathcal{N})$, while the $\mathcal{V}$-object of edges is

$$\Omega\Big((m,n),(m',n')\Big) \coloneqq \mathcal{M}\times\mathcal{N}\Big((m,n),(m',n')\Big) \coprod \Big(\coprod_{\substack{\{a\in\mathcal{A} \mid m=m'\triangleleft a,\\ a\triangleright n=n'\}}} 1_{\mathcal{V}} \Big) \coprod \Big(\coprod_{\substack{\{a\in\mathcal{A} \mid m\triangleleft a=m',\\ n=a\triangleright n'\}}} 1_{\mathcal{V}} \Big).$$

Notation 11.20. We introduce the following notation

$$\iota_{m'\triangleleft a,n}^{m',a\triangleright n} \colon 1_{\mathcal{V}} \longrightarrow \coprod_{\substack{\{a\in\mathcal{A} \mid m=m'\triangleleft a,\\ a\triangleright n=n'\}}} 1_{\mathcal{V}} \longrightarrow \Omega((m,n),(m',n'))$$

and

$$(\iota_{m\triangleleft a,n'}^{m,a\triangleright n'})^{-1} \colon 1_{\mathcal{V}} \longrightarrow \coprod_{\substack{\{a\in\mathcal{A} \mid m\triangleleft a=m',\\ n=a\triangleright n'\}}} 1_{\mathcal{V}} \longrightarrow \Omega((m,n),(m',n'))$$

for the canonical maps into the $\mathcal{V}$-object of edges for each $a \in \mathcal{A}$.

Recall from Proposition C.2 (and the surrounding discussion) the definition of the $\mathcal{V}$-category $\mathrm{Free}(\Omega)$ related to a $\mathcal{V}$-graph Ω. Moreover, in Construction C.3 it is briefly explained how to impose relations between morphisms in the enriched setting.

Definition 11.21. [**KKMP**, Definition 3.16] Let $\mathcal{A}$ be a monoidal $\mathcal{V}$-category, let $\mathcal{M}$ be a right $\mathcal{A}$-module $\mathcal{V}$-category and $\mathcal{N}$ and left $\mathcal{A}$-module $\mathcal{V}$-category. The $\mathcal{V}$-*enriched Tambara relative tensor product* is defined as

$$\mathcal{M} \times_{\mathcal{A}} \mathcal{N} \coloneqq \mathrm{Free}(\Omega)/\sim$$

where we impose the following relations between morphisms in Free(Ω):

(1) the ι's are isomorphisms:

$$(\iota^{m,a\triangleright n}_{m\triangleleft a,n})^{-1} \circ \iota^{m,a\triangleright n}_{m\triangleleft a,n} \sim \mathrm{id}_{(m\triangleleft a,n)}, \qquad \iota^{m,a\triangleright n}_{m\triangleleft a,n} \circ (\iota^{m,a\triangleright n}_{m\triangleleft a,n})^{-1} \sim \mathrm{id}_{(m,a\triangleright n)}\ ,$$

(2) naturality of ι, and
(3) compatibility of ι and unitors.

These relations are explicitly spelled out in Appendix C.2.

The enriched Tambara tensor product has the following universal property: Let $\mathcal{C}$ be a $\mathcal{V}$-category. Then there is an equivalence (of ordinary categories)

$$\mathrm{Hom}(\mathcal{M} \times_{\mathcal{A}} \mathcal{N}, \mathcal{C}) \simeq \mathrm{Fun}_{\mathcal{A}}(\mathcal{M} \times \mathcal{N}, \mathcal{C})$$

where the latter denotes the category of *$\mathcal{A}$-balanced $\mathcal{V}$-functors* $\mathcal{M} \times \mathcal{N} \to \mathcal{C}$. The definition of a $\mathcal{A}$-balanced $\mathcal{V}$-functor is a straightforward generalization of the unenriched notion, see e.g. [**KKMP**, Definition A.6] for the precise definition.

11.3.2. The truncated bar construction. Here we explain what the 2-sided bar construction amounts to in $\mathcal{V}$-Cat. Additionally, we explain in what sense it is equivalent to the Tambara relative tensor product from the previous section.

Recall that a strict 2-functor $\mathcal{D} \to \mathcal{V}$-$\mathsf{Cat}$ is called a *diagram of shape D in $\mathcal{V}$-Cat*, for $\mathcal{D}$ some 2-category. We now introduce the shape of the diagram relevant for the truncated bar construction Let $\mathcal{D}$ be the 2-category whose objects and 1-morphisms are as below

$$\begin{tikzcd} A \arrow[r, shift left=4, "G_0"] \arrow[r, "G_1"] \arrow[r, shift right=4, "G_2"'] & B \arrow[r, shift left=2, "F_0"] \arrow[r, shift right=2, "F_1"'] & C \end{tikzcd}\ ,$$

and with 2-morphisms

$$\kappa_1 \colon F_1 \circ G_0 \Rightarrow F_0 \circ G_2, \quad \kappa_2 \colon F_0 \circ G_0 \Rightarrow F_0 \circ G_1, \quad \kappa_3 \colon F_1 \circ G_2 \Rightarrow F_1 \circ G_1\ .$$

The relative tensor product used when proving excision for factorization homology with values in $\mathcal{V}$-Cat is the colimit of the truncated bar construction. That is, the following colimit of a specific diagram of shape $\mathcal{D}$ as above.

Definition 11.22. Let $\mathcal{A}$ be a monoidal $\mathcal{V}$-category and $\mathcal{M}, \mathcal{N}$ right, respectively left $\mathcal{A}$-module categories. The *truncated bar construction* is the diagram

$$\begin{tikzcd} \mathcal{M} \times \mathcal{A} \times \mathcal{A} \times \mathcal{N} \arrow[r, shift left=4, "\triangleleft \times \mathrm{id}_{\mathcal{A}\times\mathcal{N}}"] \arrow[r, "\mathrm{id}_{\mathcal{M}} \times \otimes \times \mathrm{id}_{\mathcal{N}}"] \arrow[r, shift right=4, "\mathrm{id}_{\mathcal{M}\times\mathcal{A}} \times \triangleright"'] & \mathcal{M} \times \mathcal{A} \times \mathcal{N} \arrow[r, shift left=2, "\triangleleft \times \mathrm{id}_{\mathcal{N}}"] \arrow[r, shift right=2, "\mathrm{id}_{\mathcal{M}} \times \triangleright"'] & \mathcal{M} \times \mathcal{N} \end{tikzcd}$$

in $\mathcal{V}$-Cat together with the $\mathcal{V}$-natural isomorphisms

$$\begin{aligned} \mathrm{id} &\colon (\mathrm{id}_{\mathcal{M}} \times \triangleright) \circ (\triangleleft \times \mathrm{id}_{\mathcal{A}\times\mathcal{N}}) \Rightarrow (\triangleleft \times \mathrm{id}_{\mathcal{N}}) \circ (\mathrm{id}_{\mathcal{M}\times\mathcal{A}} \times \triangleright) \\ \beta_{\mathcal{M}} &\colon (\triangleleft \times \mathrm{id}_{\mathcal{N}}) \circ (\triangleleft \times \mathrm{id}_{\mathcal{A}\times\mathcal{N}}) \Rightarrow (\triangleleft \times \mathrm{id}_{\mathcal{N}}) \circ (\mathrm{id}_{\mathcal{M}} \times \otimes \times \mathrm{id}_{\mathcal{N}}) \\ \beta_{\mathcal{N}} &\colon (\mathrm{id}_{\mathcal{M}} \times \triangleright) \circ (\mathrm{id}_{\mathcal{M}\times\mathcal{A}} \times \triangleright) \Rightarrow (\mathrm{id}_{\mathcal{M}} \times \triangleright) \circ (\mathrm{id}_{\mathcal{M}} \times \otimes \times \mathrm{id}_{\mathcal{N}}) \end{aligned}$$

where $\beta_{\mathcal{M}}$ and $\beta_{\mathcal{N}}$ are the associators for the right and left $\mathcal{A}$-action.

When we later prove that the enriched skein category $\mathbf{Sk}_{\mathcal{C}}(-)$ computes factorization homology we need that the bicolimit of the truncated bar construction agrees with the enriched Tambara relative tensor product. This has been proven in [**Coo23**, Theorem 2.27] for $\Bbbk$-linear categories and is extended to the $\mathcal{V}$-enriched setting in [**KKMP**]. Explicitly, we have

THEOREM 11.23. [**KKMP**, Theorem 3.19] *The $\mathcal{V}$-enriched Tambara relative tensor product $\mathcal{M} \times_{\mathcal{A}} \mathcal{N}$ from Definition 11.21 is equivalent to the bicolimit*

$$\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} = \operatorname{colim}(\mathcal{M} \times \mathcal{A} \times \mathcal{A} \times \mathcal{N} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathcal{M} \times \mathcal{A} \times \mathcal{N} \substack{\longrightarrow\\ \longrightarrow} \mathcal{M} \times \mathcal{N})$$

of the truncated bar construction from Definition 11.22.

For a proof of this theorem we refer the reader to [**KKMP**, Appendix A.4.1].

11.4. Enriched ribbon categories. In this section we briefly recall the notion of $\mathcal{V}$-enriched ribbon categories following [**KKMP**, §4.1]. Recall that these are examples of framed E_2-algebras in $\mathcal{V}$-Cat.

A *monoidal $\mathcal{V}$-category* $\mathcal{A}$ is a pseudomonoid in the symmetric monoidal 2-category $\mathcal{V}$-Cat [**DS97**, §4]. Explicitly, it is a $\mathcal{V}$-category $\mathcal{A}$ together with an *monoidal unit* $1_{\mathcal{A}} \in \mathcal{A}$, a $\mathcal{V}$-functor $\otimes^{\mathcal{A}} \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ giving a *monoidal product*, as well as *unitors* ρ, λ and an *associator* α which are all $\mathcal{V}$-enriched natural isomorphisms satisfying the usual axioms. A *braiding* on $\mathcal{A}$ is a suitable $\mathcal{V}$-natural transformation $\beta \colon \otimes^{\mathcal{A}} \to (\otimes^{\mathcal{A}})^{\mathrm{op}}$. Recall that if $\mathcal{A}$ is (braided/symmetric) monoidal, so is $\mathcal{A}_0$ [**Cru09**, Theorem 5.7.1].

Definition 11.24. A monoidal $\mathcal{V}$-category $\mathcal{A}$ is called *left-rigid* if for each object $a \in \mathcal{A}$ there exists a *left dual* object $a^\vee \in \mathcal{A}$ together with morphisms

$$\mathrm{ev}_a \in \mathcal{A}_0(a^\vee \otimes^{\mathcal{A}} a, 1_{\mathcal{A}}) \qquad \text{and} \qquad \mathrm{coev}_a \in \mathcal{A}_0(1_{\mathcal{A}}, a \otimes^{\mathcal{A}} a^\vee)$$

satisfying the *snake identities.* That is, such that the following two diagrams in $\mathcal{V}$ commute

$$\begin{tikzcd}
\mathcal{A}(a^\vee, a^\vee) & \mathcal{A}(a^\vee \otimes^{\mathcal{A}} 1_{\mathcal{A}}, 1_{\mathcal{A}} \otimes^{\mathcal{A}} a^\vee) \arrow[l, "\lambda_a \circ - \circ \rho_a^{-1}"'] \\
& \mathcal{A}(a^\vee \otimes^{\mathcal{A}} 1_{\mathcal{A}}, (a^\vee \otimes^{\mathcal{A}} a) \otimes^{\mathcal{A}} a^\vee) \arrow[u, "(\mathrm{ev}_a \otimes^{\mathcal{A}} \mathrm{id}) \circ -"'] \\
& \mathcal{A}(a^\vee \otimes^{\mathcal{A}} 1_{\mathcal{A}}, a^\vee \otimes^{\mathcal{A}} (a \otimes^{\mathcal{A}} a^\vee)) \arrow[u, "\alpha^{-1}_{a^\vee, a, a^\vee} \circ -"'] \\
1_{\mathcal{V}} \arrow[uuu, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & \mathcal{A}(a^\vee \otimes^{\mathcal{A}} 1_{\mathcal{A}}, a^\vee \otimes^{\mathcal{A}} 1_{\mathcal{A}}) \arrow[u, "(\mathrm{id} \otimes^{\mathcal{A}} \mathrm{coev}_a) \circ -"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}(a, a) & \mathcal{A}(1_{\mathcal{A}} \otimes^{\mathcal{A}} a, a \otimes^{\mathcal{A}} 1_{\mathcal{A}}) \arrow[l, "\rho_a \circ - \circ \lambda_a^{-1}"'] \\
& \mathcal{A}(1_{\mathcal{A}} \otimes^{\mathcal{A}} a, a \otimes^{\mathcal{A}} (a^\vee \otimes^{\mathcal{A}} a)) \arrow[u, "(\mathrm{id} \otimes^{\mathcal{A}} \mathrm{ev}_a) \circ -"'] \\
& \mathcal{A}(1_{\mathcal{A}} \otimes^{\mathcal{A}} a, (a \otimes^{\mathcal{A}} a^\vee) \otimes^{\mathcal{A}} a) \arrow[u, "\alpha_{a, a^\vee, a} \circ -"'] \\
1_{\mathcal{V}} \arrow[uuu, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & \mathcal{A}(1_{\mathcal{A}} \otimes^{\mathcal{A}} a, 1_{\mathcal{A}} \otimes^{\mathcal{A}} a) \arrow[u, "(\mathrm{coev}_a \otimes^{\mathcal{A}} \mathrm{id}) \circ -"']
\end{tikzcd}$$

A *right-rigid* structure on $\mathcal{A}$ is defined analogously. If a category is both left- and right-rigid we say that it is *rigid.*

The enriched snake identities above can be seen as the usual equality of the identity and the snake diagrams by using the enriched Reshetikin-Turaev functor. See Proposition 12.11 for details.

Remark 11.25. The category $\mathcal{A}$ is (left-)rigid if and only if $\mathcal{A}_0$ is so; this is because the snake identities can be formulated in terms of the composition in $\mathcal{A}_0$. If $\mathcal{A}$ is braided and left-rigid one can further use the braiding to construct right duals from the left duals. It follows that if $\mathcal{A}$ is braided and left-rigid it is even rigid [**JS93**, Proposition 7.2].

Definition 11.26. Let $a^\vee, b^\vee$ be left duals of $a, b \in \mathcal{A}$. The *transpose* is the morphism $\mathcal{A}(a,b) \to \mathcal{A}(b^\vee, a^\vee)$ defined as

$$\mathcal{A}\left(a,b\right) \xrightarrow{\mathrm{id}\otimes^{\mathcal{A}}(-\otimes^{\mathcal{A}}\mathrm{id})} \mathcal{A}\Big(b^\vee \otimes^{\mathcal{A}} \left(a \otimes^{\mathcal{A}} a^\vee\right), b^\vee \otimes^{\mathcal{A}} \left(b \otimes^{\mathcal{A}} a^\vee\right)\Big)$$
$$\xrightarrow{\lambda_{a^\vee}\circ(\mathrm{ev}_b\otimes^{\mathcal{A}}\mathrm{id})\circ\alpha^{-1}_{b^\vee,b,a^\vee}\circ-\circ(\mathrm{id}\otimes^{\mathcal{A}}\mathrm{coev}_a)\circ\rho^{-1}_{b^\vee}} \mathcal{A}\left(b^\vee, a^\vee\right) \ .$$

Definition 11.27. A rigid braided monoidal $\mathcal{V}$-category $\mathcal{A}$ is called *ribbon* if it comes equipped with a *twist*; i.e. a $\mathcal{V}$-natural automorphism θ of the identity functor $\mathrm{id}_{\mathcal{A}}$ satisfying

$$\theta_{a\otimes^{\mathcal{A}}b} = \beta_{b,a} \circ \beta_{a,b} \circ \left(\theta_a \otimes^{\mathcal{A}} \theta_b\right)$$

in $\mathcal{A}_0$, and for which the following diagram in $\mathcal{V}$

$$\begin{tikzcd}
1_{\mathcal{V}} \arrow[r, "\theta_a"] \arrow[dr, "\theta_{a^\vee}"'] & \mathcal{A}(a,a) \arrow[d, "\text{transpose}"] \\
& \mathcal{A}(a^\vee, a^\vee)
\end{tikzcd}$$

commutes.

Example 11.28. Recall from Example 11.17 that one interesting enriching category is $\mathcal{V} = \widehat{\mathbb{C}[[\hbar]]\text{-}\mathrm{Mod}}$, i.e. the category of complete $\mathbb{C}[[\hbar]]$-modules (where $\hbar$ is a formal variable). An example of a ribbon $\mathcal{V}$-category in this setting is given by the *Drinfeld category*, denoted $U(\mathfrak{g})\text{-}\mathrm{Mod}^{\Phi}[[\hbar]]$. Objects are given by topologically free $U(\mathfrak{g})[[\hbar]]$-modules of finite rank with associator induced by a *Drinfeld associator* Φ. We refer the reader interested in the full ribbon structure to [**KKMP**, §3.5.1] and the references therein. Moreover, the Drinfeld category is equivalent to the representation category of the quantum group $U_\hbar\mathfrak{g}$, denoted $U_\hbar(\mathfrak{g})\text{-}\mathrm{Mod}$, as ribbon $\mathcal{V}$-categories. See e.g. [**KKMP**, Theorem 3.21] and the surrounding references.

12. Basic features of enriched skein categories

We start by giving the definition of enriched skein categories in Section 12.1. This is a generalization of the usual definition (e.g. as in [**Coo23**]) in the following ways; the underlying ribbon category $\mathcal{A}$ is $\mathcal{V}$-enriched and it is not necessarily strict. Then we explain how some standard results for skein categories carry over to this setting. In Section 12.2 we get a monoidal structure on $\mathbf{Sk}_{\mathcal{A}}(C \times [0,1])$, for C some 1-manifold, and in Section 12.3 we explain how certain embeddings gives rise to module structures on enriched skein categories. The content of this section corresponds to that of [**KKMP**, §4.2].

12.1. The definition. Skein categories are categorical analogues of skein algebras. This was first defined in [**Wal, JF19**] by Walker and Johnson-Freyd, respectively. The latter definition is also the one used in [**Coo23**]. In this section we define a generalization of this, called enriched skein categories, which allow for more general enriching categories $\mathcal{V}$ than $\mathrm{V{\scriptsize ECT}}_{\Bbbk}$.

Recall from e.g [**Coo23**, Definition 1.1] that a *ribbon graph* is a finite collection of oriented[27] ribbons and coupons. Ribbon ends are allowed to be attached to coupons or they can be joined to e.g. form annuli. See Figure 28 for an example. We write $m_r^{\mathrm{or}(r)}$ to mean m_r or $m_r^\vee$ for ribbons oriented up or down. Building upon this notion we define ribbon q-graphs (after [**LM95**]) which are additionally equipped with parenthesizations of coupons.

Definition 12.1. [**KKMP**, Definition 4.5] A *ribbon q-graph* is a ribbon graph together with a choice, for each coupon, of a parenthesization of the ribbon endpoints on the top and bottom of the coupon.

Assumption 12.2. From now on we assume $\mathcal{A}$ to always mean a $\mathcal{V}$-enriched ribbon category, where $\mathcal{V}$ is as in Part 3.

The next step when defining ordinary skein categories is to "color" the ribbons by objects and the coupons by morphisms of the underlying ribbon category, c.f. [**Coo23**, Definition 1.2]. For the enriched analogue we cannot treat morphisms in a linear matter as in the $\mathrm{V{\scriptsize ECT}}_{\Bbbk}$-enriched setting, but rather need to define an object of $\mathcal{V}$ of all possible morphisms. Hence, to color a ribbon q-graph simply corresponds to the following definition, exemplified in Figure 28.

Definition 12.3. An *$\mathcal{A}$-colored ribbon q-graph* is a ribbon q-graph equipped with a choice of an object $m_r \in \mathcal{A}$ to each ribbon r.

To define the "object of all possible morphisms" of a graph we first need some more setup. Let Γ be a $\mathcal{A}$-colored ribbon q-graph. If e is the bottom or the top edge of a coupon, let

$$c_e := \otimes_{r \in e} m_r^{\mathrm{or}(r)} \in \mathcal{A} \tag{12.1}$$

be the tensor product of the objects corresponding to the decorations of the incident ribbons, parenthesized using the parenthesization of the coupon edge.

[27] That is, each ribbon has two possible orientations. In [**Coo23**] one remembers the orientation of the interval $[0,1]$ as well as the additional choice $\pm$; our orientation here should be thought of the product of these two, i.e. corresponding to the arrow in [**Coo23**, Figure 1].

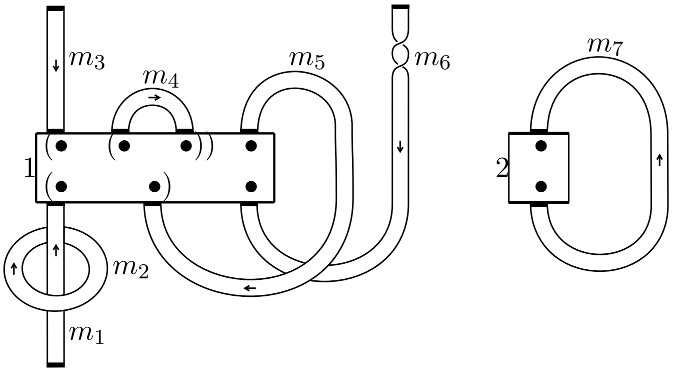


FIGURE 28. An example of a $\mathcal{A}$-colored ribbon q-graph Γ.

Definition 12.4. Let Γ be a $\mathcal{A}$-colored ribbon q-graph. We define the $\mathcal{V}$-object of morphisms $\mathcal{A}((\Gamma))$ as the unordered[28] tensor product

$$\mathcal{A}((\Gamma)) := \bigotimes_{c\in\mathrm{Coupon}(\Gamma)} \mathcal{A}(c_{\mathrm{in}}, c_{\mathrm{out}}) \ ,$$

where c_{in} and c_{out} are the objects corresponding to e in (12.1) being the incoming and outgoing edge of the coupon, respectively.

Example 12.5. Let Γ be the $\mathcal{A}$-colored ribbon q-graph of Figure 28. The corresponding object of morphisms is

$$\mathcal{A}((\Gamma)) = \mathcal{A}\Big((m_1 \otimes m_5) \otimes m_6, (m_3^\vee \otimes (m_4 \otimes m_4^\vee)) \otimes m_5\Big) \otimes^{\mathcal{V}} \mathcal{A}\Big(m_7^\vee, m_7^\vee\Big) \ .$$

The order of the two coupons is chosen as in the figure when writing down the unordered tensor product.

Analogous to [**Coo23**, Definition 1.3], a *$\mathcal{A}$-colored ribbon q-graph on a surface* Σ is an embedding of a $\mathcal{A}$-colored ribbon q-graph in $\Sigma \times [0,1]$ such that the free ends of the ribbons are sent to $\Sigma \times \{0,1\}$, and the rest of the ribbon graph lies in $\Sigma \times (0,1)$. We often abbreviate this by ribbon graph when it is clear from the context what we mean. We now assemble this kind of information into a $\mathcal{V}$-category.

Definition 12.6. Let $\mathbf{Ribbon}_{\mathcal{A}}(\Sigma)$ denote the following $\mathcal{V}$-category; objects are given by finite collections $m = \{m_{i\,x_i}^{\varepsilon_i}\}_{i\in I_m}$ of pairwise disjoint points x_i on Σ with framing, orientation ε_i and a decoration of objects m_i of $\mathcal{A}$. We often abbreviate this by simply saying a finite collection of *decorated points* on Σ and write $m = \{m_i\}_{i\in I_m}$.

Let m, n be two such objects. The corresponding Hom-object is defined as the following coproduct in $\mathcal{V}$

$$\mathbf{Ribbon}_{\mathcal{A}}(\Sigma)(m,n) := \coprod_{[\Gamma]\colon m\to n} \mathcal{A}((\Gamma)) \ ,$$

[28] i.e. for each ordering of the n coupons, a tensor product of the objects in the corresponding order modulo equivalence obtained by permuting the tensor factors and the ordering. This quotient is the coequalizer of the diagram with $n!$ objects, each being a tensor product in one order, and morphisms given by permutations. See [**MSS02**, Definition II.1.58] for more details.

where Γ runs over all isomorphism classes of $\mathcal{A}$-colored ribbon q-graphs compatible with the coloring, framing and orientation of m and n on $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$, respectively.

Composition is defined by stacking of graphs using that $\mathcal{A}((\Gamma \circ \Gamma')) \cong \mathcal{A}((\Gamma)) \otimes \mathcal{A}((\Gamma'))$ and that the monoidal product of $\mathcal{V}$ distributes over coproducts.

Example 12.7. Let $\mathcal{V} = (\mathrm{SET}, \times)$. Then an element of the set $\mathbf{Ribbon}_{\mathcal{A}}(\Sigma)(m,n)$ is an isotopy class of $\mathcal{A}$-colored ribbon graphs with coupons decorated by morphisms in $\mathcal{A}$ compatible with the decoration of the in- and outcoming ribbons.

Example 12.8. If we instead set $\mathcal{V} = (R\text{-Mod}, \otimes_R)$ we have that an element of the R-module $\mathbf{Ribbon}_{\mathcal{A}}(\Sigma)(m,n)$ is a formal R-linear combination of isotopy classes of $\mathcal{A}$-colored ribbon graphs. Further, the coupons are again decorated by suitable morphisms in $\mathcal{A}$.

Remark 12.9. In the setting of Example 12.8 a ribbon graph with a coupon decorated by $f+g$ is equal to the sum of ribbon graphs with coupons decorated by f and g. This property is not present in [**Coo23**, Definition 1.5]. The category defined there rather corresponds to $F_\star(\mathbf{Ribbon}_{U_\star\mathcal{A}}(\Sigma))$, where $U\colon \mathrm{VECT}_{\Bbbk} \leftrightarrows \mathrm{SET}\colon F$ is the usual free-forgetful adjunction. However, after passing to the corresponding skein categories this distinction vanishes.

Definition 12.10. Let $\Sigma = [0,1]^2$. Let $\mathbf{qRibbon}_{\mathcal{A}}([0,1]^2)$ denote the $\mathcal{V}$-category whose objects are *parenthesized words* m_W with letters $m_i^{\pm}$, where $m_i \in \mathcal{A}$. Given two objects m_W and $n_{W'}$ the corresponding Hom-object is defined as

$$\mathbf{qRibbon}_{\mathcal{A}}([0,1]^2)(m_W, n_{W'}) \coloneqq \mathbf{Ribbon}_{\mathcal{A}}([0,1]^2)(m_W \times \{0\}, n_{W'} \times \{0\})$$

where $m_W \times \{0\}$ is the square with the objects $m_i^{\pm}$ uniformly distributed on the interval $[0,1] \times \{0\} \subset [0,1]^2$ and with framing in the direction of $[0,1] \times \{0\}$.

Put differently, $\mathbf{qRibbon}_{\mathcal{A}}([0,1]^2)$ consists of ribbon graphs in the cube which ends at the bottom front and top front edges of the cube, together with parenthesizations. This category has a straightforward ribbon structure and affords an enriched generalization of the ordinary Reshetikin-Turaev functor from [**RT90**] and [**Tur16**], which we now explain.

Proposition 12.11. [**KKMP**, Proposition 4.9] *The category* $\mathbf{qRibbon}_{\mathcal{A}}([0,1]^2)$ *is a* $\mathcal{V}$-*enriched ribbon category. There is a ribbon* $\mathcal{V}$-*functor*

$$T\colon \mathbf{qRibbon}_{\mathcal{A}}([0,1]^2) \longrightarrow \mathcal{A}$$

such that $T(m,+) = m$, $T(m,-) = m^\vee$, T *is strict monoidal and on morphisms it satisfies the enriched analogue of the conditions of* [**Tur16**, Theorem 2.5]. *We call* T *the* enriched Reshetikin-Turaev functor.

Having established an analogue of the Reshetikin-Turaev functor we can now define the enriched version of the skein category. Normally the skein relations are invoked by a suitable quotient. In the enriched setting this is replaced by a suitable coequalizer. Let m and n be two objects of $\mathbf{Ribbon}_{\mathcal{A}}(\Sigma)$. Consider the set of all triples $(\Gamma, \mathrm{cb}, \mathrm{p})$, where $\Gamma\colon m \to n$ is a colored ribbon q-graph on Σ, $\mathrm{cb}\colon [0,1]^3 \hookrightarrow \Sigma \times [0,1]$ is an embedded cube such that Γ intersects the cube transversally and only at the bottom and top edges. Moreover, let p be the parenthesization of these intersection points on the bottom and on the top edges of the cube. From this we will form a diagram in $\mathcal{V}$ with two sets of objects:

- $\{s_\Gamma = \mathcal{A}((\Gamma))\}$ for all $\mathcal{A}$-colored ribbon q-graphs $\Gamma\colon m \to n$ on Σ, and
- $\{r_{(\Gamma,\mathrm{cb},\mathrm{p})} = \mathcal{A}((\Gamma))\}$ for all triples as above.

The arrows in this diagram are as follows: for each triple $(\Gamma, \mathrm{cb}, \mathrm{p})$ there are arrows ,

$$r_\Gamma \xrightarrow{\mathrm{id}} s_\Gamma \qquad \text{and} \qquad r_\Gamma \xrightarrow{T_{\Gamma\cap\mathrm{cb}}} s_{\Gamma'}$$

where Γ' is the ($\mathcal{A}$-colored) ribbon q-graph obtained by replacing the cube in Γ by the single parenthesized coupon given by the cube. The second arrow is given by the component of the Reshetikin-Turaev functor $T_{\Gamma\cap\mathrm{cb}}$ on the interior of the cube tensored with the identity on the exterior of the cube.

In total this gives the below diagram in $\mathcal{V}$

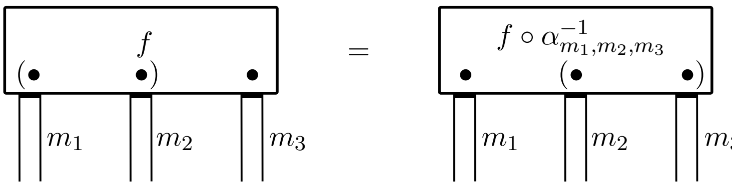
$$\{r_{(\Gamma,\mathrm{cb},\mathrm{p})} = \mathcal{A}((\Gamma))\}_{(\Gamma,\mathrm{cb},\mathrm{p})} \underset{\{T_{\Gamma\cap cb}\}_{(\Gamma,cb,p)}}{\overset{\{\mathrm{id}_{\mathcal{C}((\Gamma))}\}_{(\Gamma,cb,p)}}{\rightrightarrows}} \{s_{\Gamma'} = \mathcal{A}((\Gamma'))\}_{\Gamma'} . \tag{12.2}$$

We will use this to define the enriched skein category.

Definition 12.12 (Enriched skein category)**.** Let Σ be a surface and $\mathcal{A}$ be a ribbon $\mathcal{V}$-category. Define the *enriched skein category* $\mathbf{Sk}_\mathcal{A}(\Sigma)$ to be the following $\mathcal{V}$-category: Objects are the same as those of $\mathbf{Ribbon}_\mathcal{A}(\Sigma)$. Let m, n be two such objects. The corresponding $\mathcal{V}$-object of morphisms $\mathbf{Sk}_\mathcal{A}(\Sigma)(m, n)$ is defined to be the colimit over the diagram (12.2).

Remark 12.13. The careful reader might be puzzled that we do not consider isotopy classes of ribbon graphs Γ above. However, isotopy invariance follows automatically because the (enriched) Reshetikin-Turaev functor is isotopy invariant, and the fact that each isotopy in $\Sigma \times [0, 1]$ can be seen as a composition of isotopies inside boxes [**EK71**, Corollary 1.3].

Example 12.14. For $\mathcal{V} = \mathrm{S{\scriptsize ET}}$ or $\mathcal{V} = R$-Mod we get back the usual definition of the skein category modulo also having parenthesizations. In some more words, setting the Hom-object to be the colimit over the diagram (12.2) in those cases reduces to the usual quotient by the skein relations as in e.g. [**Coo23**, Definition 1.9]. Also observe that with the skein relations we can change the parenthesizations arbitrarily. For example, we have that



It follows that if $\mathcal{A}$ is strict the forgetful functor from the parenthesized skein category to the ordinary (i.e. non-parenthesized) skein category is an isomorphism of categories.

Example 12.15. Let $\mathcal{A}$ be a ribbon $\mathcal{V}$-category. We give an informal description of the enriched skein category $\mathbf{Sk}_\mathcal{A}(\mathbf{S}^2)$. Consider the functor

$$\mathcal{A} \cong \mathbf{Sk}_\mathcal{A}(\mathbf{D}^2) \longrightarrow \mathbf{Sk}_\mathcal{A}(\mathbf{S}^2)$$

induced by an embedding of $\mathbf{D}^2$ into $\mathbf{S}^2$ which avoids the north pole of $\mathbf{S}^2$. This functor is clearly essentially surjective and full, thus $\mathbf{Sk}_\mathcal{A}(\mathbf{S}^2)$ is a quotient of $\mathbf{Sk}_\mathcal{A}(\mathbf{D}^2)$. The additional local relations are induced by moving a ribbon through the north pole and generated by

- exchanging an overcrossing for an undercrossing, and
- moving an evaluation or coevaluation through the north pole.

We encourage the reader to draw out these relations; note that they are very much local relations and not global ones. Only imposing the first relation gives a $\mathcal{V}$-category which deserves the name *Müger cocenter*. It is unclear if there is a good name for the $\mathcal{V}$-category obtained by imposing both relations, as well as exactly how this relates to e.g. [**GJS23**, Proposition 4.4]. There is it proven that the free cocompletion of the unenriched skein category is equivalent to the Müger center of the free cocompletion of the ribbon category used.

Remark 12.16 (Skein categories introduce torsion)**.** Let $\mathcal{V} = \widehat{\mathbb{C}[[\hbar]]\text{-}\mathrm{Mod}}$, which is indeed a good choice of enriching category as explained in Example 11.17. For the ribbon $\mathcal{V}$-category we set $\mathcal{A} = U(\mathfrak{g})\text{-}\mathrm{Mod}^{\Phi}[[\hbar]]$, i.e. the Drinfeld category from Example 11.28. The relation $\beta_{X,Y} = \beta^{-1}_{X,Y}$ implies

$$0 = \beta_{X,Y} - \beta^{-1}_{X,Y} = \hbar t_{X,Y} + O(\hbar^2)\ .$$

Hence, the element $t_{X,Y} + O(\hbar)$ is torsion in $\mathbf{Sk}_{\mathcal{A}}(\mathbf{S}^2)$. It is also non-zero because its image under the canonical map to $\mathbf{Sk}_{\mathcal{A}}(\mathbf{S}^2)$ is $t_{X,Y}$, which is non-zero. This illustrates that even for a topologically free category $\mathcal{A}$ the corresponding (enriched) skein category is in general not topologically free.

If we replace $\mathbf{S}^2$ by a surface Σ with non-trivial boundary it follows from [**KKMP**, Proposition 7.11] that the corresponding enriched skein category is actually topologically free.

12.2. Monoidal structure on enriched skein categories. We now explain how, for C some 1-manifold, the $\mathcal{V}$-category $\mathbf{Sk}_{\mathcal{A}}(C \times [0,1])$ is a monoidal $\mathcal{V}$-category. This will be a direct consequence of how the construction of the enriched skein category is functorial with respect to embeddings.

Explicitly, analogous to when working with the ordinary skein category (c.f. [**Coo23**, Remark 1.6]) we have the following observation.

Observation 12.17. An embedding of surfaces $p\colon \Sigma \to \Pi$ induces a $\mathcal{V}$-functor

$$P\colon \mathbf{Sk}_{\mathcal{A}}(\Sigma) \to \mathbf{Sk}_{\mathcal{A}}(\Pi)$$

of enriched skein categories.

We explain this in some more detail. On objects m the functor sends it the points $p(x_i)$ in Π decorated by the same object m_i of $\mathcal{A}$ and with framing and orientation induced by the embedding p. This indeed defines an object in the target, i.e. $P(m) \in \mathbf{Sk}_{\mathcal{A}}(\Pi)$. We now want to explain what the functor P does on morphisms. Let m, n be two objects of $\mathbf{Sk}_{\mathcal{A}}(\Sigma)$. The $\mathcal{V}$-morphism

$$P\colon \mathbf{Sk}_{\mathcal{A}}(\Sigma)(m,n) \longrightarrow \mathbf{Sk}_{\mathcal{A}}(\Pi)(P(m), P(n))$$

is defined using the universal property of the colimit over the diagram (12.2). Explicitly, it is induced by the collections of inclusions of $\mathcal{A}((\Gamma)) = \mathcal{A}((p(\Gamma)))$ into $\mathbf{Sk}_{\mathcal{A}}(\Pi)(P(m), P(n))$.

We now explain how this induces a monoidal structure in the special case where we work with the enriched skein category of $C \times [0,1]$.

Example 12.18. Let C denote some 1-manifold. Consider the two embeddings $L, R\colon C \times [0,1] \hookrightarrow C \times [0,1]$ given by $L(c,s) = (c, \frac{s}{2})$ and $R(c,s) = (c, \frac{1}{2} + \frac{s}{2})$, respectively. In words, L embeds $C \times [0,1]$ into the *left* half of $C \times [0,1]$, while R embeds it into the *right* half. For later convenience, let l and r denote the corresponding isotopies between the identity embeddings of $C \times [0,1]$ into itself and L respectively R. Then we have an embedding

$$I\colon C \times [0,1] \sqcup C \times [0,1] \xhookrightarrow{L \sqcup R} C \times [0,1]$$

which, by Observation 12.17, induces a monoidal product on $\mathbf{Sk}_{\mathcal{A}}(C \times [0,1])$. We denote the monoidal product of two objects a, b by $a * b$. The monoidal unit is given by the empty set.

12.3. Module structure on enriched skein categories. Let C be a 1-manifold and let M be a surface with boundary $\partial M \cong C$. By using the data of an appropriate embedding of C into M we can equip $\mathbf{Sk}_{\mathcal{A}}(M)$ with the structure of a $\mathbf{Sk}_{\mathcal{A}}(C \times [0,1])$-module.

We first define what an appropriate embedding is, following [**Coo23**, Definition 1.17].

Definition 12.19. A *thickened right embedding* of C into the boundary of M consists of

(1) An embedding $\Xi \colon C \times (-\varepsilon, 1] \hookrightarrow M$, where $\varepsilon > 0$, such that its restriction to $C \times \{1\}$ gives an embedding $\xi \colon C \hookrightarrow \partial M$. We further set $\Phi := \Xi_{|[0,1]}$ and $\mu := \Xi_{|C \times \{0\}}$.
(2) An embedding $E \colon M \to M$ such that $\mathrm{im}(E)$ is disjoint from $\mathrm{im}(\Phi)$.
(3) An isotopy $\lambda \colon M \times [0,1] \to M$ from id_M to E which is trivial outside of $\mathrm{im}(\Xi)$.

The definition of a *thickened left embedding* is analogous. The only change is that Ξ is replaced by an embedding $\Xi \colon C \times [0, 1+\varepsilon) \hookrightarrow M$, for $\varepsilon > 0$, such that restricted to $C \times \{0\}$ it is an embedding $\xi \colon C \hookrightarrow \partial M$.

Example 12.20. In Figure 29 we see an illustration of the data of a thickened right embedding of $C = \mathbf{S}^1$ into the boundary of the once-punctured torus $M = \Sigma_{1,1}$. Condition 2 and 3 from Definition 12.19 implies the existence of a $\delta > 0$ such that $-\varepsilon < -\delta < 0$ as illustrated below.

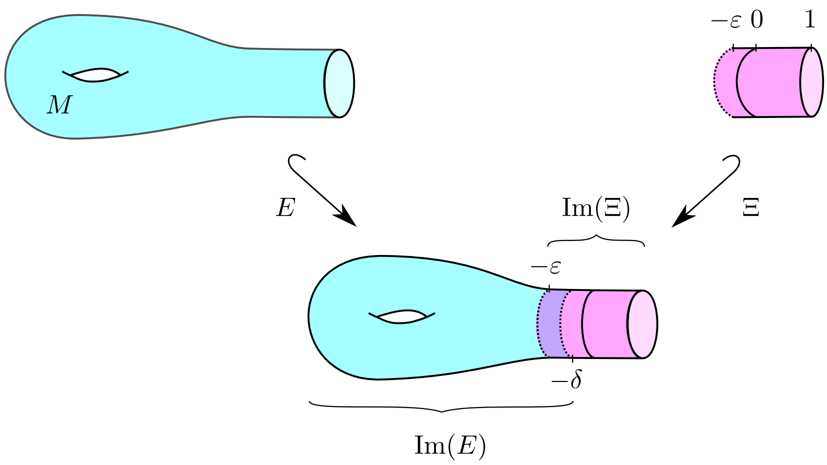


FIGURE 29. A depiction of a thickened right embedding of $\mathbf{S}^1$ into the once-punctured torus $\Sigma_{1,1}$.

Construction 12.21. Let $f, g \colon M \to M$ be two embeddings and let $\sigma \colon M \times [0,1] \to M$ be an isotopy from f to g. Let $m \in \mathbf{Sk}_{\mathcal{A}}(M)$ be any object. Then the isotopy σ induces a map

$$r_{\sigma,m} \colon 1_{\mathcal{V}} \longrightarrow \mathbf{Sk}_{\mathcal{A}}(M)(f(m), g(m)) \ .$$

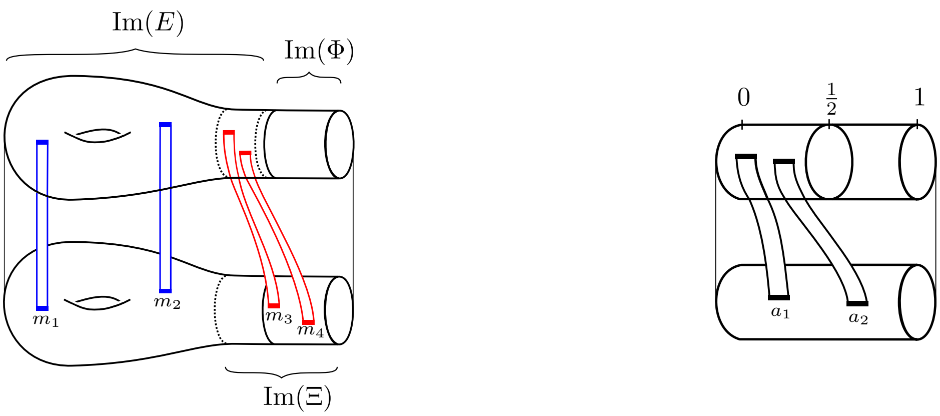
We explain in some more words how this map is obtained. Let Γ_σ be the ribbon from $f(m)$ to $g(m)$ that is traced out by the isotopy σ. Recall from Definition 12.12 that the morphism object $\mathbf{Sk}_{\mathcal{A}}(M)(f(m), g(m))$ is defined as a certain coequaliser, i.e. the colimit over (12.2). Using this, $r_{\sigma,m}$ is induced by sending $1_{\mathcal{V}} \mapsto \mathcal{A}((\Gamma_\sigma)) = 1_{\mathcal{V}}$.

Notation 12.22. We will talk about both the ribbon Γ_σ and the map $r_{\sigma,m}$ from Construction 12.21 as the *ribbon corresponding to the isotopy* σ.

We now give two examples of Construction 12.21 which will be useful later on.

Example 12.23. Let (Ξ, E, λ) be a thickened right embedding and let $m \in \mathbf{Sk}_{\mathcal{A}}(M)$ be an arbitrary object. Then the isotopy λ will give a map $r_{\lambda,m}\colon 1_{\mathcal{V}} \to \mathbf{Sk}_{\mathcal{A}}(M)(m, E(m))$. This is illustrated in Figure 30a for $C = \mathbf{S}^1$, $M = \Sigma_{1,1}$ and $m = \{m_1, m_2, m_3, m_4\}$. In words, the isotopy λ traces out the identity ribbons for the (blue) objects m_1 and m_2, while it gives ribbons that continuously moves the (red) objects m_3 and m_4 into $\mathrm{im}(E) \cap \mathrm{im}(\Xi)$.

Example 12.24. Recall the monoidal structure on $\mathbf{Sk}_{\mathcal{A}}(C \times [0,1])$ from Example 12.18, as well as the two isotopies l and r which retracts $C \times [0,1]$ into the left and right half of $C \times [0,1]$, respectively. The ribbon tracing out the action of the isotopy l on an object $a = \{a_1, a_2\}$ is depicted in Figure 30b for $C = \mathbf{S}^1$. We can think of this as the action of multiplying a with the empty set $\emptyset$ (from the right). A ribbon tracing out the isotopy r is analogous, only retracting to the right instead.



(A) The ribbon $r_{\lambda,m}$ for $m = \{m_1, m_2, m_3, m_4\}$ coming from the isotopy λ.

(B) The ribbon $r_{l,a}$ coming from the left retraction for $a = \{a_1, a_2\}$.

FIGURE 30. Two examples of ribbons arising from an isotopy.

We now explain how the module structure on $\mathbf{Sk}_{\mathcal{A}}(M)$ is constructed following the usual construction from e.g. [**Coo23**, Definition 1.19].

Construction 12.25. Let (Ξ, E, λ) be a thickened right embedding of C into the boundary of M. Then $\mathbf{Sk}_{\mathcal{A}}(M)$ is a right $\mathbf{Sk}_{\mathcal{A}}(C \times [0,1])$-module with action

$$\triangleleft\colon \mathbf{Sk}_{\mathcal{A}}(M) \times \mathbf{Sk}_{\mathcal{A}}(C \times [0,1]) \longrightarrow \mathbf{Sk}_{\mathcal{A}}(M)$$

induced from the embedding of surfaces

$$M \sqcup (C \times [0,1]) \xrightarrow{E \sqcup \Phi} M\ .$$

The corresponding associator is a $\mathcal{V}$-natural transformation; i.e. we need to specify maps

$$\beta_{a,m,b}\colon 1_{\mathcal{V}} \longrightarrow \mathbf{Sk}_{\mathcal{A}}(M)\Big((m \triangleleft a) \triangleleft b, m \triangleleft (a * b) \Big)$$

for all $a, b \in \mathbf{Sk}_{\mathcal{A}}(A)$ and $m \in \mathbf{Sk}_{\mathcal{A}}(M)$. These maps are induced by sending $1_{\mathcal{V}} \mapsto \mathcal{A}((\Gamma_\beta)) = 1_{\mathcal{V}}$, where Γ_β is the ribbon in Figure 31. Similarly, the unitor

$$\eta_m \colon 1_{\mathcal{V}} \longrightarrow \mathbf{Sk}_{\mathcal{A}}(M)\Big(m \triangleleft \emptyset, m\Big)$$

is defined through the map $1_{\mathcal{V}} \mapsto \mathcal{A}((\Gamma_\eta))$, where Γ_η is the inverse of the ribbon from Example 12.23 as illustrated in Figure 30a.

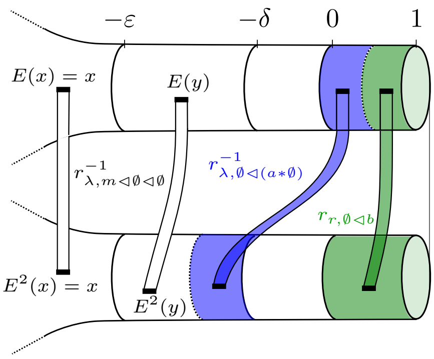


FIGURE 31. The ribbon Γ_β corresponding to the associator $\beta_{m,a,b}$ for $m = (x, y)$.

Given a thickened left embedding of C into the boundary of a surface N, there is a straightforward modification of Construction 12.25 which gives $\mathbf{Sk}_{\mathcal{A}}(N)$ a left $\mathbf{Sk}_{\mathcal{A}}(C \times [0, 1])$-module structure.

13. Enriched skeins compute factorization homology

In this section we prove that enriched skein categories satisfy excision, following [**KKMP**, §4.3]. As a consequence we get Theorem 13.1 which says that enriched skein categories compute factorization homology in $\mathcal{V}$-Cat.

THEOREM 13.1. [**KKMP**, Theorem 4.14] *Let $\mathcal{A}$ be a $\mathcal{V}$-enriched ribbon category, where $\mathcal{V}$ is a cocomplete and symmetric monoidal closed category. Then the enriched skein category computes factorization homology. Explicitly,*

$$\mathbf{Sk}_{\mathcal{A}}(\Sigma) \cong \int_{\Sigma} \mathcal{A} \in \mathcal{V}\text{-}\mathsf{Cat} \ .$$

PROOF. By Theorem 11.12 this is a direct consequence of $\mathbf{Sk}_{\mathcal{A}}(D) \cong \mathcal{A}$, Observation 12.17 and Theorem 13.2. □

The remainder of this section is dedicated to the proof of excision from [**KKMP**], which generalizes the proof in [**Coo23**]. That is, we want to prove the following theorem.

THEOREM 13.2 (Excision of enriched skein categories). [**KKMP**, Theorem 4.22] *Let C be a 1-manifold with a thickened right embedding (Ξ_M, E_m, λ_M) into the boundary of the surface M and a thickened left embedding (Ξ_N, E_N, λ_N) into the boundary of the surface N. Set $A \coloneqq C \times [0,1]$, and let*

$$M \sqcup_A N \coloneqq M \sqcup N \Big/ \left\{ \Phi_M(c,i) \sim \Phi_N(c,i) \quad \forall\, c \in C,\ i \in [0,1] \right\} .$$

Then the thickened (left and right) embeddings induce a $\mathcal{V}$-functor

$$F \colon \mathbf{Sk}_{\mathcal{A}}(M) \times_{\mathbf{Sk}_{\mathcal{A}}(A)} \mathbf{Sk}_{\mathcal{A}}(N) \xrightarrow{\simeq} \mathbf{Sk}_{\mathcal{A}}(M \sqcup_A N)$$

which is an equivalence of $\mathcal{V}$-categories. Here, $\mathbf{Sk}_{\mathcal{A}}(M) \times_{\mathbf{Sk}_{\mathcal{A}}(A)} \mathbf{Sk}_{\mathcal{A}}(N)$ is the $\mathcal{V}$-enriched Tambara relative tensor product from Definition 11.21.

Notation 13.3. To easier talk about the different regions of $M \sqcup_A N$ as above we call $\mathrm{im}(\Xi_M) \cup \mathrm{im}(\Xi_N) \times [0,1]$ for *the middle region.*

Remark 13.4. Excision of ordinary skein categories was conjectured by Johnson-Freyd [**JF19**], based on ideas of Walker [**Wal**]. A similar excision result for universal braid categories in SET is proven by Yetter in [**Yet92**]. The topological parts of this proof carries over to the proof of excision for ordinary skein categories in [**Coo23**, Theorem 1.22]. We also rely on the same topological arguments here.

PROOF OF THEOREM 13.2. We first need to define the $\mathcal{V}$-functor F. From the definition of the Tambara relative tensor product we know that objects of $\mathbf{Sk}_{\mathcal{A}}(M) \times_{\mathbf{Sk}_{\mathcal{A}}(A)} \mathbf{Sk}_{\mathcal{A}}(N)$ are given by pairs of objects (m, n), where $m \in \mathbf{Sk}_{\mathcal{A}}(M)$ and $n \in \mathbf{Sk}_{\mathcal{A}}(N)$. Here, m is a finite collection of disjoint, oriented and decorated (by objects of $\mathcal{A}$) points in M, and similarly for n. Using this, on objects we define the functor by

$$F(m,n) \coloneqq E_M(m) \sqcup E_N(n) \ ,$$

which is again a finite set of disjoint, oriented and decorated points in $M \sqcup_A N$, i.e. indeed an object of $\mathbf{Sk}_{\mathcal{A}}(M \sqcup_A N)$.

Next, we define F on $\mathcal{V}$-objects of morphisms. Let (m,n) and (m',n') be two objects of $\mathbf{Sk}_{\mathcal{A}}(M)\times_{\mathbf{Sk}_{\mathcal{A}}(A)}\mathbf{Sk}_{\mathcal{A}}(N)$. Recall from Section 11.3.1 that the $\mathcal{V}$-object of edges of the graph Ω underlying the Tambara relative tensor product $\mathcal{V}$-category is

$$\Omega\Big((m,n),(m',n')\Big) = \tag{13.1}$$
$$\mathbf{Sk}_{\mathcal{A}}(M)\times\mathbf{Sk}_{\mathcal{A}}(N)\Big((m,n),(m',n')\Big)\coprod\Big(\coprod_{\substack{\{a\,|\,m=m'\triangleleft a,\\ a\triangleright n=n'\}}}1_{\mathcal{V}}\Big)\coprod\Big(\coprod_{\substack{\{a\,|\,m\triangleleft a=m',\\ n=a\triangleright n'\}}}1_{\mathcal{V}}\Big)\,,$$

where in both the big coproducts $a\in\mathbf{Sk}_{\mathcal{A}}(A)$. From this we want to produce a $\mathcal{V}$-object of edges from $E_M(m)\sqcup E_N(n)$ to $E_M(m')\sqcup E_N(n')$. Let Γ_M be a ribbon from m to m' in M, and analogously Γ_N a ribbon from n to n' in N. The map on Hom-objects is induced by sending $\Gamma_M\mapsto E_M(\Gamma_M)$ respectively $\Gamma_N\mapsto E_N(\Gamma_N)$. In addition, the copies of $1_{\mathcal{V}}$ for $a\in\mathbf{Sk}_{\mathcal{A}}(A)$ in the first big coproduct in (13.1), i.e. corresponding to $\iota^{m',a\triangleright n}_{m'\triangleleft a,n}$, are sent to $\mathcal{A}((\Gamma_\iota))=1_{\mathcal{V}}$, where Γ_ι is the ribbon in Figure 32. Similarly, for the copies of $1_{\mathcal{V}}$ appearing in the second big coproduct of (13.1), i.e. corresponding to $\left(\iota^{m,a\triangleright n'}_{m\triangleleft a,n'}\right)^{-1}$, we instead use $\Gamma_{\iota^{-1}}$ which is the inverse ribbon of Γ_ι.

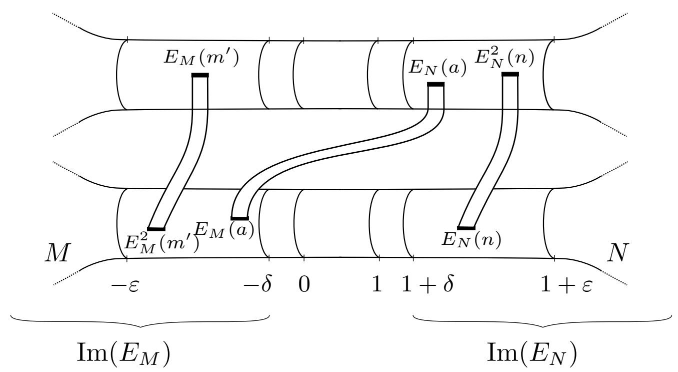


FIGURE 32. The ribbon corresponding to $F\left(\iota^{m',a\triangleright n}_{m'\triangleleft a,n}\right):1_{\mathcal{V}}\to\mathbf{Sk}_{\mathcal{A}}(M\sqcup_A N)(F(m'\triangleleft a,n),F(m',a\triangleright n))$. Here we have $m=m'\triangleleft a$ and $n'=a\triangleright n$.

F is well-defined. We first recall how the Hom-object of $\mathbf{Sk}_{\mathcal{A}}(M)\times_{\mathbf{Sk}_{\mathcal{A}}(A)}\mathbf{Sk}_{\mathcal{A}}(N)$ is constructed. That is, one uses $\mathrm{Free}(\Omega((m,n),(m',n'))$, where $\Omega((m,n),(m',n'))$ is given in (13.1), and then imposes essentially three relations, i.e. 1, 2 and 3 from Definition 11.21. We need to ensure that F respects these relations. For example, the first condition (1) (explicitly spelled out in Equation (C.2)) imposes relations such that $\iota^{m',a\triangleright n}_{m'\triangleleft a,n}$ is an isomorphism with inverse given by $\left(\iota^{m,a\triangleright n'}_{m\triangleleft a,n'}\right)^{-1}$. Pictorially, this is exactly the fact that composing the ribbon in Figure 32 corresponding to $F\Big(\iota^{m',a\triangleright n}_{m'\triangleleft a,n}\Big)$ with the ribbon corresponding to $F\Big(\left(\iota^{m,a\triangleright n'}_{m\triangleleft a,n'}\right)^{-1}\Big)$ is isotopic to the identity ribbon on $F(m\triangleleft a,n)$. The other relations can be checked by analogous topological arguments, and we conclude that F is indeed well-defined.

F is essentially surjective. Any point in $E_M(M)\sqcup E_N(N)\subset M\sqcup_A N$ is in the image of F by construction. Consider a decorated point x which is not in $E_M(M)\sqcup E_N(N)$. Then there exists a ribbon which continuously moves x to the left into the image of E_M[29]. This ribbon is an isomorphism. Hence, every decorated point in $M \sqcup_A N$ is isomorphic to any point in the image of the functor F, making it essentially surjective.

F **is fully faithful.** Let (m, n) and (m', n') be two objects of $\mathbf{Sk}_{\mathcal{A}}(M) \times_{\mathbf{Sk}_{\mathcal{A}}(A)} \mathbf{Sk}_{\mathcal{A}}(N)$. For F to be fully faithful we need to prove that $F_{(m,n),(m',n')}$ provides isomorphisms in $\mathcal{V}$ between the Hom-objects, i.e.

$$F_{(m,n),(m',n')} \colon \mathbf{Sk}_{\mathcal{A}}(M) \times_{\mathbf{Sk}_{\mathcal{A}}(A)} \mathbf{Sk}_{\mathcal{A}}(N)((m,n),(m',n')) \xrightarrow{\cong} \mathbf{Sk}_{\mathcal{A}}(M \sqcup_A N)(F(m,n), F(m',n')) \,. \tag{13.2}$$

We split this into two parts; first we construct an inverse, and then we prove that the inverse is well-defined.

Construction of inverse. Let $\widetilde{\Gamma}$ denote a ribbon diagram from $\mathsf{E}_M(m) \sqcup E_N(n)$ to $E_M(m') \sqcup E_N(n')$ in $M \sqcup_A N$. In other words, it comes with a map

$$\mathcal{A}((\widetilde{\Gamma})) \longrightarrow \mathbf{Sk}_{\mathcal{A}}(M \sqcup_A N)\Big(F(m,n), F(m',n')\Big) \,.$$

Here we will construct a ribbon Ψ appearing in the colimit of the Hom-object of the enriched Tambara tensor product, i.e. with a map

$$\mathcal{A}((\Psi)) \longrightarrow \mathbf{Sk}_{\mathcal{A}}(M) \times_{\mathbf{Sk}_{\mathcal{A}}(A)} \mathbf{Sk}_{\mathcal{A}}(N)\Big((m,n),(m',n')\Big)$$

such that F sends Ψ to Γ for some Γ that is isotopic to $\widetilde{\Gamma}$ as above. Given this, the assignment $\widetilde{\Gamma} \mapsto \Psi$ induces an inverse $F^{-1}_{(m,n),(m',n')}$, i.e. an inverse of the map in (13.2).

We first explain the overall idea. The ribbon $\widetilde{\Gamma}$ might be non-trivial in the middle region. Meanwhile, ribbons in the image of F only have strands of the form $F(\iota^{m',a\triangleright n}_{m'\triangleleft a,n})$ and its inverse crossing the middle region. Put differently, ribbons in the image of F has *no* (non-trivial) coupons, braidings and twists in the middle region. Thus, we need to produce a ribbon Γ that, up to isotopy, captures the parts of the ribbon $\widetilde{\Gamma}$ which are non-trivial across the middle region. We do this by reverse-engineering Γ from $\widetilde{\Gamma}$ by a sequence of isotopies in six steps. [30]

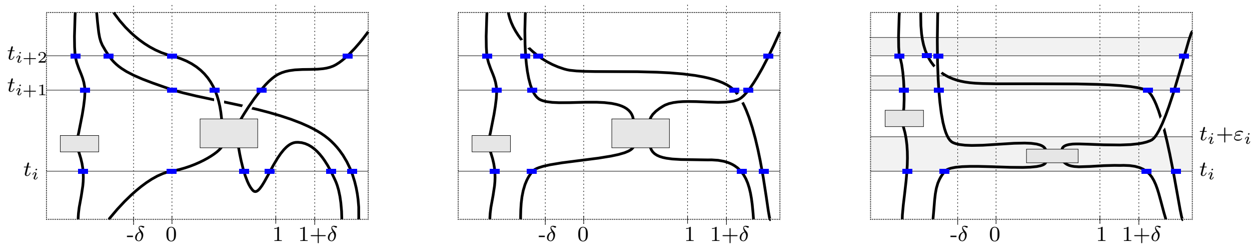
(1) *Transverse intersections.* We first apply an isotopy fixed outside of the middle region to $\widetilde{\Gamma}$ to obtain Γ with strands that only intersect $C \times \{0\} \times [0,1]$ by a finite number of transverse strands. Let $t_i \in [0,1]$, $i \in I$, be these different intersection points which we refer to as the different *levels* of Γ. Moreover, assume that the chosen isotopy ensures that we only have decorated points and no coupons of the ribbon Γ intersecting the hyperplane at each level t_i. We denote these decorated points by Γ_{t_i}. This is illustrated for a small part of a ribbon Γ in Figure 33a.

(2) *Move points of* Γ_{t_i} *to* $F(M,N)$. Choose a isotopy in the t-direction, i.e. the $[0,1]$-direction of $M \sqcup_A N \times [0,1]$, that is fixed outside of the middle region such that each Γ_{t_i} only has points lying in $F(M,N)$. This is illustrated in Figure 33b for the same local picture.

(3) *Choose small* ε_i *for each* t_i. Choose a small $\varepsilon_i > 0$ for each t_i such that the ribbon Γ can be isotoped in the t-direction to to only have non-trivial ribbons in the region $C \times (-\delta, 1+\delta) \times [0,1]$ in between the hyperplanes at t_i and $t_i + \varepsilon_i$. We also choose

[29] or similarly, we can choose to move x to the right into the image of E_N.

[30] by slight abuse of notation we call the ribbon obtained from isotoping $\widetilde{\Gamma}$ for Γ in all the steps below, even though we are really considering several isotopic versions of the ribbon Γ.



(A) Step 1 (B) Step 2 (C) Step 3

FIGURE 33. A depiction of a local part of a ribbon and the action of isotopies satisfying the conditions of the first 3 steps on the ribbon.

the ε_i's such that $t_i+\varepsilon_i < t_{i+1}$ for all $i \in I$. Moreover, the isotopy should also ensure that the intersection of Γ at each hyperplane t_i and $t_i+\varepsilon_i$ only contains decorated points and no coupons. This is depicted in Figure 33c for the running local example.

(4) *Moving points of* Γ_{t_i} *and* $\Gamma_{t_i+\varepsilon_i}$ *to* $F(M \triangleleft A, A \triangleright N)$. We now apply an isotopy fixed in $C \times (0,1) \times [0,1]$ which makes the decorated points of Γ_{t_i} and $\Gamma_{t_i+\varepsilon_i}$ be contained in $F(M \triangleleft A, A \triangleright N)$. With this we can assume that $\Gamma_{t_i} = (m \triangleleft (a \sqcup \bar{b}), (\bar{c} \sqcup d) \triangleright n)$, where in the time interval $[t_i, t_i+\varepsilon_i]$ only the points $\bar{b}$ and $\bar{c}$ intersect $C \times \{-\delta\} \times [0,1]$ respectively $C \times \{1+\delta\} \times [0,1]$. Analogously we can also write $\Gamma_{t_i+\varepsilon_i} = (m' \triangleleft (a' \sqcup \bar{b}'), (\bar{c}' \sqcup d') \triangleright n')$.

(5) *Isotopy in t-direction for non-crossing ribbons.* We now isotope in the t-direction such that any part of the ribbon not crossing the middle region is the identity in the intervals $[t_i, t_i+\varepsilon_i]$. Denote the part of the ribbon Γ in between $[t_i, t_i+\varepsilon_i]$ by $\Gamma_{[t_i,t_i+\varepsilon_i]}$. This step ensures that $\Gamma_{[t_i,t_i+\varepsilon_i]} = \mathrm{id}_{m\triangleleft a, d\triangleright n} \sqcup \Theta_{[t_i,t_i+\varepsilon_i]}$, where $\Theta_{[t_i,t_i+\varepsilon_i]} \colon E_M(\bar{b} \sqcup E_N(\bar{c}) \to E_M(\bar{b}') \sqcup E_N(\bar{c}')$ is the part of the ribbon straddling the middle region.

(6) *Moving straddling coupons to the side.* The middle region is homeomorphic $C \times [0,1]$ (at any fixed time t). Since $[0,1]$ is topologically trivial there exists, for each $i \in I$, a ribbon tangle $\bar{\Theta}_{[t_i,t_i+\varepsilon_i]} \colon \bar{b} \triangleright \bar{c} \to \bar{b}' \triangleright \bar{c}'$ in $\mathbf{Sk}_{\mathcal{A}}(N)$ such that $\Theta_{[t_i,t_i+\varepsilon_i]}$ is isotopic to $\bar{\Theta}_{[t_i,t_i+\varepsilon_i]}$ pre- and postcomposed with ribbons obtained from "dragging" $\Theta_{[t_i,t_i+\varepsilon_i]}$ to the right into $F(N)$. [31] This procedure is illustrated in [**Coo23**, Figure 7].

To summarize, given the arbitrary ribbon $\widetilde{\Gamma}$ we have produced an isotopic ribbon Γ which moreover is in the image of F; this is because Γ consists of ribbons in $E_M(M)$ and $E_N(N)$ composed with ribbons of the form $F(\iota^{m',a\triangleright n}_{m'\triangleleft a,n})$ and $F(\left(\iota^{m,a\triangleright n'}_{m\triangleleft a,n'}\right)^{-1})$. Put differently, there exists some ribbon diagram Ψ such that F sends Ψ to Γ. The inverse of (13.2), i.e. $F^{-1}_{(m,n),(m',n')}$, is induced by sending $\widetilde{\Gamma} \mapsto \Psi$.

The inverse is well-defined. First of all, note that since $F(\Gamma_M, \Gamma_N)$ and $F(\iota^{m',a\triangleright n}_{m'\triangleleft a,n}) \mapsto \iota^{m',a\triangleright n}_{m'\triangleleft a,n}$ this map will automatically be the inverse. However, we still need to argue that the assignment $\widetilde{\Gamma} \mapsto \Psi$ is well-defined; i.e. that it sends equivalent ribbons to equivalent

[31] The reason we drag the ribbon to the right here is to avoid changing the levels which are defined by the intersection as $C \times \{-\delta\} \times [0,1]$. However, we could equally well work with the levels being given by intersections as $C \times \{1+\delta\} \times [0,1]$ and modify this step by dragging the ribbon to the left into $F(M)$ instead.

ribbons. Pick some open cover of $(M \sqcup_A N) \times [0,1]$ consisting of (opens homeomorphic to) boxes. As explained in Remark 12.13, any equivalence of ribbon diagrams in $\mathbf{Sk}_{\mathcal{A}}(M \sqcup_A N)$ can be decomposed into equivalences which are fixed outside one of the opens of the cover. Thus, checking well-definedness can be done by considering the different local situations of equivalences of ribbon diagrams separately. Morally, this directly follows from how the above six steps are compatible with identifying ribbons locally. For more details including explicit isotopies we refer the interested reader to [**Coo23**, Section 1.4]. $\square$

Remark 13.5. The diligent reader might wonder why we never mentioned the parenthesization of coupons a single time in this section. Every single step of the above proof only isotopes coupons, i.e. moves them around in $M \sqcup_A N$, but it never changes the coupons themselves. Hence, the coupons and their parenthesization stays the same, and it was never necessary to explicitly mention the latter.

Part 4

Appendices

A. Background on ∞-operads and ∞-categories

For the convenience of the reader we assemble some basics from the theory of ∞-operads and ∞-categories here. Explicitly, we first consider ∞-operads à la Lurie in Appendix A.1. In Appendix A.2 we introduce dendroidal sets as a model for ∞-operads and recall that these two notions are (Quillen) equivalent. An important tool in this thesis is that of colimit cofinality which we review in Appendix A.3. In Appendix A.4 we recall a useful localization criterion, and in Appendix A.5 we give a brief result useful for computing the groupoid-core of a slice category. Another key tool is that of operadic left Kan extensions, and we give a brief specialization of the general theory to our setting in Appendix A.6. Lastly we explain how to give a topology on a category, i.e. a Grothendieck topology, in Appendix A.7.

A.1. Colored operads and ∞-operads. Colored operads, and the ∞-operads they give rise to play a central role in many of the constructions and arguments in this thesis. For completeness we recall some of the basic definitions, all coming from [**Lur17**, Section 2], as well as some relevant results.

Definition A.1. [**Lur17**, Def. 2.1.1.1] A *colored operad* $\mathcal{O}$ consists of the following data:

(1) A collection $\{x, y, z, ...\}$ of *objects* or *colors* of $\mathcal{O}$.
(2) For every finite set I, every I-indexed collection of objects $\{x_i\}_{i\in I}$ in $\mathcal{O}$, and every object $y \in \mathcal{O}$, a set $\mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I}, y)$, which we call the *set of morphisms* from $\{x_i\}_{i\in I}$ to y.
(3) For every map of finite sets $I \to J$ having fibers $\{I_j\}_{j\in J}$, every finite collection of objects $\{x_i\}_{i\in I}$, every finite collection of objects $\{y_j\}_{j\in J}$ and every object $z \in \mathcal{O}$, a *composition map*
$$\prod_{j\in J} \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I_j}, y_j) \times \mathrm{Mul}_{\mathcal{O}}(\{y_j\}_{j\in J}, z) \longrightarrow \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I}, z)\ .$$
We further require the composition to be associative.
(4) A collection of morphisms $\{\mathrm{id}_x \in \mathrm{Mul}_{\mathcal{O}}(\{x\}, x)\}_{x\in\mathcal{O}}$ which are both *left and right units* for the composition on $\mathcal{O}$.

Recall from Construction 2.13 how any posets of marked opens gives rise a to colored operad. Before we explain how to turn this into an ∞-operad we first need to define the latter. For this we start by introducing some language. Throughout we let $\mathrm{F{\scriptsize IN}}_*$ denote the category of finite pointed sets and maps of finite sets that preserve the basepoint.

Definition A.2. A morphism $\alpha : I_+ \to J_+$ in $\mathrm{F{\scriptsize IN}}_*$ is called

- *inert* if $\alpha^{-1}(j) \simeq *$ for all $j \in J$, and
- *active* if $\alpha^{-1}(+) = \{+\}$.

Remark A.3. The inert and active morphisms gives rise to a *factorization system* on $\mathrm{F{\scriptsize IN}}_*$. That is, any morphism $\alpha\colon I_+ \to J_+$ can be factored as
$$I_+ \xrightarrow{\alpha_{\mathrm{inert}}} \widetilde{I}_+ \xrightarrow{\alpha_{\mathrm{act}}} J_+,$$
where the first arrow α_{inert} is inert and the second arrow α_{act} is an active morphism.

Definition A.4. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. A morphism $f\colon d \to d'$ in $\mathcal{D}$ is called *F-coCartesian* if there exists an initial filler for each solid diagram of categories

$$\begin{tikzcd} * \arrow[r, "\langle c\rangle"] \arrow[d, "\langle s\rangle"'] & \mathcal{C} \arrow[d, "F"] \\ \{s<t\} \arrow[r, "\langle f\rangle"] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd} .$$

The lift of f is denoted $f_!$.

Definition A.5. Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ and an F-coCartesian morphism in $\mathcal{D}$, $f\colon d \to d'$, we can consider the *coCartesian monodromy functor of f*

$$f_!\colon \mathcal{C}_{|_d} \to \mathcal{C}_{|_{d'}}$$

that sends $c \in \mathcal{C}_{|_d}$ to $f_!(c)$, the coCartesian lift of f evaluated at c.

Definition A.6. A functor $F\colon \mathcal{C} \to \textsc{Fin}_*$ is called an *inert-coCartesian fibration* if each inert morphism in $\textsc{Fin}_*$ is F-coCartesian.

We are now ready to define what an ∞-operad is.

Definition A.7. [**Lur17**, Definition 2.1.1.10] An ∞*-operad* is an ∞-category $\mathcal{O}^{\otimes}$ and a functor $p\colon \mathcal{O}^{\otimes} \to \textsc{Fin}_*$ such that

(1) p is an inert-coCartesian fibration. We refer to the inert-cocartesian lifts as the *inert* morphisms in $\mathcal{O}^{\otimes}$.
(2) If $x \in \mathcal{O}^{\otimes}{}_{|_{I_+}}$ and $x \to x_i$ is an inert morphism over $\rho_i\colon I_+ \to \{i\}_+$, then for any $y \in \mathcal{O}^{\otimes}{}_{|_{J_+}}$ the commutative square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{O}^{\otimes}}(y,x) \arrow[r] \arrow[d] & \prod\limits_{1\le i\le I} \mathrm{Map}_{\mathcal{O}^{\otimes}}(y,x_i) \arrow[d] \\ \mathrm{Hom}_{\textsc{Fin}_*}(J_+,I_+) \arrow[r] & \mathrm{Hom}_{\textsc{Fin}_*}(J_+,1_+) \end{tikzcd}$$

is a pullback in $\textsc{Spaces}$.
(3) Given objects $x_1, \dots, x_I \in \mathcal{O}^{\otimes}{}_{|_{1_+}}$, there exists an object $x \in \mathcal{O}^{\otimes}{}_{|_{I_+}}$ with cocartesian morphisms $x \to x_i$ over ρ_i.

Observe that the above definition should be thought of as the ∞-categorical version of a *colored* ordinary operad. Just as important as knowing what an ∞-operad is, we also need to understand maps between them.

Definition A.8. Let $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ be two ∞-operads. A *morphism of* ∞*-operads* from $\mathcal{O}^{\otimes}$ to $\mathcal{P}^{\otimes}$ is a commutative triangle

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "F"] \arrow[dr] & & \mathcal{P}^{\otimes} \arrow[dl] \\ & \textsc{Fin}_* & \end{tikzcd}$$

such that the functor F preserves inert morphisms. We also refer to such a functor F as an $\mathcal{O}^{\otimes}$*-algebra in* $\mathcal{P}^{\otimes}$, and denote by

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \subset \mathrm{Fun}_{\textsc{Fin}_*}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes})$$

the full ∞-subcategory spanned by the $\mathcal{O}^{\otimes}$-algebras in $\mathcal{P}^{\otimes}$.

We now return to the original quest, namely explaining how to get an ∞-operad from a colored operad arising from a poset of marked opens. This is the main source of ∞-operads in this thesis.

Construction A.9. [**Lur17**, Construction 2.1.1.7] Let $\mathcal{O}$ be a colored operad. Define a category $\mathcal{O}^{\otimes}$ as follows:

(1) The objects of $\mathcal{O}^{\otimes}$ are finite sequences of colors $x_1, \ldots, x_n \in \mathcal{O}$.
(2) Given two sequences of objects $x_1, \ldots, x_m \in \mathcal{O}$ and $y_1, \ldots, y_n \in \mathcal{O}$, a morphism from $\{x_i\}_{1\leq i\leq m}$ to $\{y_j\}_{1\leq j\leq n}$ is given by a map $\alpha\colon M_+ \to N_+$ in $\textsc{Fin}_*$ together with a collection of morphisms
$$\{\phi_j \in \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in\alpha^{-1}(j)}, y_j)\}_{1\leq j\leq n} \tag{A.1}$$
in $\mathcal{O}$.
(3) Composition of morphisms in $\mathcal{O}^{\otimes}$ is determined by the composition laws of $\textsc{Fin}_*$ and of the colored operad $\mathcal{O}$.

Note that the category $\mathcal{O}^{\otimes}$ comes equipped with a forgetful functor $\pi\colon \mathcal{O}^{\otimes} \to \textsc{Fin}_*$. In fact, one can reconstruct the colored operad $\mathcal{O}$ from this functor up to canonical equivalence.

Notation A.10. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. In general, an object of $\mathcal{O}^{\otimes}$ is denoted by $(I_+, (o_i)_{i\in I})$, where $I_+ \in \textsc{Fin}_*$ and $o_i \in \mathcal{O}$. To simplify notation we often abbreviate this by only the finite tuple, i.e. $\bar{o} := (o_i)_{i\in I}$, and leave the object $I_+ \in \textsc{Fin}_*$ implicit. If the object is in the fiber over 1_+ we simply write $o := (1_+, (o))$.

Remark A.11. Let $\mathcal{O}$ be a colored operad, and $\mathcal{O}^{\otimes}$ be the corresponding (ordinary) category from Construction A.9. Then the forgetful functor $N(\mathcal{O}^{\otimes}) \longrightarrow N(\textsc{Fin}_*)$ is an ∞-operad [**Lur17**, Example 2.1.1.21].

In general we suppress the nerve from the notation. Hence in the situation of Remark A.11 above we simply say "the ∞-operad $\mathcal{O}^{\otimes}$".

Example A.12. We start with a colored operad coming from a poset of marked opens as in Construction 2.13, and want to give the corresponding ∞-operad $\mathfrak{U}^{\otimes}$. Objects are finite sequences of marked opens in $\mathfrak{U}$ which we denote by $\overline{u} = (u_i)_{i\in I}$. Let $\bar{v} = (v_j)_{j\in J}$ be a second object. A morphism $f\colon \bar{u} \to \bar{v}$ is given by a map $f\colon I_+ \to J_+$ in $\textsc{Fin}_*$ such that for every $j \in J$ the marked opens $(u_i)_{i\in f^{-1}(j)}$ are all pairwise disjoint and come with a marked inclusion $\bigcup_{i\in f^{-1}(j)} u_i \hookrightarrow^{\mathrm{m}} v_j$.

The conditions on the opens to get a morphism is exactly the conditions for the collection of maps in Equation (A.1) to all be the singleton set (instead of the empty set).

Definition A.13. [**Lur17**, Definition 2.3.1.1] We say that an ∞-operad $\mathcal{O}^{\otimes}$ is *unital* if, for every object $X \in \mathcal{O}$, the space $\mathrm{Mul}_{\mathcal{O}}(\emptyset, X)$ is contractible.

This definition is the ∞-analogue of asking for an ordinary (i.e. discrete, colored) operad to have a unique nullary operation. Unitality in the sense of having a distinguished unique unary operation, i.e. identity operation, is already built into the definition of an ∞-operad.

Example A.14. If one is working with ∞-operads arising from ordinary (unmarked) opens and inclusions, i.e. $\mathfrak{U}^{\otimes}$ for $\mathfrak{U} \subseteq \mathrm{open}(X)$ some full subposet, one obtains a unital ∞-operad. This is because $\emptyset$ is an initial object.

Remark A.15. In the marked setting $\emptyset$ is only an initial object if the global space X is trivially marked, i.e. if there are no marked points. This is because for any open $U \in \mathrm{open}(X)_{\mathrm{mrk}}$ that contains at least one marked point there is *no* marked inclusion of $\emptyset$ into U. We have that $\mathfrak{U}^{\otimes}$, for $\mathfrak{U} \subseteq \mathrm{open}(X)_{\mathrm{mrk}}$ some full subposet, is *not* a unital ∞-operad in general.

Symmetric monoidal ∞-categories can be defined as ∞-operads satisfying some additional conditions. To give the precise statement we first need the following definition.

Definition A.16. Let $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be an ∞-operad. We say that a morphims f in $\mathcal{O}^{\otimes}$ is

- *inert* if $p(f)$ is inert in Fin_* and f is a p-coCartesian morphism;
- *active* if $p(f)$ is active in Fin_*.

Definition A.17. [**Lur17**, Definition 2.0.0.7] A *symmetric monoidal ∞-category* is a coCartesian fibration of simplicial sets $q \colon \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ with the following property:

- For each $N \geq 0$, the maps $\{\rho^i \colon N_+ \to 1_+\}_{1 \leq i \leq N}$ induce functors $\rho^i_! \colon \mathcal{C}^{\otimes}_{|N_+} \to \mathcal{C}^{\otimes}_{|1_+}$ which determine an equivalence $\mathcal{C}^{\otimes}_{|N_+} \simeq \left(\mathcal{C}^{\otimes}_{|1_+}\right)^N$.

Moreover, in Part 1 and 2 we always assume our target category to be $\otimes$-presentable. We give the precise definition of this here.

Definition A.18. A symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is *$\otimes$-presentable* if it is presentable, and if, for each $C \in \mathcal{C}$, the functor $C \otimes - \colon \mathcal{C} \to \mathcal{C}$ takes colimit diagrams to colimit diagrams.

In particular, $\mathcal{C}^{\otimes}$ being presentable means that all (small) colimits exist. See for example [**Lur09a**, Definition 5.5.0.1] for more details.

Example A.19. The ∞-category of chain complexes over a fixed ring is $\otimes$-presentable. This is a commonly used target in the literature on factorization algebras and encompasses e.g. all examples of Costello-Gwilliam from [**CG17, CG21**].

A.2. Dendroidal sets. In this subsection we give definitions, some examples and recollect facts about dendroidal sets relevant for Section 7.1.1. We refer the reader to e.g. [**CM11**] or [**HM22**] for more details. Dendroidal sets has a model category structure, briefly recalled below, which is Quillen equivalent to that of Lurie's ∞-operads, hence it presents the ∞-category of ∞-operads.

Notation A.20. We use the notation Opd' for the $(1,1)$-category of operads (and isomorphisms), while Opd is the $(2,1)$-category of operads and operad maps.

Definition A.21. Let Ω denote the *symmetric tree category*, i.e. the full subcategory $\Omega \hookrightarrow \mathrm{Opd}'$ whose objects are the symmetric operads that are free on finite rooted non-planar trees. The category of *dendroidal sets* is the presheaf category

$$\mathrm{dSet} := \mathrm{Fun}(\Omega^{\mathrm{op}}, \mathrm{Set}) .$$

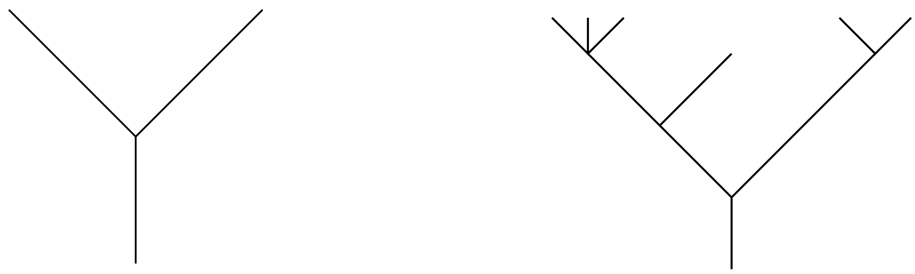


FIGURE 34. Two examples of (finite rooted non-planar) trees.

Definition A.22. The *dendroidal nerve functor* $\mathrm{N_d}\colon \mathrm{OPD}' \to \mathrm{dSET}$ is defined by

$$\mathrm{N_d}(\mathcal{O})(T) \coloneqq \mathrm{OPD}'(T,\mathcal{O}) \ .$$

That is, it selects operations of shape T in the operad $\mathcal{O}$.

Example A.23. Let $\mathfrak{U}$ be a poset of marked opens, considered as a (colored) operad. Moreover, let T be the left tree of Figure 34. The dendroidal nerve of $\mathfrak{U}$ evaluates as

$$\mathrm{N_d}(\mathfrak{U})(T) = \mathrm{OPD}'(T,\mathfrak{U}) = \{U_1, U_2, V \in \mathfrak{U} \mid U_1 \uplus U_2 \hookrightarrow^{\mathrm{m}} V\} \ .$$

That is, it picks out triples of objects in $\mathfrak{U}$ coming with a marked inclusion of the shape of the tree T. Explicitly, a marked inclusion of two disjoint marked opens U_1, U_2 into another marked open V. For more complicated trees one simply gets more complicated (possibly nested) marked inclusions of more opens.

We now briefly give the model structure on the category of dendroidal sets. Recall that a model category is called *left proper* if all objects are cofibrant.

THEOREM A.24. [**CM11**, Theorem 2.4 and Proposition 2.6] *The category of dendroidal sets is endowed with a model category structure for which the cofibrations are the normal monomorphisms. Moreover, the model structure is left proper.*

We omit spelling out the fibrant objects, fibrations and weak equivalences since they are not explicitly needed here. However, in Section 7.1.1 we need to understand the cofibrations, i.e. the normal monomorphisms, so we now consider these in some more detail. Note that for a dendroidal set B and a tree T in Ω, the group $\mathrm{Aut}(T)$ acts on the set B_T.

Definition A.25. We say that a dendroidal set B is *normal* if the action of $\mathrm{Aut}(T)$ on B_T is free for any tree T.

We now consider the class of examples of normal dendroidal sets relevant to us.

Example A.26. Let $\mathfrak{U}$ be a poset of marked opens with $\emptyset \notin \mathfrak{U}$. Consider $\mathrm{N_d}(\mathfrak{U})(T)$ as in Example A.23. The automorphism group $\mathrm{Aut}(T)$ in this case is simply the symmetric group on 2 elements, i.e. $\mathrm{Aut}(T) = \Sigma_2 = \{\mathrm{id}, \tau\}$. Acting with the non-trivial permutation element on an arbitrary object gives

$$\tau\Big((U_1, U_2, V), U_1 \uplus U_2 \hookrightarrow^{\mathrm{m}} V\Big) = \Big((U_2, U_1, V), U_2 \uplus U_1 \hookrightarrow^{\mathrm{m}} V\Big) \ . \tag{A.2}$$

Even though the marked inclusions clearly are the same because $\uplus$ is symmetric, the triples are in general not the same. The only way the triples on both sides of (A.2) would be equal is if $U_1 = U_2$ in which case we do not have a marked inclusion because the two opens would not be disjoint.

To conclude that $\mathrm{N_d}(\mathfrak{U})$ is normal we need to consider arbitrary trees. The general automorphism group of a tree is given by an iterated semidirect product of symmetric groups (see e.g. [**HM22**, Section 3.1]). However, the core argument above works the same, and we leave the details to the interested reader.

Remark A.27. If we instead work with a poset of marked opens $\mathfrak{U}$ which contains the empty set the corresponding dendroidal set $\mathrm{N_d}(\mathfrak{U})$ is *not* normal anymore. For example, if in the triple $((U_1, U_2, V), U_1 \uplus U_2 \hookrightarrow^{\mathrm{m}} V)$ we set $U_1 = \emptyset = U_2$ it follows that both sides of (A.2) are equal. Hence, it is crucial to exclude the emptyset from the poset of marked opens to obtain normal dendroidal sets.

In a similar fashion to Definition A.25 one can define what it means for a map of dendroidal sets to be normal. However, in the case of normal monomorphisms we have the following result which is sufficient for our use.

Remark A.28. [**CM11**, Corollary 1.8] Any monomorphism $i\colon A \to B$ of dendroidal sets with B normal is a normal monomorphism.

In Section 7.1.1 we exactly work with dendroidal sets which are all normal exactly to be able to employ the above result. The model category of dendroidal sets is used as a presentation of the ∞-category of ∞-operads. This is justified by a series of Quillen equivalences existing in the literature, namely:

Remark A.29.
- Barwick has a model for ∞-operads which is Quillen equivalent to Lurie's model [**Bar18**].
- Moreover, Barwick's model is shown to be Quillen equivalent to the model of complete dendroidal Segal spaces by Chu, Haugseng and Heuts in [**CHH18**].
- Lastly, the three dendroidal models, and also dendroidal sets are compared to simplicial operads by Cisinski and Moerdijk in [**CM13a, CM13b**].

In total we learn that even though we do an explicit computation in Section 7.1.1 using the model of dendroidal sets the homotopy theory thereof is equivalent to that of Lurie's ∞-operads [**CHH18**, Corollary 1.2]. This lets us extract a model-independent consequence, and justifies not picking one specific model throughout.

A.3. Colimit cofinality. In this subsection we first give the definition of a colimit cofinal functor. Then we recall some useful results for proving colimit cofinality where the most notable one is Quillen's theorem A. Lastly, we recall a colimit cofinality criterion which is used in Section 7.3.

Definition A.30. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is *colimit cofinal* if for each functor $\mathcal{D} \to \mathcal{E}$ to another ∞-category the canonical morphism

$$\operatorname{colim}\left(\mathcal{C} \xrightarrow{F} \mathcal{D} \to \mathcal{E}\right) \longrightarrow \operatorname{colim}\left(\mathcal{D} \to \mathcal{E}\right)$$

is an equivalence given that both colimits exists.

It is straightforward from the definition that the composition of two colimit cofinal functors is again colimit cofinal. The dual notion is that of *limit cofinal* functors. However, since we mostly manipulate colimits coming from (operadic) left Kan extensions in this thesis we only talk about colimit cofinal functors here.

Remark A.31. We warn the reader that the terminology "colimit cofinal" is *not* universal. For example in Lurie's work this is simply called a final functor, while the dual notion (i.e. limit cofinal here) is called an initial functor. Meanwhile, e.g. Cisinski calls a colimit cofinal functor for an initial functor in [**Cis19**]. We choose to say colimit or limit cofinal here to circumvent this confusing terminology all together.

We now recall Quillen's theorem A which is a very useful tool to check if a functor is colimit cofinal. For more details on the ∞-categorical treatment of this theorem we refer the reader for example to [**Lur09a**, §4.1.3].

THEOREM A.32 (Quillen's theorem A). *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. The functor F is colimit cofinal if, and only if, for each $d \in \mathcal{D}$ the classifying space*

$$\left|\mathcal{C}_{d_{/}}\right| \simeq * \tag{A.3}$$

is weakly contractible.

Explicitly, in Part 1 of this thesis we often check weak contractibility of the undercategory (A.3) by proving that the undercategory is cofiltered or that it has an initial or terminal object. Moreover, we also frequently use the results in the below (non-exhaustive) list of situations which ensures colimit cofinality.

- Localizations of ∞-categories are both colimit and limit cofinal ([**Cis19**, Proposition 7.1.10]).
- Every right adjoint functor is colimit cofinal ([**AF20b**, Corollary 5.12]).
- A cartesian fibration with weakly contractible fibers is colimit cofinal ([**Lur09a**, Lemma 4.1.3.2]).

Lastly, in Section 7.3 we use the following lemma, which gives a useful colimit cofinality criterion. The pure category theoretic statement below was extracted from [**AF20a**, Proposition 2.22].

Lemma A.33. [**KSW24**, Lemma A.4] *Let $\mathcal{D}$ be an ∞-category and*

$$F\colon P^{\triangleright} \to \mathrm{RFib}(\mathcal{D}), \qquad p \mapsto (F(p) \to \mathcal{D})$$

a cone of right fibrations over $\mathcal{D}$. The following are equivalent

(1) The map

$$\operatorname*{colim}_{p\in P} F(p) \to F(\infty)$$

of ∞-categories is colimit cofinal.

(2) The map

$$\operatorname*{colim}_{p\in P} F(p)^{\simeq} \to F(\infty)^{\simeq} \tag{A.4}$$

of ∞-groupoids is an equivalence.

(3) For every object $d \in \mathcal{D}$, the induced map

$$\operatorname*{colim}_{p\in P} F(p)_d \to F(\infty)_d$$

of ∞-groupoids is an equivalence.

A.4. A localization criterion. Here we briefly recall the definition of ∞-categorical localizations, before recalling a general criterion to identify localizations which is used in Section 7.2.

Recall that the inclusion of $\textsc{Spaces}$ into $\textsc{Cat}_\infty$ has a left adjoint

$$\begin{tikzcd} \textsc{Spaces} \arrow[r, hook] & \textsc{Cat}_\infty \arrow[l, bend right, "|-|"'] \end{tikzcd} .$$

For an ∞-category $\mathcal{D}$ the value of the left adjoint $|\mathcal{D}|$ is called the *classifying space of* $\mathcal{D}$. Using the model of complete Segal spaces one has that $|\mathcal{D}| = \operatorname{colim} \mathcal{D}$, while in the setting of quasi-categories the classifying space is given by the geometric realization.

Definition A.34. Let $\mathcal{D}$ be an ∞-category and let $\mathcal{W} \subset \mathcal{D}$ be an ∞-subcategory which contains all weak equivalences of $\mathcal{D}$. The *localization of* $\mathcal{D}$ *at* $\mathcal{W}$ is then defined to be the pushout

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ |\mathcal{W}| \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{D}[\mathcal{W}^{-1}] \end{tikzcd} .$$

We now give a lemma which provides a convenient criterion used to check if something is a localization. We use the notation $\mathcal{D}^{\simeq}$ for the groupoid-core of an ∞-category $\mathcal{D}$, i.e. the maximal groupoid inside $\mathcal{D}$.

Lemma A.35. [**KSW24**, Lemma A.2] *Let $F\colon \mathcal{D} \to \mathcal{E}$ be a functor between ∞-categories. Define*

$$\mathcal{W} := F^{-1}\{\textit{equivalences in } \mathcal{E}\}$$

to be the wide subcategory of $\mathcal{D}$ containing those arrows which become invertible in $\mathcal{E}$. Assume that

(1) the map of ∞-groupoids

$$|\mathcal{W}| \xrightarrow{\simeq} \mathcal{E}^{\simeq}$$

induced by F is an equivalence, and

(2) for each $d \in \mathcal{D}$ the map

$$\left|\mathcal{W} \times_{\mathcal{D}} \mathcal{D}_{/d}\right| \xrightarrow{\simeq} (\mathcal{E}_{/F(d)})^{\simeq}$$

induced by F is an equivalence.

Then F induces an equivalence

$$\mathcal{D}[\mathcal{W}^{-1}] \xrightarrow{\simeq} \mathcal{E} .$$

Remark A.36. The above lemma is taken from [**KSW24**] where the authors learned about the argument from Berry's thesis [**Ber21**] and private communication with David Ayala. Originally this argument, although with a major gap, appeared in [**AFT17a**]. The gap was later closed using a result of Mazel-Gee from [**MG19**]. This argument also appears in [**Cep22**]. Moreover, at the same time as [**KSW24**] was written essentially the same result was extracted by Arakawa in [**Ara24**, Proposition 2.27].

A.5. Groupoid-core of overcategories. Here we give a short lemma which identifies the groupoid core of an overcategory with a certain coproduct used in Section 6.3. This result seems to be implicitly present e.g. in the proof of [**AFT17a**, Lemma 2.21], but since we do not know a reference for this we have included a short proof.

Lemma A.37. *Let $\mathcal{D} \hookrightarrow \mathcal{C}$ be an inclusion of two ∞-categories, and let $x \in \mathcal{C}$ be an object. Then we have an equivalence*

$$\left(\mathcal{D}_{/x}\right)^{\simeq} \simeq \coprod_{y \in \pi_0(\mathcal{D}^{\simeq})} \mathcal{C}(y,x)_{\mathrm{Aut}_{\mathcal{D}}(y)} \ . \tag{A.5}$$

Proof. Recall that the map $\mathcal{D}_{/x} \to \mathcal{D}$ is a right fibration. The same is true after passing to the groupoid-cores, i.e. $\left(\mathcal{D}_{/x}\right)^{\simeq} \to \mathcal{D}^{\simeq}$ is also a right fibration. By straightening, the corresponding functor is

$$\mathcal{C}(-,x) \colon \mathcal{D}^{\simeq} \longrightarrow \text{Spaces} \ . \tag{A.6}$$

We can tautologically write the groupoid-core of $\mathcal{D}$ as a coproduct over its isomorphism classes. Explicitly, we have $\mathcal{D}^{\simeq} = \coprod_{y \in \pi_0(\mathcal{D}^{\simeq})} \langle y \rangle$, where $\langle y \rangle$ is the full subcategory spanned by the object y. Restricting the functor in (A.6) to one connected component of the groupoid-core, i.e.

$$\mathsf{B}\mathrm{Aut}_{\mathcal{D}}(y) \hookrightarrow \mathcal{D}^{\simeq} \longrightarrow \text{Spaces} \ , \tag{A.7}$$

unstraightens to the left vertical map of the below pullback square

$$\begin{tikzcd} \mathcal{C}(y,x)_{\mathrm{Aut}_{\mathcal{D}}(y)} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \left(\mathcal{D}_{/x}\right)^{\simeq} \arrow[d] \\ \mathsf{B}\mathrm{Aut}_{\mathcal{D}}(y) \arrow[r, hook] & \mathcal{D}^{\simeq} \end{tikzcd}$$

where we have also computed the total space corresponding to the functor in (A.7) as the colimit

$$\operatorname*{colim}_{y \in \mathsf{B}\mathrm{Aut}_{\mathcal{D}}(y)} \mathcal{C}(y,x) = \mathcal{C}(y,x)_{\mathrm{Aut}_{\mathcal{D}}(y)} \ ,$$

i.e. gives exactly the $\mathrm{Aut}_{\mathcal{D}}(y)$-equivariant maps in $\mathcal{C}$ from y to x. This in turn implies equation (A.5). ☐

A.6. Operadic left Kan extension. The theory of operadic colimits and operadic left Kan extensions is developed in [**Lur17**, Section 3.1] in much greater generality than what we need in this thesis. Here we briefly recollect the necessary definitions and results simplified to our setting following [**KSW24**, Appendix B].

Recall that we assume our target ∞-category $(\mathcal{C}, \otimes)$ to be $\otimes$-presentable (see Definition A.18), so it is in particular symmetric monoidal. Let $q \colon \mathcal{C}^{\otimes} \to \text{Fin}_*$ be the cocartesian fibration defining the symmetric monoidal structure (see Definition A.17), and let $\mathcal{C} \coloneqq \mathcal{C}^{\otimes}_{1_+}$ denote the underlying ∞-category. Recall from Notation A.10 that objects of $\mathcal{C}^{\otimes}$ are abbreviated by $(c_i)_i$ or simply $\bar{c}$ while a 1-tuple is abbreviated by $c = (1_+, (c))$.

Notation A.38. We write *operadic colimit diagram* for "operadic q-colimit diagram" (as in [**Lur17**, Definition 3.1.1.2]) when q is the structure-defining cocartesian fibration to Fin_*, since this is always the case in this thesis. When we have an operadic colimit cone $F^{\triangleright} \colon K^{\triangleright} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ whose value at the cone point $\infty \in K^{\triangleright}$ lies in the fiber over $I_+ \in \text{Fin}_*$ we write

$$\operatorname*{ocolim}_{k \in K} F(k) \xrightarrow{\simeq} F^{\triangleright}(\infty) \quad \in \mathcal{C}^{\otimes}_{I_+} \ .$$

The latter is a mild abuse of notation since for an operadic colimit one needs to specify both the base diagram $F\colon K \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ as well as a cone $F_0^{\triangleright}\colon K^{\triangleright} \to \mathrm{FIN}_*^{\mathrm{act}}$ of $F_0 \coloneqq q \circ F$. In this notation the cone $F_0^{\triangleright}$ is left implicit.

Proposition A.39. [**Lur17**, Proposition 3.1.1.15 (2) & Proposition 3.1.1.20] *Let $F^{\triangleright}\colon K^{\triangleright} \to \mathrm{FIN}_*^{\mathrm{act}}$ be a cone, and let $\otimes F^{\triangleright}\colon K^{\triangleright} \to \mathcal{C}$ be the transferred cone along the terminal transformation $qF^{\triangleright}(-) \Rightarrow 1_+$. Then $F^{\triangleright}$ is an operadic colimit cone if and only if $\bigotimes F^{\triangleright}$ is an (ordinary) colimit cone in $\mathcal{C}$. Explicitly, we have*

$$\operatorname*{ocolim}_{k\in K} F(k) \xrightarrow{\simeq} F^{\triangleright}(\infty) \qquad \in \mathcal{C}^{\otimes}_{qF^{\triangleright}(\infty)}$$

if and only if we have

$$\operatorname*{colim}_{k\in K} \bigotimes F(k) \xrightarrow{\simeq} \bigotimes F^{\triangleright}(\infty) \qquad \in \mathcal{C}\ .$$

Remark A.40. By the above proposition we see that every diagram $F\colon K \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ admits an operadic colimit with cone point in $\mathcal{C} = \mathcal{C}^{\otimes}_{1_+}$ by setting

$$\operatorname*{ocolim}_{k\in K} F(k) \coloneqq \operatorname*{colim}_{k\in K} \bigotimes F(k) \qquad \in \mathcal{C}\ .$$

This lies above the (unique) cone $F_0^{\triangleright}\colon qF \Rightarrow 1_+$ which is left implicit.

Remark A.41. Precomposition with colimit cofinal maps also preserves operadic colimit cones [**Lur17**, Remark 3.1.1.4]. That is, we have

$$\operatorname*{ocolim}_{h\in K} F(\alpha(h)) \simeq \operatorname*{ocolim}_{k\in K} F(k) \qquad \in \mathcal{C}^{\otimes}_{F_0^{\triangleright}(\infty)}$$

whenever $\alpha\colon H \to K$ is a colimit cofinal map.

Just like in the setting of ordinary or ∞-categories there is a notion of left Kan extension also for ∞-operads. Moreover, this also gives rise to an adjunction [**Lur17**, Corollary 3.1.3.5]. Explicitly, given a map $\iota\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ of small ∞-operads we have an adjunction

$$\iota_!\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \leftrightarrows \mathrm{Alg}_{\mathcal{P}}(\mathcal{C})\colon \iota^*$$

where the left adjoint $\iota_!$ is *operadic left Kan extension* and the right adjoint ι^* is given by restriction. The operadic left Kan extension is characterized uniquely by the pointwise operadic colimit formula

$$\operatorname*{ocolim}_{\bar{o}\in\mathcal{O}^{\otimes}_{/\bar{p}}{}^{\mathrm{act}}} \mathcal{A}(\bar{o}) \xrightarrow{\simeq} (\iota_!\mathcal{A})(\bar{p})\ . \tag{A.8}$$

Corollary A.42. [**KSW24**, Corollary B.1] *If $\iota\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is fully faithful, then $\iota_!$ is also fully faithful and induces an equivalence*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\simeq} \left\{ \mathcal{A} \in \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \;\middle|\; \forall p \in \mathcal{P}\colon \operatorname*{ocolim}_{\bar{o}\xrightarrow{\mathrm{act}}p} \mathcal{A}_{|\mathcal{O}}(\bar{o}) \xrightarrow{\simeq} \mathcal{A}(p) \right\}$$

Corollary A.43. [**KSW24**, Corollary B.2] *An algebra $\mathcal{A} \in \mathrm{Alg}_{\mathcal{P}}(\mathcal{C})$ is the operadic left Kan extension of its restriction to $\mathcal{O}$ if and only if it induces an equivalence*

$$\operatorname*{colim}_{\bar{o}\xrightarrow{\mathrm{act}}p} \bigotimes \mathcal{A}(\bar{o}) \xrightarrow{\simeq} \mathcal{A}(p) \qquad \in \mathcal{C}$$

for all $p \in \mathcal{P}$.

In special settings we are able to relate the operadic and ordinary left Kan extension.

Lemma A.44. [**KSW24**, Lemma B.3] *Let $\iota\colon \mathcal{O}^{\otimes} \hookrightarrow \mathcal{P}^{\otimes}$ be a fully faithful inclusion of ∞-operads. Assume that for every object $\bar{o} \in \mathcal{O}^{\otimes}_{I_+}$ and every active map $\iota(\bar{o}) \to p$, i.e. with codomain $p \in \mathcal{P}$, there exists a cocartesian arrow $\bar{o} \to o$ in $\mathcal{O}^{\otimes}$ lifting the active map $I_+ \to 1_+$ which remains cocartesian in $\mathcal{P}^{\otimes}$. Then the (a priori lax) square*

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, hook, "\mathrm{oLKE}"] \arrow[d, "(-)_{1_+}"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \arrow[d, "(-)_{1_+}"] \\ \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \arrow[r, hook, "\mathrm{LKE}"] & \mathrm{Fun}(\mathcal{P}, \mathcal{C}) \end{tikzcd}$$

commutes and is a pullback. Here, the upper and lower horizontal functors are operadic and ordinary left Kan extension, respectively.

Example A.45. Let $\iota\colon \mathfrak{V} \hookrightarrow \mathfrak{U}$ be an inclusion of posets of marked opens. The associated ∞-operad inclusion $\iota\colon \mathfrak{V}^{\otimes} \hookrightarrow \mathfrak{U}^{\otimes}$ satisfies the assumption of Lemma A.44 if and only if $\mathfrak{V}$ is closed under disjoint unions subordinate to $\mathfrak{U}$. That is, for $U, V \in \mathfrak{V}$ two disjoint marked opens such that $U \sqcup V \in \mathfrak{U}_{\downarrow_{\mathrm{m}}}$ it follows that $U \sqcup V \in \mathfrak{V}$.

We also use the following definition, which is a natural extension of Lurie's terminology.

Definition A.46. [**KSW24**, Definition B.5] Given the solid outer commutative diagram

$$\begin{tikzcd} K \arrow[r, "F"] \arrow[d, hook] & \mathcal{C}^{\otimes}_{\mathrm{act}} \arrow[d, "q"] \\ L \arrow[r, "\overline{F_0}"'] \arrow[ur, dashed, "\overline{F}"] & \mathrm{FIN}^{\mathrm{act}}_{*} \end{tikzcd}$$

where the left vertical map is fully faithful, we say that a dashed lift $\overline{F}$ is the *operadic left Kan extension of* F if for each $l \in L$ we have

$$\operatorname*{ocolim}_{k \to l} F(k) \xrightarrow{\simeq} \overline{F}(l) \ .$$

Remark A.47. Using Proposition A.39 we can equivalently say that $\overline{F}$ is a left Kan extension of F if the transferred diagram $\bigotimes \overline{F}\colon L \to \mathcal{C}$ is an ordinary left Kan extension of the transferred diagram $\bigotimes F\colon K \to \mathcal{C}$. In particular, it follows that operadic left Kan extensions are transitive and preserve the operadic colimits of a diagram.

A.7. Grothendieck topology and cosheaves. Pointless factorization algebras satisfies a local-to-global property that is encoded by a cosheaf condition. In this section we give a more detailed explanation underlying the ad-hoc approach in Section 3.2. In particular, we explain what a topology on a category, i.e. a Grothendieck topology, is before giving more details on the marked Weiss topology. With this we also define what a cosheaf with respect to a Groethendieck topology is, and tie this to the previous definition of marked Weiss cosheaves.

A crucial concept for equipping a category with a (Grothendieck) topology is to specify the allowable ways one can access an object.

Definition A.48. For $d \in \mathcal{D}$, a *sieve on* d is a fully faithful functor $\mathcal{U} \hookrightarrow \mathcal{D}_{/d}$ such that for each $(d' \xrightarrow{f} d) \in \mathcal{U}$ and $(e \xrightarrow{g} d') \in \mathcal{D}(e, d')$ we have $(e \xrightarrow{g} d' \xrightarrow{f} d) \in \mathcal{U}$.

In other words, a sieve is closed under precomposition with *any* morphism of $\mathcal{D}$.

Example A.49. Let $\mathcal{D} = \mathfrak{U}$ be some poset of marked opens, and let $U \in \mathfrak{U}$ be a marked open. We immediately see that the downward closure $U_{\downarrow_{\mathrm{m}}} = \mathfrak{U}_{/U}$, i.e. a sieve in the sense of Definition 3.6, is indeed a sieve in the sense of Definition A.48. This justifies the ad-hoc naming convention in Section 3.

We now go directly to the main definition in this subsection.

Definition A.50. A *Grothendieck topology*, τ, on $\mathcal{D}$ is for each $d \in \mathcal{D}$, a collection of *covering sieves* for d, denoted $\tau(d)$, such that

(1) for each $d \in \mathcal{D}$, $\mathcal{D}_{/d} \overset{=}{\hookrightarrow} \mathcal{D}_{/d}$ is in $\tau(d)$,
(2) for $\mathcal{U} \in \tau(d)$, and morphism $f : e \to d$ in $\mathcal{D}$, we have $f^*\mathcal{U} \in \tau(e)$,
(3) if $\mathcal{U}$ is any sieve on $d \in \mathcal{D}$ such that the sieve
$$\bigcup_e \{f : e \to d \,|\, f^*\mathcal{U} \in \tau(e)\} \in \tau(d)$$
then $\mathcal{U} \in \tau(d)$.

A category $\mathcal{C}$ equipped with a Grothendieck topology τ is called a *site*.

Before we consider (marked) Weiss covers we give the most standard example:

Example A.51. For a topological space X there is a standard Grothendieck topology on $\mathrm{open}(X)$. For an open $V \in \mathrm{open}(X)$, a sieve $\mathcal{U} = \{U_i \hookrightarrow V\}$ is a covering sieve iff for each $x \in V$ there exists some U_i such that $x \in U_i$. In other words, the covering sieves are exactly the sieves corresponding to standard open covers.

Example A.52. Entirely analogously to the above example we can equip $\mathrm{open}(X)_{\mathrm{mrk}}$ with a marked version of the standard Grothendieck topology. For a marked open $V \in \mathrm{open}(X)_{\mathrm{mrk}}$, a sieve $\mathcal{U} = \{U_i \hookrightarrow^{\mathrm{m}} V\}$ is a covering sieve iff for each $x \in V$ there exists an U_i such that $x \in U_i$. That is, the covering sieves are exactly the sieves corresponding to the *marked* open covers as in Definition 3.1.

Just like how a set can be equipped with several different topologies, one can also assign several distinct Grothendieck topologies to a given category. As an illustration of this we now give the Grothendieck topology on $\mathrm{open}(X)_{\mathrm{mrk}}$ relevant to our setting.

Example A.53. Let X be a marked topological space, and consider the category $\mathrm{open}(X)_{\mathrm{mrk}}$. Let $V \in \mathrm{open}(X)_{\mathrm{mrk}}$ be a marked open. The *marked Weiss topology* is given as follows: a sieve $\mathcal{W} = \{W_i \hookrightarrow^{\mathrm{m}} V\}$ is a covering sieve iff it is a marked Weiss cover, i.e. for every finite subset $S \subset V$ there exists a marked open $W_i \in \mathcal{W}$ such that $S \subset W_i$.

To be able to formulate the cosheaf condition, we recall the *right cone* of a category $\mathcal{W}$
$$\mathcal{W}^{\triangleright} := \mathcal{W} \times \{0,1\} \coprod_{\mathcal{W}\times\{1\}} \star \,.$$
That is, one freely adjoins a terminal object to $\mathcal{W}$. If $\mathcal{W} \subset \mathcal{D}_{/d}$, we have a functor
$$\mathcal{W}^{\triangleright} \to \mathcal{D}_{/d}$$
that sends the terminal (co)cone point $\star$ to $(d \xrightarrow{\mathrm{id}} d)$. Each morphism $(d' \xrightarrow{f} d) \to \star$ is sent to the obvious square.

Definition A.54. Let $(\mathcal{D}, \tau)$ be a site and $\mathcal{F}\colon \mathcal{D} \to \mathcal{C}$ a functor. We say that $\mathcal{F}$ is a ($\mathcal{C}$-valued) cosheaf (w.r.t τ) if for all objects $d \in \mathcal{D}$ and all covering sieves $\mathcal{W} \subset \mathcal{D}_{/d}$, the composite

$$\mathcal{W}^{\triangleright} \to \mathcal{D}_{/d} \xrightarrow{\text{fgt}} \mathcal{D} \xrightarrow{\mathcal{F}} \mathcal{C} \tag{A.9}$$

is a colimit diagram.

There is a terminal object in the diagram (A.9), namely $\mathcal{F}(d)$. By the diagram being a colimit diagram we mean that this object is the colimit of the diagram with the terminal object removed, c.f. [**Lur09a**, Remark 1.2.13.5].

Example A.55. Let $\mathfrak{U}$ be a poset of marked opens and $\mathcal{A}$ a copresheaf defined on $\mathfrak{U}$. Asking for $\mathcal{A}$ to be a cosheaf for the marked Weiss topology as defined above exactly boils down to the cosheaf-condition in Definition 3.21. In particular, the colimit diagram condition is exactly that of Equation (3.4).

B. Background on (∞, n)-categories and dualizability

In Appendix B.1 of this appendix we first recall how n-fold Segal objects model higher categories. This is relevant for the pointless higher Morita categories where we construct n-fold Segal objects in CAT_∞, and extract its underlying $(\infty, n+1)$-category. Then we recall dualizability in this setting in Appendix B.2. From dualizability in VECT and adjunctability in CAT we motivate the definitions of duals and adjoints of objects, respectively morphisms of an (∞, n)-category. We also recall results which drastically reduce the dualizability-data one needs to provide.

B.1. Higher categories as n-fold Segal objects. We first give the definitions of n-uple category objects and n-fold Segal objects in an ∞-category $\mathcal{D}$ with finite limits. Then we recall a result of Haugseng which produces n-fold Segal objects in $\mathcal{D}$ from n-uple category objects in $\mathcal{D}$. Together with the localization functor from n-fold Segal spaces to their complete version this gives Construction B.8 which is used in Section 9.1 when constructing the pointless higher Morita categories.

Let $\mathcal{D}$ be an ∞-category with finite limits. Recall that for a simplicial object $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathcal{D}$ of $\mathcal{D}$ to satisfy the *Segal condition* one needs to show that there is an equivalence

$$X_{n+k} \longrightarrow X_n \times_{X_0} X_k \ ,$$

where the maps are induced by the inclusions of $0 < n$, respectively $n < n+k$ into $0 < \cdots < n+k$ in Δ. We say that $X_\bullet$ is a *Segal* (or *category) object* in $\mathcal{D}$.

Definition B.1. Let $\mathcal{D}$ be an ∞-category with finite limits. An n-uple simplicial object, i.e. a functor

$$X_{\bullet,\ldots,\bullet}\colon (\Delta^{\mathrm{op}})^n \to \mathcal{D}$$

is called an *n-uple category object* in $\mathcal{D}$ if, for each $1 \le i \le n$ and every tuple $k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n$ in $(\Delta^{\mathrm{op}})^{n-1}$, the simplicial object

$$X_{k_1,\ldots,k_{i-1},\bullet,k_{i+1},\ldots,k_n}\colon \Delta^{\mathrm{op}} \longrightarrow \mathcal{D}$$

satisfies the Segal condition. Denote by $\mathrm{Cat}^n(\mathcal{D}) \subseteq \mathrm{Fun}((\Delta^{\mathrm{op}})^n, \mathcal{D})$ the full subcategory spanned by the n-uple category objects.

For $n = 2$ this gives a double ∞-category, and in general we get n-uple ∞-categories.

Example B.2. We make this a bit more explicit for $n = 2$ and $\mathcal{D} = \mathrm{SPACES}$. Then we get

- a space $X_{0,0}$ of objects,
- a space $X_{1,0}$ of horizontal 1-morphisms and a space $X_{0,1}$ of vertical morphisms,
- as well as a space of commutative squares $X_{1,1}$.

Moreover, we also get units and coherent composition laws for all of these morphisms.

To instead get (∞, n)-categories we need two additional conditions to hold. First of all we only want one type of 1-morphisms, so some of the spaces from the above definition needs to be trivial. This is captured by the following definition:

Definition B.3. Let $\mathcal{D}$ be an ∞-category with limits. An *n-fold Segal object* in $\mathcal{D}$ is an n-uple category object X of $\mathcal{D}$ which satisfies the *essential constancy* condition. That is, for each $1 \le i \le n-1$ and for every tuple $k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n$ in $(\Delta^{\mathrm{op}})^{n-1}$ the degeneracy map

$$X_{k_1,\ldots,k_{i-1},0,\ldots,0} \longrightarrow X_{k_1,\ldots,k_{i-1},\bullet,k_{i+1},\ldots,k_n} \tag{B.1}$$

is a weak equivalence. The full subcategory spanned by the n-fold Segal objects is denoted by $\mathrm{Seg}_n(\mathcal{D}) \subseteq \mathrm{Cat}^n(\mathcal{D})$. In the special case of n-fold Segal objects of $\mathcal{D} = \textsc{Spaces}$ we call them *n-fold Segal spaces.*

Example B.4. In contrast to Example B.2, a 2-fold Segal space X corresponds to

- a space $X_{0,0}$ of objects,
- a space $X_{1,0}$ of objects, and
- a space $X_{1,1}$ of 2-morphisms.

Again, this comes with units and coherent compositions laws.

Note that for $n = 1$ there is no difference between the two notions because the condition (B.1) is empty. For $n \geq 2$ the essential constancy condition exactly corresponds to getting rid of the unwanted 1-morphisms present in n-uple category objects.

In Section 9.1 the formal construction of the (pointless) higher Morita categories is given in detail for $n = 1$ and sketched for general n. In particular, for $n = 1$ the construction leverages that constructible pointless factorization algebras forms an ∞-category to construct a category object in $\textsc{Cat}_\infty$, c.f. Corollary 9.14. Similarly, for general n one constructs an n-fold Segal object in $\textsc{Cat}_\infty$, see e.g. (9.2) and the surrounding discussion. From this we want to extract an $(\infty, 2)$-category, respectively $(\infty, n+1)$-category.

Observation B.5. Using the model of complete Segal spaces for ∞-categories we have an inclusion $\textsc{Cat}_\infty \hookrightarrow \mathrm{Cat}^1(\textsc{Spaces})$. This in turn induces an inclusion $\mathrm{Cat}^n(\textsc{Cat}_\infty) \hookrightarrow \mathrm{Cat}^{n+1}(\textsc{Spaces})$. Hence, from an n-uple category object in $\textsc{Cat}_\infty$ one gets an $(n+1)$-uple category object in $\textsc{Spaces}$.

Moreover, there is a construction to go from n-uple category objects of $\mathcal{D}$ to n-fold Segal objects. We make this more explicit for $n = 2$.

Construction B.6. Let $n = 2$, and let $X_{\bullet,\bullet}\colon (\Delta^{\mathrm{op}})^2 \to \mathcal{D}$ denote a 2-uple category object of $\mathcal{D}$. From this one extracts a 2-fold Segal space $\widetilde{X}$ as follows:

- Set $\widetilde{X}_{0,\bullet}$ to be the constant simplicial object at $X_{0,0}$.
- For $n > 0$ one defines $\widetilde{X}_{n,\bullet}$ to be the pullback

$$\begin{tikzcd} \widetilde{X}_{n,\bullet} \arrow[r] \arrow[d] & X_{n,\bullet} \arrow[d] \\ \widetilde{X}_{0,\bullet} \arrow[r] & X_{0,\bullet} \end{tikzcd} .$$

  The bottom horizontal map is induced by the degeneracies.

In the second step above we forget about the objects of $X_{0,1}$ that are not in the image of the degeneracy map $X_{0,0} \to X_{0,1}$. In other words, we exactly forget about the non-trivial 1-morphisms in one direction. The construction for general n essentially iterates the construction above, and yields the following result:

Proposition B.7. [**Hau18**, Proposition 4.12] *Let $\mathcal{D}$ be an ∞-category with finite limits. The inclusion $\mathrm{Seg}_n(\mathcal{D}) \hookrightarrow \mathrm{Cat}^n(\mathcal{D})$ has a right adjoint $\mathrm{U}_{\mathrm{Seg}}\colon \mathrm{Cat}^n(\mathcal{D}) \to \mathrm{Seg}_n(\mathcal{D})$.*

The above produces an n-fold Segal space, but to get an honest (∞, n)-category we need a *complete* n-fold Segal space. This corresponds to localizing at a suitable class of morphisms

[**Rez01**]. We denote the corresponding localization functor by

$$\mathrm{L}_n \colon \mathrm{Seg}_n(\text{SPACES}) \longrightarrow \mathrm{CSS}_n(\text{SPACES}) ,$$

where the latter denotes the full subcategory of complete n-fold Segal spaces.

In total, by putting together the different results above we get the following construction:

Construction B.8. Let X be an n-uple category object in CAT_∞. By Observation B.5 we view this as an $(n+1)$-uple category object in SPACES instead. From this we can produce an $(\infty, n+1)$-category

$$\mathrm{L}_n \mathrm{U}_{\mathrm{Seg}} X \in \mathrm{CSS}_{n+1}(\text{SPACES}) .$$

We refer to this as the *the underlying* $(\infty, n+1)$*-category of* X.

B.2. Dualizability in higher categories. The main motivation for constructing the pointless higher Morita category in Part 2 of this thesis is that it has advantages for dualizability. In this section we give a short review of the notions of dualizability and adjointability in a symmetric monoidal (∞, n)-category. We mostly follow [**Lur09b**] where more details can be found. Another reference is [**Hau18**] which works within the model of n-fold Segal spaces. Lastly, we also recall some results from [**Ara17**] which drastically reduces the data needed to exhibit (full) dualizability of objects.

As a gentle introduction we start by recalling the corresponding notions in ordinary category theory. First out is what it means for an object of an ordinary symmetric monoidal category to have a dual.

Definition B.9. Let $(\mathcal{D}, \otimes, \mathbb{1})$ be a symmetric monoidal category and let $\mathcal{R}$ be an object in $\mathcal{D}$. A *dual* of $\mathcal{R}$ is an object $\mathcal{R}^\vee$ in $\mathcal{D}$ together with an *evaluation* map $\mathrm{ev}_\mathcal{R} : \mathcal{R}^\vee \otimes \mathcal{R} \to \mathbb{1}$ and a *coevaluation* map $\mathrm{coev}_\mathcal{R} : \mathbb{1} \to \mathcal{R} \otimes \mathcal{R}^\vee$ such that the snake identities

$$(\mathrm{id}_\mathcal{R} \otimes \mathrm{ev}_\mathcal{R}) \circ (\mathrm{coev}_\mathcal{R} \otimes \mathrm{id}_\mathcal{R}) = \mathrm{id}_\mathcal{R} \qquad \text{and} \qquad (\mathrm{ev}_\mathcal{R} \otimes \mathrm{id}_\mathcal{R}) \circ (\mathrm{id}_\mathcal{R} \otimes \mathrm{coev}_\mathcal{R}) = \mathrm{id}_\mathcal{R}$$

holds. If an object $\mathcal{R}$ has a dual we say that $\mathcal{R}$ is *(1-)dualizable.*

Remark B.10. Note that above we assume the monoidal category $\mathcal{D}$ to be symmetric, so there is no need to distinguish between left and right duals. This will always be the case for the higher categories considered in this thesis.

One way of thinking about dualizability is that it often is a finiteness-condition of some sort. An explicit example of this is the following:

Example B.11. Consider $\mathcal{D} = \text{VECT}_\Bbbk$. An object $V \in \text{VECT}_\Bbbk$ is dualizable in the sense of Definition B.9 exactly if it is finite-dimensional. In that case the dual space is given by $V^\vee = \mathrm{Hom}(V, \Bbbk)$.

Now we turn towards the analogous notion for morphisms, i.e. asking for adjoints.

Definition B.12. Let $\mathcal{D}$ be a bicategory and let $R : \mathcal{R} \to \mathcal{S}$ and $L : \mathcal{S} \to \mathcal{R}$ be two 1-morphisms in $\mathcal{D}$. We say that L is a *left adjoint* of R and R is a *right adjoint* of L if there are 2-morphisms

$$\eta : \mathrm{id}_\mathcal{S} \Rightarrow R \circ L \qquad \text{and} \qquad \varepsilon : L \circ R \Rightarrow \mathrm{id}_\mathcal{R}$$

called the *unit* and *counit* of the adjunction, respectively, such that

$$L = L \circ \mathrm{id}_\mathcal{S} \xRightarrow{\mathrm{id}_L \circ \eta} L \circ R \circ L \xRightarrow{\varepsilon \circ \mathrm{id}_L} \mathrm{id}_\mathcal{R} \circ L \qquad \text{and}$$

$$R = \mathrm{id}_\mathcal{S} \circ R \xRightarrow{\eta \circ \mathrm{id}_R} R \circ L \circ R \xRightarrow{\mathrm{id}_R \circ \varepsilon} R \circ \mathrm{id}_\mathcal{R}$$

are identities. These are also referred to as snake identities.

If one sets $\mathcal{D} = \textsc{Cat}$, i.e. the 2-category of categories, functors and natural transformations, one gets back the usual notion of left and right adjoint functors from the above. Analogous to how dualizability data often boils down to a finiteness condition of some sort this is also often the case when asking for adjoints. For example, let $\mathcal{D} = \textsc{Alg}_1^{\mathrm{pl}}(\textsc{Vect}_{\Bbbk})$. A 1-morphism, i.e. an $(\mathbf{A}, \mathbf{B})$-bimodule $\mathbf{M}$, has a left adjoint if and only if it is finitely-generated and projective over $\mathbf{B}$.

We use the terms "objects of $\mathcal{D}$" and "0-morphisms of $\mathcal{D}$" interchangeably here. We now turn towards the generalization of the above two definitions to the higher categorical setting. For this we will exploit that from a higher category $\mathcal{D}$ one can extract an ordinary category called the *homotopy category* $h\mathcal{D}$ as explained e.g. in [**Lur09b**, Remark 2.1.19] in the setting of Segal spaces. Moreover, fixing two $(k-2)$-morphisms, for $k \geq 2$ one can extract a certain bicategory called the *homotopy bicategory*, c.f. [**Lur09b**, Definition 2.3.13]. Using this we have:

Definition B.13. Let $\mathcal{D}$ be an (∞, n)-category.

- A 1-morphism $A : \mathcal{R} \to \mathcal{S}$ has a *left* (or *right*) *adjoint* if it has a left (or right) adjoint in the homotopy bicategory $h_2\mathcal{D}$.
- For $k \geq 2$ and $\mathcal{R}, \mathcal{S}$ two fixed $(k-2)$-morphisms in $\mathcal{D}$, let A and B be two $(k-1)$-morphisms from $\mathcal{R}$ to $\mathcal{S}$ in $\mathcal{D}$. We say that a k-morphism $f : A \to B$ has a *left* (or *right*) *adjoint* if it has a left (or right) adjoint in the homotopy bicategory $h_2\mathrm{Hom}_{\mathcal{D}}(\mathcal{R}, \mathcal{S})$.
- If $\mathcal{D}$ is symmetric monoidal we say that an object $\mathcal{R}$ of $\mathcal{D}$ is *(1-)dualizable* if it admits a dual in the homotopy category $h\mathcal{D}$ of $\mathcal{D}$.

Remark B.14. At a first glance it might be puzzling that the above definition, say for adjoints, is meaningful using the homotopy bicategory instead of extracting an $(\infty, 2)$-category of k-morphisms (between two fixed $(k-1)$-morphisms) and $(k+1)$-morphisms. However, this is justified by results of e.g. [**RV16**] which says that every adjunction in the homotopy bicategory of an $(\infty, 2)$-category extends to an adjunction in the $(\infty, 2)$-category itself. Thus, working within the homotopy bicategory is indeed sufficient for detecting adjoints.

Remark B.15. If $\mathcal{D}$ is a Segal space which is not necessarily complete it is not actually necessary to complete it before checking for adjoints. The homotopy bicategory can be defined without completing $\mathcal{D}$ ([**Hau17**, Lemma 5.50]).

Remark B.16. Sometimes, like in the case of the pointed higher Morita categories it is instead convenient to establish the snake identities in the (∞, n)-category $\mathcal{D}$ up to equivalence. This in turn implies that in the homotopy (bi-)category they are identities.

Definition B.17. We say that a symmetric monoidal (∞, n)-category $\mathcal{D}$ is *fully m-dualizable*, for some $m \leq n$, if every object is dualizable and for every $1 \leq k < m$, every k-morphism has a left and right adjoint.

Remark B.18. We warn the reader about ambiguous notation in the literature: If a symmetric monoidal (∞, n)-category $\mathcal{D}$ is fully n-dualizable (i.e. $m = n$) in the above sense it is said to "have duals" in [**Lur09b**]. Conversely, for $m \leq n$ we have that $\mathcal{D}$ is fully m-dualizable in our sense if the underlying (∞, m)-category of $\mathcal{D}$ (obtained by discarding all non-invertible higher morphisms) "has duals" in Lurie's sense.

The property of "having duals" or full dualizability heavily depends on m, which is why we choose to include this dependence in our notation. For example, as recalled in Section 10 we know that the pointed higher Morita category $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$, in particular a $(\infty, n+1)$-category, is fully n-dualizable but far from being fully $(n+1)$-dualizable. See also [**Lur09b**, Warning 2.3.15 & Warning 2.3.22] for another example.

Another important concept is knowing when only certain objects of a higher category are higher dualizable, without the entire higher category being so. For this we follow [**Ara17**]. We start with some definitions.

Definition B.19. [**Ara17**, Definition 4.1.8] Let $\mathcal{D}$ be an (∞, n)-category and let f be a k-morphism in $\mathcal{D}$. We say that f *has all adjoints* in $\mathcal{D}$ if there exists a tower of adjunctions

$$\cdots \dashv f^{LL} \dashv f^{L} \dashv f \dashv f^{R} \dashv f^{RR} \dashv \cdots$$

in $\mathcal{D}$. If we discard all k-morphisms of $\mathcal{D}$ that does not have all adjoints in $\mathcal{D}$ we obtain an (∞, n)-category that is denoted by $\mathcal{D}^{(k)}$.

Definition B.20. Let $\mathcal{R}$ be an object in a symmetric monoidal (∞, n)-category $\mathcal{D}$. We say that $\mathcal{R}$ is *1-dualizable* in $\mathcal{D}$ if it has a dual in $\mathcal{D}$. For $k \geq 2$ we say that $\mathcal{R}$ is *k-dualizable* in $\mathcal{D}$ if it is $(k-1)$-dualizable as an object in $\mathcal{D}^{(k-1)}$.

Definition B.21. [**Ara17**, Definition 4.1.11] Let $\mathcal{R}$ be an object of a symmetric monoidal (∞, n)-category $\mathcal{D}$. A *complete set of 1-dualizability data for* $\mathcal{R}$ is a choice of dual $\mathcal{R}^\vee$, together with evaluation and coevaluation 1-morphisms. For $k \geq 2$, a *complete set of k-dualizability data for* $\mathcal{R}$ consists of a complete set of $(k-1)$-dualizability data for $\mathcal{R}$, together with the choice of a tower of left and right adjoints for each $(k-1)$-morphism in this set, and choices of unit and counit k-morphisms witnessing these adjunctions.

Proposition B.22. [**Ara17**, Proposition 4.1.14] *Let $\mathcal{R}$ be an object in a symmetric monoidal (∞, n)-category $\mathcal{D}$. Then $\mathcal{R}$ is k-dualizable if and only if there exists a complete set of k-dualizability data for $\mathcal{R}$.*

From the above proposition we see that the data and conditions one must check for n-dualizability of an object involves an increasing number of (infinite) towers of adjoints as n increases. However, it is possible to drastically reduce the data needed to ensure full n-dualizability of an object. For this we need the following definition.

Definition B.23. [**Ara17**, Definition 4.1.15] Let $\mathcal{R}$ be an object in a monoidal (∞, n)-category $\mathcal{D}$. A *partial set of 1-dualizability data for* $\mathcal{R}$ is a choice of a dual $\mathcal{R}^\vee$, together with evaluation and coevaluation 1-morphisms. For $k \geq 2$, a *partial set of k-dualizability data for* $\mathcal{R}$ consists of a partial set of $(k-1)$-dualizability data for $\mathcal{R}$, together with choices of left (or right) adjoints for all $(k-1)$-morphisms in this set, together with unit and counit k-morphisms witnessing these adjunctions.

As the reader might anticipate, we then have the following very useful result.

THEOREM B.24. [**Ara17**, Corollary 4.1.20] *Let $\mathcal{R}$ be an object in a symmetric monoidal (∞, n)-category $\mathcal{D}$. Then $\mathcal{R}$ is fully n-dualizable if and only if there exists a partial set of full n-dualizability data for $\mathcal{R}$.*

Lastly, one can transfer dualizability-data along a symmetric monoidal functor. We use this to translate the known n-dualizability of $\mathrm{ALG}_n^{\mathrm{ptd}}(\mathcal{C})$ to $\mathrm{ALG}_n^{\mathrm{pl}}(\mathcal{C})$. As we do not know a concrete reference for this fact we also provide a (sketch of) proof tailored to our situation.

Proposition B.25. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor between two symmetric monoidal $(\infty, n+1)$-categories. If a k-morphism f, for $0 \leq k \leq n$, of $\mathcal{C}$ has a left (right) adjoint, it follows that its image $F(f)$ in $\mathcal{D}$ also has a left (right) adjoint.*

Sketch of proof. We model $\mathcal{C}$ and $\mathcal{D}$ as functors $\mathcal{C}_{\vec{\bullet}}, \mathcal{D}_{\vec{\bullet}}\colon \mathrm{Fin}_* \times (\Delta^{\mathrm{op}})^n \to \mathrm{Cat}_\infty$ and $F\colon \mathcal{C}_{\vec{\bullet}} \Rightarrow \mathcal{D}_{\vec{\bullet}}$ as a natural transformation. Let us for simplicity assume $k \geq 2$. For $k \in \{0,1\}$ the argument is an easier version of the below. Let f be a k-morphism of $\mathcal{C}$, whose source and target $(k-1)$-morphisms go from $\mathcal{R}$ to $\mathcal{S}$ two fixed $(k-2)$-morphisms of $\mathcal{C}$. The symmetric monoidal $(\infty, k+2)$-category of morphisms is given by:

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{R}, \mathcal{S})_{\vec{\bullet}} := \{\mathcal{R}\} \times_{\mathcal{C}_{(k-2),0,\vec{\bullet}}} \mathcal{C}_{(k-2),1,\vec{\bullet}} \times_{\mathcal{C}_{(k-2),0,\vec{\bullet}}} \{\mathcal{S}\}$$

where we use Notation 10.23. The natural transformation F induces a natural transformation

$$F\colon \mathrm{Hom}_{\mathcal{C}}(\mathcal{R}, \mathcal{S})_{\vec{\bullet}} \Rightarrow \mathrm{Hom}_{\mathcal{D}}(F(\mathcal{R}), F(\mathcal{S}))_{\vec{\bullet}}\colon \mathrm{Fin}_* \times (\Delta^{\mathrm{op}})^{n-k+2} \to \mathrm{Cat}_\infty\ .$$

Upon truncating and applying h_2, i.e. going to the corresponding homotopy bicategory, we in turn get a symmetric monoidal functor between the corresponding homotopy bicategories. The k-morphism f we started with is a 1-morphism in the corresponding homotopy bicategory $h_2\tau\mathrm{Hom}_{\mathcal{C}}(\mathcal{R}, \mathcal{S})$, which by assumption has a left adjoint f^L witnessed by unit and counit maps η and ε. In this setting it is straightforward to verify that the corresponding image $F(f)$ has left adjoint $F(f^L)$ witnessed by the unit and counit maps $F(\eta)$ and $F(\varepsilon)$ in $h_2\tau\mathrm{Hom}_{\mathcal{D}}(F(\mathcal{R}), F(\mathcal{S}))$. The statement for right adjoints is entirely analogous. $\square$

C. Enriched category theory

We first give a brief recollection of how one can construct "free" $\mathcal{V}$-categories from $\mathcal{V}$-graphs in Appendix C.1. This is the first step of the definition of the enriched Tambara tensor product in Section 11.3.1. The second step of the definition corresponds to imposing relations on a $\mathcal{V}$-category, a procedure which is briefly explained in Construction C.3. The complete list of relations imposed for the Tambara relative tensor product is given in Appendix C.2. Throughout this appendix we assume that $\mathcal{V}$ is a cocomplete symmetric monoidal closed category.

C.1. $\mathcal{V}$-categories from $\mathcal{V}$-graphs. Here we first define the notion of a $\mathcal{V}$-graph before explaining how to obtain certain "free" $\mathcal{V}$-categories from such $\mathcal{V}$-graphs. The reference for this is [**Wol74**].

Definition C.1. A *$\mathcal{V}$-graph* Ω consists of a set of objects $\mathrm{Ob}(\Omega)$ and for every pair $A, B \in \mathrm{Ob}(\Omega)$ an object $\Omega(A,B) \in \mathcal{V}$. Given two $\mathcal{V}$-graphs Ω_1 and Ω_2, a *morphism of $\mathcal{V}$-graphs* $\mathcal{F}\colon \Omega_1 \to \Omega_2$ consists of a function $\mathcal{F}\colon \mathrm{Ob}(\Omega_1) \to \mathrm{Ob}(\Omega_2)$ together with maps

$$\mathcal{F}_{A,B}\colon \Omega_1(A,B) \longrightarrow \Omega_2(\mathcal{F}(A), \mathcal{F}(B))$$

in $\mathcal{V}$ for all $A, B \in \mathrm{Ob}(\Omega_1)$. We write $\mathcal{V}\text{-}\mathsf{Gph}$ for the (1-)category of $\mathcal{V}$-graphs and $\mathcal{V}$-graph morphisms

There is a natural functor

$$\mathcal{V}\text{-}\mathsf{Cat} \xrightarrow{U} \mathcal{V}\text{-}\mathsf{Gph}$$

which simply forgets composition and identities. This forgetful functor turns out to have a left adjoint.

Proposition C.2. [**Wol74**, Proposition 2.2] *The forgetful functor U has a left adjoint*

$$\mathrm{Free}\colon \mathcal{V}\text{-}\mathsf{Gph} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$$

sending a $\mathcal{V}$-graph Ω to the free $\mathcal{V}$-category $\mathrm{Free}(\Omega)$ *defined as follows:*

- $\mathrm{Ob}(\mathrm{Free}(\Omega)) = \mathrm{Ob}(\Omega)$
- *For $A, B \in \mathrm{Ob}(\Omega)$ let $\Delta_{A,B}$ be the set of all finite sequences $\{A, X_1, \ldots, X_n, B\}$ with $X_i \in \mathrm{Ob}(\Omega)$ for $1 \le i \le n$. Define:*

$$\mathrm{Free}(\Omega)(A,B) = \begin{cases} \coprod_{\Delta_{A,B}} \Omega(A,X_1) \otimes^{\mathcal{V}} \Omega(X_1,X_2) \otimes^{\mathcal{V}} \cdots \otimes^{\mathcal{V}} \Omega(X_n,B), & \text{if } A \neq B \\ \left(\coprod_{\Delta_{A,B}} \Omega(A,X_1) \otimes^{\mathcal{V}} \Omega(X_1,X_2) \otimes^{\mathcal{V}} \cdots \otimes^{\mathcal{V}} \Omega(X_n,B) \right) \coprod 1_{\mathcal{V}}, & \text{if } A = B \end{cases}$$

We explain how the composition in $\mathrm{Free}(\Omega)$ is defined. First, note that by assumption we have that $V \otimes^{\mathcal{V}} -$ preserves colimits. Thus, for $A \neq B$, $B \neq C$ we have:

$$\begin{aligned} &\mathrm{Free}(\Omega)(A,B) \otimes^{\mathcal{V}} \mathrm{Free}(\Omega)(B,C) \\ &\qquad \cong \coprod_{\Delta_{A,B} \times \Delta_{B,C}} \Big(\Omega(A,X_1) \otimes^{\mathcal{V}} \cdots \otimes^{\mathcal{V}} \Omega(X_n,B) \Big) \otimes^{\mathcal{V}} \Big(\Omega(B,Y_1) \otimes^{\mathcal{V}} \cdots \otimes^{\mathcal{V}} \Omega(Y_m,C) \Big) \,, \end{aligned}$$

and similarly if $A = B$ or $B = C$ (including an extra copy of $1_{\mathcal{V}}$). Denote by $\iota^{\tau}_{A,B}$ the canonical maps into the coproduct for each $\tau \in \Delta_{A,B}$. With this we have that the composition $\circ_{A,B,C}\colon \mathrm{Free}(\Omega)(A,B) \otimes^{\mathcal{V}} \mathrm{Free}(\Omega)(B,C) \to \mathrm{Free}(\Omega)(A,C)$ is defined on components by

$$(\circ_{A,B,C}) \circ (\iota^{\tau}_{A,B} \otimes \iota^{\tau'}_{B,C}) = \iota^{\tau\tau'}_{A,C} \,.$$

Here $\tau\tau'$ denotes the concatenation of the sequences τ and τ' over their common target, respectively source.

Define $j_A\colon 1_{\mathcal{V}} \to \mathrm{Free}(\Omega)(A,A)$ to be $\iota^{1_{\mathcal{V}}}_{A,A}$. In the case that $A = B$, define

$$(\circ_{A,A,B}) \circ (\iota^{1_{\mathcal{V}}}_{A,A} \otimes \iota^{\tau}_{A,B})) = \iota^{\tau}_{A,B} \circ l^{\tau}_{A,B}$$

where $l^{\tau}_{A,B}$ is the left unitor

$$1_{\mathcal{V}} \otimes \big(\mathcal{G}(A,X_1) \otimes \cdots \otimes \mathcal{G}(X_n,B)\big) \xrightarrow{l^{\tau}_{A,B}} \big(\mathcal{G}(A,X_1) \otimes \cdots \otimes \mathcal{G}(X_n,B)\big)$$

in $\mathcal{V}$. The case $B = C$ is analogous.

The following construction explains how one can impose relations on morphisms in the setting of enriched categories.

Construction C.3. Let $\mathcal{C}$ be a $\mathcal{V}$-category and let $f, g\colon 1_{\mathcal{V}} \to \mathcal{C}(x,y)$ be two morphisms that we wish to identify in $\mathcal{C}$. We can do so by defining the following $\mathcal{V}$-category

$$\mathcal{C}/f \sim g = \begin{cases} \mathrm{Ob}(\mathcal{C}/f \sim g) & = \ \mathrm{Ob}(C) \\ (\mathcal{C}/f \sim g)(a,b) & = \ \text{coequalizer in } \mathcal{V} \text{ of the diagram (C.1)} \end{cases}$$

$$\mathcal{C}(y,b) \otimes^{\mathcal{V}} 1_{\mathcal{V}} \otimes^{\mathcal{V}} \mathcal{C}(a,x) \underset{\mathrm{id}_{\mathcal{V}} \otimes^{\mathcal{V}} f \otimes^{\mathcal{V}} \mathrm{id}_{\mathcal{V}}}{\overset{\mathrm{id}_{\mathcal{V}} \otimes^{\mathcal{V}} g \otimes^{\mathcal{V}} \mathrm{id}_{\mathcal{V}}}{\rightrightarrows}} \mathcal{C}(y,b) \otimes^{\mathcal{V}} \mathcal{C}(x,y) \otimes^{\mathcal{V}} \mathcal{C}(a,x) \underset{- \circ - \circ -}{\overset{- \circ - \circ -}{\rightrightarrows}} \mathcal{C}(a,b) \quad . \tag{C.1}$$

C.2. Enriched Tambara tensor products. We here include the complete list of relations imposed in Definition 11.21, i.e. in the definition of the enriched Tambara relative tensor product.

- Isomorphism:

$$\begin{array}{c} 1_{\mathcal{V}} \cong 1_{\mathcal{V}} \otimes^{\mathcal{V}} 1_{\mathcal{V}} \\ \big\downarrow {\scriptstyle (\iota^{m,a \rhd n}_{m \lhd a,n})^{-1} \otimes^{\mathcal{V}} \iota^{m,a \rhd n}_{m \lhd a,n}} \\ \mathrm{Free}(\Omega)((m, a \rhd n),(m \lhd a,n)) \otimes^{\mathcal{V}} \mathrm{Free}(\Omega)((m \lhd a,n),(m,a \rhd n)) \\ \big\downarrow {\scriptstyle - \circ -} \\ \mathrm{Free}(\Omega)((m \lhd a,n),(m \lhd a,n)) \\ \\ \sim \\ \\ 1_{\mathcal{V}} \\ \big\downarrow {\scriptstyle \mathrm{id}_{(m \lhd a,n)}} \\ \mathrm{Free}(\Omega)((m \lhd a,n),(m \lhd a,n)) \end{array} \tag{C.2}$$

Similarly, we impose the relation $\iota^{m,a \rhd n}_{m \lhd a,n} \circ (\iota^{m,a \rhd n}_{m \lhd a,n})^{-1} \sim \mathrm{id}_{(m,a \rhd n)}$.

- Naturality:

$$
\begin{array}{c}
1_{\mathcal{V}} \cong 1_{\mathcal{V}} \otimes^{\mathcal{V}} 1_{\mathcal{V}} \otimes^{\mathcal{V}} 1_{\mathcal{V}} \otimes^{\mathcal{V}} 1_{\mathcal{V}} \\
\downarrow{\scriptstyle f \otimes^{\mathcal{V}} u \otimes^{\mathcal{V}} g \otimes^{\mathcal{V}} \iota^{m, a \triangleright n}_{m \triangleleft a, n}} \\
\mathcal{M}(m, m') \otimes^{\mathcal{V}} \mathcal{A}(a, a') \otimes^{\mathcal{V}} \mathcal{N}(n, n') \otimes^{\mathcal{V}} \mathrm{Free}(\Omega)((m \triangleleft a, n), (m, a \triangleright n)) \\
\downarrow{\scriptstyle \mathrm{id}_{\mathcal{V}} \otimes^{\mathcal{V}} \triangleright_{(a,n),(a',n')} \otimes^{\mathcal{V}} \mathrm{id}_{\mathcal{V}}} \\
\mathrm{Free}(\Omega)((m, a \triangleright n), (m', a' \triangleright n')) \otimes^{\mathcal{V}} \mathrm{Free}(\Omega)((m \triangleleft a, n), (m, a \triangleright n)) \\
\downarrow{\scriptstyle - \circ -} \\
\mathrm{Free}(\Omega)((m \triangleleft a, n), (m', a' \triangleright n'))
\end{array}
$$

$$\sim$$

$$
\begin{array}{c}
1_{\mathcal{V}} \cong 1_{\mathcal{V}} \otimes^{\mathcal{V}} 1_{\mathcal{V}} \otimes^{\mathcal{V}} 1_{\mathcal{V}} \otimes^{\mathcal{V}} 1_{\mathcal{V}} \\
\downarrow{\scriptstyle \iota^{m', a' \triangleright n'}_{m' \triangleleft a', n'} \otimes^{\mathcal{V}} f \otimes^{\mathcal{V}} u \otimes^{\mathcal{V}} g} \\
\mathrm{Free}(\Omega)((m' \triangleleft a', n'), (m', a' \triangleright n')) \otimes^{\mathcal{V}} \mathcal{M}(m, m') \otimes^{\mathcal{V}} \mathcal{A}(a, a') \otimes^{\mathcal{V}} \mathcal{N}(n, n') \\
\downarrow{\scriptstyle \mathrm{id}_{\mathcal{V}} \otimes^{\mathcal{V}} \triangleleft_{(m,a),(m',a')} \otimes^{\mathcal{V}} \mathrm{id}_{\mathcal{V}}} \\
\mathrm{Free}(\Omega)((m' \triangleleft a', n'), (m', a' \triangleright n')) \otimes^{\mathcal{V}} \mathrm{Free}(\Omega)((m \triangleleft a, n), (m' \triangleleft a', n')) \\
\downarrow{\scriptstyle - \circ -} \\
\mathrm{Free}(\Omega)((m \triangleleft a, n), (m', a' \triangleright n'))
\end{array}
$$

- Compatibility with the associator:

$$
\begin{array}{c}
1_{\mathcal{V}} \cong 1_{\mathcal{V}} \otimes^{\mathcal{V}} 1_{\mathcal{V}} \otimes^{\mathcal{V}} 1_{\mathcal{V}} \\
\downarrow{\scriptstyle (\mathrm{id}_m \otimes^{\mathcal{V}} \beta^{\mathcal{N}}_{a,b,n}) \otimes^{\mathcal{V}} \iota^{m, a \triangleright b \triangleright n}_{m \triangleleft a, b \triangleright n} \otimes^{\mathcal{V}} \iota^{m \triangleleft a, b \triangleright n}_{m \triangleleft a \triangleleft b, n}} \\
\mathrm{Free}(\Omega)((m, a \triangleright (b \triangleright n)), (m, (a \otimes b) \triangleright n)) \otimes^{\mathcal{V}} \mathrm{Free}(\Omega)((m \triangleleft a, b \triangleright n), (m, a \triangleright (b \triangleright n))) \\
\otimes^{\mathcal{V}} \mathrm{Free}(\Omega)((m \triangleleft a) \triangleleft b, n), (m \triangleleft a, b \triangleright n)) \\
\downarrow{\scriptstyle - \circ - \circ -} \\
\mathrm{Free}(\Omega)(((m \triangleleft a) \triangleleft b, n), (m, (a \otimes b) \triangleright n))
\end{array}
$$

$$\sim$$

$$
\begin{array}{c}
1_{\mathcal{V}} \cong 1_{\mathcal{V}} \otimes^{\mathcal{V}} 1_{\mathcal{V}} \\
\downarrow{\scriptstyle \iota^{m, a \otimes b \triangleright n}_{m \triangleleft a \otimes b, n} \otimes^{\mathcal{V}} (\beta^{\mathcal{M}}_{m,a,b} \otimes^{\mathcal{V}} \mathrm{id}_n)} \\
\mathrm{Free}(\Omega)((m \triangleleft (a \otimes b), n), (m, (a \otimes b) \triangleright n)) \otimes^{\mathcal{V}} \mathrm{Free}(\Omega)(((m \triangleleft a) \triangleleft b, n), (m \triangleleft (a \otimes b), n)) \\
\downarrow{\scriptstyle - \circ -} \\
\mathrm{Free}(\Omega)(((m \triangleleft a) \triangleleft b, n), (m, (a \otimes b) \triangleright n))
\end{array}
$$

- Compatibility with unitors:

$$\begin{array}{c}
1_{\mathcal{V}} \cong 1_{\mathcal{V}} \otimes^{\mathcal{V}} 1_{\mathcal{V}} \\
\Big\downarrow{\scriptstyle (\mathrm{id}_m \otimes^{\mathcal{V}} \eta^L_n) \otimes^{\mathcal{V}} \iota^{m,1 \triangleright n}_{m \triangleleft 1,n}} \\
\mathrm{Free}(\Omega)((m, 1_{\mathcal{A}} \triangleright n), (m,n)) \otimes^{\mathcal{V}} \mathrm{Free}(\Omega)((m \triangleleft 1_{\mathcal{A}}, n), (m, 1_{\mathcal{A}} \triangleright n)) \\
\Big\downarrow{\scriptstyle - \circ -} \\
\mathrm{Free}(\Omega)((m \triangleleft 1_{\mathcal{A}}, n), (m,n)) \\
\sim \\
1_{\mathcal{V}} \\
\Big\downarrow{\scriptstyle \eta^R_m \otimes^{\mathcal{V}} \mathrm{id}_n} \\
\mathrm{Free}(\Omega)((m \triangleleft 1_{\mathcal{A}}, n), (m,n))
\end{array}$$

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