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L

(Quick) Introduction to stacks I

Motivation:
(1)

$$\begin{tikzcd}
\text{(Pre)Stacks} \arrow[rr] \arrow[d] \arrow[dd, bend right] & & \text{Derived version} \\
\text{Quotient stacks} \arrow[d] & & \\
\text{Application: Moduli problem} & &
\end{tikzcd}$$

(2) <u>Stacks</u> (= "2-sheaves")

For sheaves: Fct of Points : Objects ⟶ Sets
stacks: Gpds

<u>Note</u>: Set ⟼ Gpd discrete groupoid!

<u>Stackification</u>: Presheaves ⟶ sheaves

Prestacks $\xrightarrow[\text{descent data}]{}$ stacks

Stacks ⟶ Ho(Stacks)

⇝ Classification (classifying stacks)
can be modelled w/ stacks.

Homotopy theory of groupoids

Gpd ob = gpds
mor = functors =: Gpd_1 $\overset{?}{/\!/}$ W = equivalences equiv. of cat.

$Ho(Gpd) := W^{-1} Gpd$

$Ho(Gpd) =: [Gpd]$ ob = gpds
mor = iso classes of morphisms

Homotopy theory of I diagrams

Let I be a category, C = Hom(I, Gpds)

Def'n: An <u>equiv. of I-diagrams</u> is a morphism $f: F \to G$ s.t. $\forall$ $i \in I_0$, $f_i: F(i) \longrightarrow G(i)$ is an equiv. of gpds.

Example I :

$$\begin{tikzcd} & 1 \arrow[d] \\ 2 \arrow[r] & 0 \end{tikzcd}$$

$\lim_{I} \underline{Hom}(I, Gpd)$ is fiber product of gpds.

Def'n. $Ho(\underline{Hom}(I, Gpd)) := W^{-1}\underline{Hom}(I, Gpd)$

Rem: $\neq \underline{Hom}(I, Ho(Gpd))$

Ex: I as above $\rightsquigarrow$ homotopy fiber product
$F \in \underline{Hom}(I, Gpd)_0 \Rightarrow \operatorname{Holim}_I F$

Category of prestacks and its homotopy version

Def'n $\mathcal{C}$ Grothendieck site
$Ho(PreSt(\mathcal{C})) := Ho(\underline{Hom}(\mathcal{C}^{op}, Gpd))$

$PreSt(\mathcal{C}) = \underline{Hom}(\mathcal{C}^{op}, Gpd)$

site = category w/ choice of coverings

e.g. schemes w/ étale
Zariski
fppf
⋮

From prestacks to stacks

Def'n: A stack is a prestack with effective descent data.
Let $F \in \underline{Hom}(\mathcal{C}^{op}, Gpd)$, $X \in \mathcal{C}_1$

Given a covering $\{U_i \to X\}$ we have

$$F(X) \to \prod_i F(U_i) \begin{smallmatrix}\xrightarrow{d_0}\\ \xleftarrow{s_0}\\ \xrightarrow{d_1}\end{smallmatrix} \prod_{i,j} F(U_{i,j}) \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \leftarrow\end{smallmatrix} \prod_{i,j,k} F(U_{i,j,k})$$

$$U_{i,j} = U_i \times_X U_j$$

Def'n: A prestack $F \in PrSt(\mathcal{C})$ is a stack if $\forall X \in \mathcal{C}$ and $\forall \{U_i \to X\}$ the natural maps

$$F(X) \to \lim\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{i,j}) \begin{smallmatrix}\leftarrow\\ \rightarrow\\ \leftarrow\end{smallmatrix} \prod_{i,j,k} F(U_{i,j,k})\right)$$

is an iso in gpds (= "effective" descent data)

Prop: A prestack F is a stack if
(1) $\forall X \in \mathcal{C}$, $(a,b) \in F(X)$ the presheaf

$$\underline{Iso}(a,b) : (\mathcal{C}/X)^{op} \longrightarrow Set$$

$$(u : Y \to X) \longmapsto Hom_{F(Y)}(u^*(a), u^*(b))$$

is a sheaf

and (2) %

(2) $\forall X \in \mathcal{C}$, $\forall \{\mathcal{U}_i \longrightarrow X\}$ covering, $\forall a_i \in F(\mathcal{U}_i)$
$\forall$ isom $\in F(\mathcal{U}_{ij})$
families of

$$\Phi_{i,j} : (a_i)|_{\mathcal{U}_{ij}} \simeq (a_j)|_{\mathcal{U}_{ij}}$$

satisfying
$$\left.\begin{aligned} &(\Phi_{jk})|_{\mathcal{U}_{i,j,k}} \circ (\Phi_{i,j})|_{\mathcal{U}_{i,jk}} = (\Phi_{ik})|_{\mathcal{U}_{ij,k}} \\ &\Phi_{i,i}|_{\mathcal{U}_i} = \mathrm{id} \end{aligned}\right\} (*)$$

$\Rightarrow \exists\, a \in F(X)$ and isomorphisms $\alpha_i : a|_{\mathcal{U}_i} \xrightarrow{\sim} a_i$
s.t. $\Phi_{i,j} = (\alpha_j)|_{\mathcal{U}_{i,j}} \circ (\alpha_i)|_{\mathcal{U}_{ij}}^{-1}$

Remark: $\{a_i \in F(\mathcal{U}_i), \Phi_{ij}\}$ as above is called descent data
(*) = "the descent data is effective"

Extra Reference: "Stacks for everybody" Fantecchi

Examples Stacks:

① Vector bundles
② Sheaves
③ (qc) Coherent sheaves
④ Algebraic spaces, schemes

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① Vect: $\mathcal{C} = \mathrm{Aff}_{\mathbb{Z}}$ we fix a topology (étale) Zariski

$\mathrm{Aff}_{\mathbb{Z}}^{op} \longrightarrow$ Gpd

$\mathrm{Vect}(A) =$ Gpd of locally free (projective) A-mod of finite rank

ob = proj mod. of finite rk mor = isomorphisms

$$\mathrm{Vect}(A) = \coprod_n \mathrm{Vect}_n(A)$$

open s/stacks

② $Sh(A)$ = Gpd of sheaves on Spec A (arrows = isos)

③ $QCoh(A)$ = Gpd of A-modules

$Coh(A)$ = — " — finitely presented A-mod.

<u>Observation:</u> All these examples have effective descent.

① cocycle condition (proj. mod)

② glueing of sheaves

Pf using

<u>Prop:</u> A presheaf F regarded as a prestack is a stack iff F is a sheaf.

~~<u>Idea</u>: Stackification:~~

~~Def~~ ~~(local equivalence)~~

~~Recall that there exists a~~ ~~fully faithful~~ ~~functor~~

$$\Pi_0 : \begin{cases} PrSt(\mathcal{C}) \longrightarrow PrSh(\mathcal{C}) \\ (X \longmapsto F(X)) \longmapsto (X \longmapsto F(X)_0) \end{cases}$$

~~<u>Def'n</u> A morphism $F \to G$ of prestacks is a local equiv.~~
~~if it satisfies~~

~~(1) $\pi_0(F) \longrightarrow \pi_0(G)$ is an isom. of presheaves~~

~~(2) If we define the presheaf~~ $s \in$

$$\pi_1^{pr}(F,s) : (\mathcal{C}/_X)^{op} \longrightarrow Gpd$$
$$(u : Y \to X) \longmapsto Aut_{F(Y)}(u^*(s))$$

<u>Prop 1.1 in Toën.</u>

Def^n Let F be a prestack. The associated stack is given by a stack a(F) and a local equivalence

$$F \longrightarrow a(F)$$

Outlook: additional examples

Ex: ① PreVect free instead of loc. free

a(PreVect) = Vect

② $\boxed{\text{Quotient stacks}}$ scheme + action → $[X//G]$

Stacks II - Jon Skowera

	(Big) Sites:	objects	topology $\{U_i \xrightarrow{f_i} X\}$
	(Aff/S)	$X = \operatorname{Spec} A \longrightarrow S$	$\{U_i \xrightarrow{f_i} X\}$ f_i étale
	$(\mathrm{Sch}/S)_{fpqc}$	$X \longrightarrow S$	f_i flat & $\forall \operatorname{Spec} A \subset X$ $\exists$ fin. many $\operatorname{Spec} B_i \subset \mathcal{U}_i$ s.t. $\bigcup_{i_k} f_{i_k}(\operatorname{Spec} B_{i_k}) = \operatorname{Spec}$
	$(\mathrm{Sch}/S)_{fppf}$	$X \longrightarrow S$	f_i flat & locally of finite presentation
strong ↑	$(\mathrm{Sch}/S)_{sm}$	"	f_i smooth
	$(\mathrm{Sch}/S)_{et}$		étale
weak	$(\mathrm{Sch}/S)_{Zar}$		open immersions

Def'n $F : (\mathrm{Aff}/S)^{op} \longrightarrow \mathrm{Set}$ is an <u>algebr. space</u>/S sch

if

(1) F is a sheaf

(2) $\Delta : F \longrightarrow F \times_S F$ is schematic and quasi-cpct

(3) $\exists$ scheme $\mathcal{U}$, $h_{\mathcal{U}} \xrightarrow{\pi} F$ which is étale & surjective

where $h : (\mathrm{Sch}/S) \longrightarrow \mathrm{Hom}((\mathrm{Sch}/S)^{op}, \mathrm{Set})$

$\mathcal{U} \longmapsto h_{\mathcal{U}} : \left(\begin{matrix} T \\ \downarrow \\ S \end{matrix} \longmapsto \mathrm{Hom}_S(T, \mathcal{U}) \right)$

is the (fully faithful) Yoneda embedding

Rem: A natural transformation $f : F \to G$ of sheaves on (Aff/S) is <u>schematic</u> (= <u>representable</u>) if for all $\operatorname{Spec} A \to S$, the pullback

$$\begin{tikzcd} F \times_G h_{\operatorname{Spec} A} \arrow[r] \arrow[d] & h_{\operatorname{Spec} A} \arrow[d] \\ F \arrow[r] & G \end{tikzcd}$$

has $F \times_G h_{\operatorname{Spec} A} \simeq h_{\mathcal{U}}$ for some scheme $\mathcal{U}$ (alg space) stable under pullback

A schem./repr. morphism is étale, smooth, open immersion, --- if every pullback of the morphism to a scheme / alg sp. morphism is resp. étale, smooth, ---

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<u>2-Yoneda lemma</u>: Let $\mathcal{X}$ be a stack on $\mathcal{C}$, and $T \in \mathcal{C}$.
There is an equivalence of groupoids

$$\mathrm{Hom}(T, \mathcal{X}) \simeq \mathcal{X}(T)$$

<u>Def'n</u> A stack $\mathcal{X}$ on (Aff/S) is <u>algebraic</u> if

(1) The diagonal

$$\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X} \qquad (\rightsquigarrow \mathrm{Auts})$$

is representable, quasi-cpct & separated

(2) There exists a scheme $\mathcal{U}$ and morphism

$$\mathcal{U} \longrightarrow \mathcal{X}$$

which is smooth & surjective.

<u>2-Fiber products</u> Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be stacks on a site $\mathcal{C}$ with morphisms

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{X} \arrow[r, "F"] & \mathcal{Z} \end{tikzcd}$$

then the 2-fiber product is the category

$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}\ (T)$: <u>objects</u>: $(x, y) \in (\mathcal{X}(T), \mathcal{Y}(T)\}$, $\varphi : F(x) \xrightarrow{\sim} G(y)$
$\in \mathrm{Hom}_{\mathcal{Z}(T)}(F(x), G(y)$

<u>morphisms</u>: $(\alpha, \alpha') : (x, y, \varphi) \longrightarrow (x', y', \varphi')$ s.t.

$$\begin{tikzcd} F(x) \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & G(y) \arrow[d, "\alpha'"] \\ F(x') \arrow[r] & G(y') \end{tikzcd} \qquad \text{commutes}$$

<u>Rem</u>:

$$\begin{tikzcd} \mathrm{Aut}_S(pt) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ S \arrow[r] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

$\boxed{\text{Stacks}}$: ① $F : (\mathrm{Aff}/S) \longrightarrow (\mathrm{Gpd})$

lax sheaf s.t. ......

e.g: $(T \to S) \longmapsto$ curves over T

② Category fibered in gpds:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi\ \text{functor}"] \\ (\mathrm{Aff}/S) \end{tikzcd}$$

s.t.

1)

relative fiber products exist in $\mathcal{C}$

2)

$$\begin{tikzcd} & & & u \arrow[dr, "h"] & \\ & s \arrow[urr, dashed, "\exists! \tilde{g}"] \arrow[rrr, "f"] \arrow[dd] & & & t \arrow[dd] \\ \pi & & & U \arrow[dr, "h"] \arrow[from=uu, crossing over] & \\ & S \arrow[urr, "g"] \arrow[rrr, "f"] & & & T \end{tikzcd}$$

s.t. $\pi(\tilde{g}) = g$

② ⇒ ① ⇒ $\mathcal{C}(T) := \begin{cases} \text{obj} : c \in \mathcal{C} \quad \pi(c) = T \\ \text{mor} : \mathrm{Hom}_{\mathcal{C}(T)}(c, c') = \{ f \in \mathrm{Hom}_{\mathcal{C}}(c, c') \mid \pi(f) = id_T \end{cases}$

For ② e.g. $\begin{pmatrix} C & C\ \text{curve} \\ \downarrow & \\ T & \\ \downarrow & \\ S & \end{pmatrix} \xrightarrow{\pi} \begin{pmatrix} T \\ \downarrow \\ S \end{pmatrix}$

Quotient Stacks:

Let S be a scheme and X be a noetherian S-scheme. Let G be a smooth, affine group S-scheme acting on X. Then

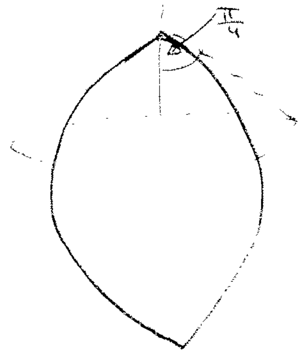
$$[X/G](T) := \begin{cases} \text{Obj} = \left(\begin{tikzcd} P \arrow[r, "G\text{-equiv.}"] \arrow[d, "\substack{\text{princ.}\\ G\text{-bundle}}"'] & X \\ T & \end{tikzcd} \right) \\ \text{Mor} = \begin{tikzcd} & P' \arrow[dr, bend left] \arrow[dd] & \\ P \arrow[ur] \arrow[rr, bend left=10] \arrow[d] & & X \\ T \arrow[r] & T' & \end{tikzcd} \end{cases}$$

pullback square

is an algebraic stack.

Example: $\mathbb{Z}/2 \circlearrowright \mathbb{CP}^1$ by rotation

$$[\mathbb{CP}^1/\mathbb{Z}/2] =$$



is an orbifold.

Proof. cat. fibered in gpds:

1) pullbacks of principal G-bundles are principal G-bundles

2) Exercise: use universal prop. of fiber products

Isom is a sheaf: Let $U \xrightarrow[u]{} [X/G]$, $U' \xrightarrow[u']{} [X/G]$

Then $\underline{\text{Isom}}_{U \times_S U'}(p_1^* u, p_2^* u') = \text{Hom}_S(T, U \times U' \times G \times_{X \times X} X)$

where
$$\begin{tikzcd} U \times_S U' \arrow[r, "p_2"] \arrow[d, "p_1"] & U' \\ U & \end{tikzcd}$$

(use trivializing covers of $\begin{smallmatrix} P \\ \downarrow \\ U \end{smallmatrix}$, $\begin{smallmatrix} P' \\ \downarrow \\ U' \end{smallmatrix}$)

So it is a schematic $\overset{\text{stack}}{\text{sheaf}}$ represented by a quasi-affine $\Big\{$ scheme over $U \times_S U'$

for algebr. stack

quasi-aff → qu.-cpt + sep

Descent data is effective:

- $P \to T$ is affine
- if X is affine, $P \to X$ is affine

fppf Descent of affine morphisms is effective

$\Rightarrow$ étale.

<u>Atlas</u>: $\pi: X \longrightarrow [X/G]$ is given by (⇝ cond (2) for alg. stack)

$$\begin{tikzcd} X \times G \arrow[r, "\text{action}"] \arrow[d, "p_1"] & X \\ X & \end{tikzcd} \in [X/G](X)$$

for X affine

fiber product:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\text{``}/G\text{''} \ \text{sm}"'] & X \arrow[d, "\pi"] \\ T \arrow[r, "f"] & {[X/G]} \end{tikzcd}$$

f is the morphism given by (using Yoneda 2-lemma)

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \\ T & \end{tikzcd}$$

A <u>moduli problem</u> roughly consists of a choice of

(1) geometric objects

(2) families of objects - how they vary

(3) Morphisms of families: which objects are equiv.

A <u>moduli space</u> is a solution to a moduli problem. It must satisfy

1) points correspond to equiv. classes of objects
2) paths through the space correspond to families

The moduli space will often be a stack:

Define the mod. sp to be

eg
$$\begin{tikzcd} \mathcal{M} \arrow[d] \\ Aff/S \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}(T) \arrow[d, mapsto] \\ T \end{tikzcd}$$
$\mathcal{M}(T) =$ families over T (2) and their Boms (3)

<u>Trade-off</u> for this easy def'n is complexity of proving properties of $\mathcal{M}$.

<u>Example</u>: Moduli space of marked triangles

(1) top. space w/ metric isometric to triangle + labelling of the vertices

(2) fam: $$\begin{tikzcd} X \arrow[r] & S \arrow[l, bend left, "\alpha"] \arrow[l, bend left=60, "\beta"] \arrow[l, bend left=90, "\gamma"] \end{tikzcd}$$ continuous with continuously varying metric on X s.t. im α, im β, im γ are always vertices

(3) <u>morphisms</u>. $$\begin{tikzcd} X' \arrow[r] \arrow[d, "((("'] & X \arrow[d, ")))"] \\ S' \arrow[r] & S \end{tikzcd}$$ pull back comm. w/ sections

Stacks II

$$\mathcal{M}'_\Delta := \left\{(a,b,c) \;\middle|\; \begin{array}{l} a+b \geq c \\ b+c \geq a \\ a+c \geq b \end{array} \right\} \subset \mathbb{R}^3$$

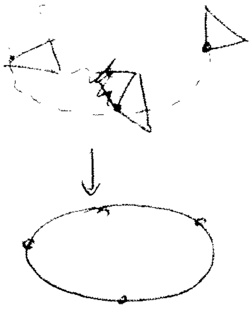
mod sp.
is a __top. space__

open cone w/ vert.
(0,1,1),(1,0,1),(1,1,0)

Problem for unmarked triangles

Try $\mathcal{M} := \mathcal{M}'/S_3$ ↖ acts by relabelling α, β, γ

__BUT:__
$$\begin{tikzcd} X \arrow[d] \\ S^1 \end{tikzcd}$$



$\frac{2\pi}{3}$ twist of equilat. Δ

should come from a path

$$\begin{tikzcd} S^1 \arrow[r] & \mathcal{M} \\ x \arrow[r, mapsto] & \text{modulus of equilat } \Delta \text{ (a pt.)} \end{tikzcd}$$

But the pullback of a universal family

$$\begin{tikzcd} S^1 \times \Delta \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ S^1 \arrow[r] & \mathcal{M} \end{tikzcd}$$

would have to then be trivial ↯

__SOL'N:__ $\mathcal{M} :=$ __stack__ of triangles $\begin{smallmatrix} X \\ \downarrow \\ S \end{smallmatrix}$ over

$$\mathcal{M} \cong [\mathcal{M}'/S_3]$$

$$\begin{tikzcd} X \arrow[d] & & \tilde{S}_X \arrow[r] \arrow[d] & \mathcal{M}_1 \\ S \arrow[r, mapsto, shorten=8mm] & & S & \end{tikzcd}$$

$\tilde{S}_x = \{(s \in S, \text{ ordering of sides of } X_s)\}$

Rem: site is
(Top) w/ open inclusions

1) Vector bundles on a projective scheme X over an alg closed field k w/ fixed Chern classes c_i and rank r form an algebraic stack

$$V_{r,c_i}(T) = \left(\begin{tikzcd} E \arrow[d] \\ T \times X \end{tikzcd} \text{ v.b. st } \quad , \quad \begin{tikzcd} E \arrow[r] \arrow[d] & F \arrow[d] \\ S \times X \arrow[r] & T \times X \end{tikzcd} \text{ pullback} \right)$$

(Pf uses descent for qu-coh. $\mathcal{O}_X$-modules)

2) Moduli curves. genus $g \geq 0$ curves (1-dim'l, smooth, proper reduced, irred. schemes/k field)

families: $X \to S$ smooth, proper
$\forall$ geom. pts s, X_s is a curve

morphism:
$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \quad \text{pullback}$$

is an algebraic stack.

Let $\mathfrak{X}$ be a stack on (Aff/k).
The <u>tangent space</u> at $x \in \mathfrak{X}$ forms a <u>groupoid</u>

$$\begin{tikzcd} T_{\mathfrak{X},x} \arrow[r] \arrow[d] & \mathrm{Hom}(D, \mathfrak{X}) \arrow[d] \\ \{*\} \arrow[r, "x"] & \mathrm{Hom}(\mathrm{Spec}\, k, \mathfrak{X}) \end{tikzcd}$$

where $D = \mathrm{Spec}\ k[\varepsilon]/\varepsilon^2$.

It has an <u>addition</u> given by

$$\begin{tikzcd} \mathrm{Spec}\ k \arrow[rr] \arrow[dd] & & D \arrow[dd, "\text{univ}"] \arrow[rddd, bend left, "v"] & \\ & D \arrow[rd, "+"'] & & \\ D \arrow[rr] \arrow[rrrd, bend right, "u"'] & & D \sqcup_{\mathrm{Spec}} D \arrow[rd, dashed, "\exists!"] & \\ & & & \mathfrak{X} \end{tikzcd}$$

where $+$ comes from the ring morphism

$$\mathbb{C}[\varepsilon_1, \varepsilon_2]/_{(\varepsilon_1, \varepsilon_2)^2} \longrightarrow \mathbb{C}[\varepsilon]/_{\varepsilon^2}$$
$$\varepsilon_i \longmapsto \varepsilon$$

and scalar multiplication

$$\begin{tikzcd} D \arrow[r, "\lambda"] \arrow[rr, bend right, "\lambda v"'] & D \arrow[r, "v"] & \mathcal{X} \end{tikzcd}$$

coming from $\mathbb{C}[\varepsilon]/\varepsilon^2 \longrightarrow \mathbb{C}[\varepsilon]/\varepsilon^2$

$\varepsilon \longmapsto \lambda\varepsilon$

This corresponds to a 2-term complex of vector spaces

$$C^{-1} \xrightarrow{d} C^0$$

where $C^0 =$ ~~objects~~ tangents at x $(D \longrightarrow \mathcal{X})$

$C^{-1} =$ morphisms

$$\begin{tikzcd} D \arrow[rr, "\varphi"] \arrow[dr, "u"'] & & D \arrow[dl, "v"] \\ & \mathcal{X} & \end{tikzcd}$$

$$d : (\varphi : u \to v) \longmapsto v - u$$

$H^0 = C^0/\operatorname{im} d =$ isom classes of tangents

$H^1 = \ker d = \operatorname{Aut}(0 : D \longrightarrow \mathcal{X}) \underset{(*)}{=} \operatorname{Lie}(\operatorname{Aut}(x))$

To recover the groupoid of tangents, take the stack quotient

$$[C^0/C^{-1}]$$

DEFORMATION THEORY

What is a deformation problem?

Running

Example ① Deformation of an associative algebra structure (alg. ex)

Let V be a finite dim'l vsp. / k a field of dim n
$\{e_i\}$ basis (or free k-module of rk n)

An associative algebra structure ^{A} on V is determined by

$$e_i \cdot e_j = \sum_k \underset{\substack{\cap \\ k}}{a_{ij}^k} e_k \quad \longleftrightarrow \quad \mu : V \otimes V \longrightarrow V$$

a_{ij}^k structure constants

associativity: $\mu(e_i, \mu(e_j, e_k)) = \mu(\mu(e_i, e_j), e_k)$

$$\Longleftrightarrow \quad \sum_\ell a_{i\ell}^r a_{jk}^\ell = \sum_\ell a_{ij}^\ell a_{\ell k}^r \qquad (1)$$

for $i, j, k, r = 1, \dots, n$

Deformation of the structure (a_{ij}^k) w/ parameters $(x_1, \dots, x_s) \in k^s$ is ...

- replace $a_{ij}^k \rightsquigarrow a_{ij}^k(x_1, \dots, x_s)$ fcts in $(x_1, \dots, x_s)$

  s.t. 1. $\forall (x_1, \dots, x_s)$, $a_{ij}^k(x_1, \dots, x_s)$ satisfy (1)

  2. $a_{ij}^k(0, \dots, 0) = a_{ij}^k$

In other words, w/ R = ring of fcts on k^s,
$R \longrightarrow k$ evaluation at 0,

- ... an associative algebra structure $\mathcal{A}$ on the free R-module $R \otimes_k V$

  s.t. $\mathcal{A} \otimes_R k \simeq A$

② Deformation of a vector bundle structure (geom. ex)

Let X be a manifold, V a (locally trivial) vector bundle on X of rank r, $p: V \to X$

This is determined by the following data:

Choose a trivialization of the v.b., i.e. an open cover $\{U_i\}$ of X s.t. $p|_{U_i}: V|_{U_i} \longrightarrow U_i$ is trivial.

$$V|_{U_i} \overset{\shortparallel}{\cong} U_i \times V_i$$

Choose basis $(e_i)_\alpha$ of V_i

$$\Rightarrow \quad \phi_i^j : \underset{\underset{U_i \cap U_j}{\shortparallel}}{U_{ij}} \longrightarrow GL_r(k) \quad \text{transition functions}$$

satisfying cocycle condition

$$\forall i, j, k: \quad \phi_i^k = \phi_i^j \phi_j^k \qquad (2)$$

Deformation of the vector bundle $p: V \to X$ with parameters in the space (B, b), $b \in B$ is

- a vector bundle W of rank r on $B \times X$
- an isomorphism btw V and the restriction of W to $\{b\} \times X$

$$V \cong W|_{\{b\} \times X}$$

Why?

Assuming that W can be trivialized along $B \times U_i$ (e.g. if B, U_i are contractible) can rephrase as deformation of the (ϕ_i^j)

Deformation of the structure cocycles (ϕ_i^j) with parameters (B, b) is

$$\left[\begin{array}{l} \text{functions } \Phi_i^j : B \times U_{ij} \longrightarrow GL_r(k) \\ \text{s.t. } 1.\ \Phi_i^k = \Phi_i^j \Phi_j^k \quad \text{on } B \times U_{ijk} \\ \qquad 2.\ \Phi_i^j(b, x) \equiv \phi_i^j(x) \end{array}\right.$$

%

If A = ring of fcts on B, $A \to k$ ev. at $b \in B$
and A of finite dimension (rank over k?)

$$\Phi_i^j : B \times U_{ij} \longrightarrow Gl_n(k) \quad \longleftrightarrow \quad \Phi_i^j : U_{ij} \longrightarrow GL_n(A)$$

Axiomatization

Axiom	alg	geom
	Ex	Ex

(B, b) pointed space "parameters"
X some structure
A deformation of X w/ parameters in (B, b) is

1, $\forall p \in B$, a structure $X(p)$ of same type depending on the parameter $p \in B$
"family of structures parametrized by b"

2, "identification" between X and $X(b)$.

algebraic structures: i.e. underlying v.sp / ab gp / module over $\Bbbk$

A deformation of an algebraic structure A over $\Bbbk$ with parameters in (B, b) is

1, an alg. str. $\mathcal{A}$ of the same kind over $\mathbf{R} = \mathcal{O}(B)$

2, an isom. of the structure between A and $\mathcal{A} \otimes_{\mathbf{R}} \Bbbk$,
where $\mathbf{R} \longrightarrow \Bbbk$ is evaluation at $b \in B$

= family of algebr. structures, def. at $p \in B$ is $\mathcal{A} \otimes_{\mathbf{R}} \Bbbk$, where $\mathbf{R} \xrightarrow[\text{at } b]{} \Bbbk$ is ev.

geometric structures i.e. underlying topological space

A deformation of an geometric structure/object V with parameters in (B, b) is

1, a geom. object W of the same kind with a projection over B

2, an isomorphism between the fiber $W(p)$ and V

= family of geom. objects over base B,
deformation at $p \in B$ is $W(p)$ = fibe of proj at $p \in B$

EXAMPLES:

algebraic	a) ①	associative algebras	
	b)	Lie algebras	Lie alg coh
	c)	module structures over a given ring	Ext gps
	d)	group representations	gp coho
	e)	Passon algebras	Poisson coho
geometric	α) ②	vector bundles	
	β)	point in a variety	tangent cplx
	γ)	manifold structure	coh of tang bundle
	δ)	principal G-bundle	coh. of adjt bundle
	ε)	local system	coh of π_1 in $\mathrm{Mat}_n(K)$

Formal deformations $\left\{ \begin{array}{l} R = k[[t]] \longrightarrow k \\ \quad\quad t \longmapsto 0 \end{array} \right.$

Infinitesimal deformations $\left\{ \begin{array}{l} R = k[\varepsilon]/\varepsilon^2 \longrightarrow k \\ \quad\quad \varepsilon \longmapsto 0 \end{array} \right.$

n(th order) deformation $\left\{ \begin{array}{l} R = k[\varepsilon]/\varepsilon^{n+1} \longrightarrow k \\ \quad\quad \varepsilon \longmapsto 0 \end{array} \right.$

Def'n Two deformations are ~~isomorphic~~ equivalent if there is a structure preserving isomorphism between them compatible with the restriction isos to fibers

Hochschild complex

Let A be an associative algebra, M an A-module

Consider the Hochshild complex

$$0 \to M \xrightarrow{\delta_{Hoch}} C^1(A,M) \xrightarrow{\delta_H} C^2(A,M) \to \cdots,$$

where $C^n(A,M) := \mathrm{Lin}(A^{\otimes n}, M)$ linear maps

$$(\delta_H f)(a_0 \otimes \cdots \otimes a_n) = a_0 f(a_1, \ldots, a_n)$$
$$+ \sum_{i=1}^{n} (-1)^i f(a_0, \ldots, a_{i-1}a_i, \ldots, a_n)$$
$$+ (-1)^{n+1} f(a_0, \ldots, a_{n-1})\, a_n$$

Check: $\delta_H \circ \delta_H = 0$.

Hochshild Cohomology denoted by $H^\bullet(A,M)$ ~~H~~

$M = A$ $\qquad HH^\bullet(A) := H^\bullet(A,A)$

$HC^\bullet(A)$ Hochshild complex

Interpretation of n-cocycles for small n:

n = 0: $H^0(A,M) = \{ m \in M : \forall a \in A,$ ~~am=ma~~ $am - ma = 0 \} = M^A$

$HH^0(A) = Z(A)$ center of A

n = 1: $Z^1(A,M) = \{ \ell : A \to M$ linear s.t. $\ell(ab) = a\ell(b) + \ell(a)b \ \ \forall a,b \in A \}$

$= \mathrm{Der}(A,M)$

$B^1(A,M) = \{ \ell_m : A \to M : \ell_m(a) = ma - am, \ m \in M \}$

= inner derivations of A w/ values in M

n=2: **Recall** infinitesimal deformations:

$$R = k[\varepsilon]/\varepsilon^2 \longrightarrow k$$

algebra structure on $R \underset{k}{\otimes} A \cong A[\varepsilon]/(\varepsilon^2)$ s.t. $A[\varepsilon]/\varepsilon^2 \underset{A}{\otimes} k \cong A$

ε-linear product $*$ on $A[\varepsilon]/\varepsilon^2$ s.t.

$$a * b = ab \mod \varepsilon$$

$$\iff a * b = ab + \mu(a,b)\varepsilon,$$

$$\mu : A \otimes A \longrightarrow A$$

$*$ associative $\iff a\mu(b,c) + \mu(a,bc) = \mu(a,b)c + \mu(ab,c)$

$\iff \mu \in \underset{HZ^2(A)}{Z^2(A,A)}$ is 2-cocycle.

$$\iff (\delta_H \mu)(a,b,c) = a\mu(b,c) - \mu(ab,c) + \mu(a,bc) - \mu(a,b)c = 0$$

Recall: 2 def equiv. $*, *'$ iff

$\exists$ iso of $k[\varepsilon]/\varepsilon^2$-algebras

$$\varphi : (A[\varepsilon]/\varepsilon^2, *) \longrightarrow (A[\varepsilon]/\varepsilon^2, *')$$

which is identity mod ε.

$\iff \exists\, \ell : A \longrightarrow A$ linear s.t.

$$a \longmapsto a + \ell(a)\varepsilon$$

φ morphism of alg $\iff \mu(a,b) + \underline{\ell(ab)} = \mu'(a,b) + \underline{a\ell(b) + \ell(a)b}$

$$\iff \mu - \mu' = \delta_H(\ell)$$

So, $HH^2(A)$ = infinitesimal def / equivalence.

Similarly, can show

Prop: The obstruction of extending deformations (from order n to order n+1) lies in $HH^3(A)$.

Associative algebra structures as a stack:

Let V be an n-dim'l v.sp. $/k$

Recall: algebra structure determined by a_{ij}^{k} $i,j,k=1,\dots,n$ s.t.

$$\sum_{\ell} a_{i\ell}^{r}\, a_{jk}^{\ell} = \sum_{\ell} a_{ij}^{\ell}\, a_{\ell k}^{r} \qquad i,j,k,r=1,\dots,n$$

$\longrightarrow$ defines affine algebraic variety $\mathrm{Ass}(k^n)$

$G L_n = GL(k^n) \circlearrowright \mathrm{Ass}(k^n)$ change of basis

Situation: $G \circlearrowright X$

G: smooth affine gp scheme; X: noetherian scheme

$\longrightarrow$ quotient stack $[X/G]$

Can compute tangent complex to this stack:

~~Prop: $\mathbb{T}_{[X/G],x}$ for $x: Y=\mathrm{Spec}(R)\longrightarrow X$ is quasi-iso to the complex of R-modules~~

$$\begin{tikzcd} T_{G\times X\times_X Y,x} \arrow[r, "b^{\circ}"] & T_{X,x} \end{tikzcd}$$

~~where b° is the differential of the target map $G\times X \longrightarrow X$ restricted to $G\times X\times_X Y$~~

$T_{G\times X\times_X Y,x} = T_{G\times Y,x}$

$x: \mathrm{Spec}\,R \longrightarrow X=\mathrm{Ass}(V) \longleftrightarrow R$-algebra structure on V

groupoid $G \times X \overset{s}{\underset{t}{\rightrightarrows}} X$

$$\begin{tikzcd} \cdots & G_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \boxed{G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0} \end{tikzcd}$$

$G_2 = G_1 \times_{G_0} G_1$

$$\begin{tikzcd} T_x G_{0/1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G_{0/1}(k[\varepsilon]/\varepsilon^2) \arrow[d] \\ 1 \arrow[r, "x"] & G_{0/1}(k) \end{tikzcd}$$

$\rightsquigarrow$ pull back of sets

Jon $\rightsquigarrow$

$$\begin{tikzcd} \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & T_x G_1 \arrow[r, shift left] \arrow[r, shift right] & T_x G_0 \end{tikzcd}$$

simplicial abelian group

$\wr$ Dold-Kan

chain complex coming from groupoid $\rightsquigarrow$ $C_0 \xrightarrow{d} C_1$

Explicitly, $T_{(e,x)} G \times X \overset{s^*}{\underset{t^*}{\rightrightarrows}} T_x X$

$$\left\{\begin{array}{lcl} \mathfrak{g} \times T_x X & \rightrightarrows & T_x X \\ (g,v) & \mapsto & v + \chi(g) \\ & \mapsto & v \end{array}\right.$$

(with $T_{(e,x)} G \times X = \mathfrak{g} \times T_x X$)

determined by (this is chain cplx coming from D-K)

$$\begin{array}{ccl} \mathfrak{g} & \longrightarrow & T_x X \\ g & \longmapsto & \chi(g) = (s^* - t^*)(g, v) \end{array}$$

$X = \mathrm{Ass}(k^n)$, $G = GL_n(k)$ $\qquad x : \mathrm{Spec}(k) \longrightarrow \mathrm{Ass}(k^n)$

$\longleftrightarrow$ k-algebra str. A on k^n

$$T_{X,x} = \text{infinitesimal } (= \text{first order}) \text{ deformations of } A$$
$$= HZ^2(A)$$

$$\mathfrak{g} = \mathfrak{gl}_n = \mathrm{Hom}_k(A, A) = HC^1(A)$$

<u>Computation of the differential:</u>

$h \in GL_n \circlearrowright X$ induces $h \in GL_n \circlearrowright HC^2(A) \ni (M : A \otimes A \to A)$

$$\downarrow$$

$$h\,M(h^{-1}(-), h^{-1}(\ ))$$

Differentiating the action gives

$\alpha \in \mathfrak{gl}_n = HC^1(A)$ acts by

$$M \longmapsto \alpha M(-,-) - M(\alpha(-), -) - M(-, \alpha(-))$$

which <u>is</u> Hochschild differential.

So, $$\left(HC^1(A) \xrightarrow{\delta_H} HZ^2(A)\right) = T_{[\mathrm{Ass}\,k^n / GL_n], A}$$

Stacks ++
7.10.2013

Yoneda 2-Lemma:

$\mathrm{Set} \subset \mathrm{Gpd}$

$\mathrm{Aff}^{op} \longrightarrow \mathrm{Set} \longrightarrow \mathrm{Gpd}$

$$\begin{tikzcd}
\mathrm{Aff} \arrow[r, hook, "\mathrm{Yoneda}", "f.f."'] \arrow[dr, hook, bend right, "f.f"'] & \mathrm{Pr}(\mathrm{Aff}) \arrow[r, hook, "f.f"'] & \mathrm{Pr}(\mathrm{Aff}, \mathrm{Gpd}) \\
& \mathrm{Sh}(\mathrm{Aff}) \arrow[u, phantom, "\cup"] \arrow[r, hook, "f.f."'] & \mathrm{St}(\mathrm{Aff}, \mathrm{Gpd}) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

X stack (can think as fibered cat.)

scheme $S \rightsquigarrow (\overset{\mathrm{Aff}}{\mathrm{Sch}}/S)(T) = \{ T \longrightarrow S \}$

$\mathrm{Hom}(\underset{\mathrm{Aff}}{\mathrm{Sch}}/S, X) = X(S)$ What are morphisms of this gpd?

~~Hom~~

$$\begin{tikzcd}
\mathcal{C} = \mathrm{Aff}/X \arrow[r] \arrow[d] & \mathcal{D} = \mathrm{Aff}/Y \arrow[dl, bend left] \\
\mathrm{Aff} &
\end{tikzcd}$$

morph of stacks
= natural
transf

$$\begin{tikzcd}
\mathcal{C} \arrow[d] & \\
\mathrm{Aff} \arrow[r, "F"'] & (\mathrm{Gpd})^{op}
\end{tikzcd}$$

$\mathrm{FibCat}/_{\mathrm{Aff}} \simeq \mathrm{Pr}(\mathrm{Aff}, \mathrm{Gpd})$
as bicat

Stacks++
7.10.2013
$\boxed{2}$

tangent complex

Dold-Kan equivalence

simplicial abelian groups sAb

chain cplxes (of ab. gps) $C_{\geq 0}(\mathbb{Z})$

$$\cdots \to E_2 \to \overset{4}{E_1} \to E_0$$

$$\mathrm{sAb} \longrightarrow C_{\geq 0}(\mathbb{Z})$$

$$(X_n) \longmapsto (\cdots \to X_2 \xrightarrow{\Sigma(-1)^i d_i} X_1 \xrightarrow{d_1 - d_0} X_0) = AX$$

Too big b/c of degenerate faces!

$$(X_n) \longmapsto NX = \left(NX_n = \bigcap_{0 < i \leq n} \ker d_i \subset X_n\right)$$

$$\cdots \longrightarrow NX_2 \xrightarrow{d_0} NX_1 \xrightarrow{d_0} NX_0$$

$$NX \hookrightarrow AX \quad \text{q-iso.}$$

("Get rid of deg. n-cells")

deg. n-cells $\rightsquigarrow$ $DX \hookrightarrow AX$

Claim: $AX = \underset{\substack{\uparrow \\ \text{no homology!}}}{DX} \oplus NX$ splitting of cplxes

Prop. $\mathrm{sAb} \xrightarrow{N} C_{\geq 0}(\mathbb{Z})$ is an equivalence of categories!

Pf. $\begin{cases} \Delta \longrightarrow C_{\geq 0}(\mathbb{Z}) \\ \Delta[n] \longmapsto C_*(\Delta[n], \mathbb{Z}) \end{cases}$

So, given $E \in C_{\geq 0}(\mathbb{Z})$, $\quad \Delta^{\circ} \longrightarrow \mathrm{Ab}$

$\Delta[n] \longmapsto \mathrm{Hom}$ [illegible] $(C_*(\Delta[n], E))$

defines $\mathrm{sAb} \underset{\leftarrow}{\xrightarrow{N}} C_{\geq 0}$

$$\begin{tikzcd}
0 \arrow[r] & 1 \arrow[r, shift left] \arrow[r, shift right] & 2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & 3 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \\
E_0 & E_0 & E_0 & E_0 & E_0 & \cdots \\
 & \oplus & \oplus & \oplus & \oplus & \\
 & E_1 \arrow[luu, "d"'] & E_1 \oplus E_1 & E_1^{\oplus 3} & E_1^{\oplus 4} & \cdots \\
 & & \oplus & \oplus & \oplus & \\
 & & E_2 \arrow[luu, "d"'] & E_2^{\oplus\binom{3}{2}} & E_2^{\oplus\binom{4}{2}} & \cdots \\
 & & & \oplus & \oplus & \\
 & & & E_3^{\oplus\binom{3}{3}} \arrow[luu, "d"'] & E_3^{\oplus\binom{4}{3}} & \cdots \\
 & & & & \oplus & \\
 & & & & E_4 \arrow[luu, "d"'] &
\end{tikzcd}$$

} given by combinatorial ("data") structure + linear structure

← all information is here!

From this it is easy to construct $K(A,2)$
$K(A,n)$

<u>References</u>: $\underbrace{\text{Jardin}}_{\text{Goerss-}}$, Simplicial Homotopy Theory

Joyal, Notes on quasi-categories

<u>Remark</u>: •) If $S \in sAb$ is the nerve of a gpd,
2-cells are pairs of composable arrows,
so $0 = E_2 = E_3 = \cdots$

In terms of NX, $X = N\mathcal{G} \Rightarrow 0 = NX_2 = NX_3 = \cdots$

•) Conversely, $E_1 \xrightarrow{d_1} E_0$

$$\begin{tikzcd}
 & & E_1 \arrow[r, "d_1"] \arrow[d, rightsquigarrow] & E_0 \\
E_0 \oplus E_1 \oplus E_1 \arrow[r, shift left=2, "p_2"] \arrow[r, "+" description] \arrow[r, shift right=2, "p_2"'] & E_0 \oplus E_1 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "d"'] & E_0 \arrow[l, bend left=60, "i"] &
\end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Aff}^{op} \arrow[r, "X"] \arrow[rr, bend right, "X_\bullet"'] & \mathrm{Gpd} \arrow[r, "N"] & \mathrm{SSet} \end{tikzcd}$$

$$\mathrm{Aff}^{op} \xrightarrow{X_n} \mathrm{Set}$$

$x \in X(A)$

$$\begin{tikzcd} T_x X_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_n(A[\varepsilon]) \arrow[d] \\ * \arrow[r, "x"] & X_n(A) \end{tikzcd}$$

$\rightsquigarrow$ $T_x X_\bullet$ simplicial ~~v.sp.~~ A-modules

$\Downarrow$

$T_x X$ is a chain complex
=: tangent cplx of X at x.

b/c $X_\bullet$ is $N(X)$, get $T_x X_\bullet$ is 2-term chain cplx by the above.

Need regularity assumptions to prove that $T_x X_n$ is an A-module.

$\boxed{7.10.2013}$

Introduction to model categories

① Localizations

$W \subset \mathcal{C}$ class of maps in the cat $\mathcal{C}$

The localization of $\mathcal{C}$ along W is the initial object in the category

ob: $\mathcal{C} \xrightarrow{F} \mathcal{D}$ s.t. $F(W) \subset \mathrm{iso}\,\mathcal{D}$

map:
$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[dr] & \mathcal{D} \arrow[d] \\
 & \mathcal{D}'
\end{tikzcd}$$

is defined up to isom of categories ⇝ is object in 1-category of categories
(not equiv.!!)

notation: $\mathcal{C}[W^{-1}]$

$$\begin{tikzcd}
 & \mathcal{C} \arrow[r] \arrow[dr] & \mathcal{C}[W^{-1}] \arrow[d, dashed, "\exists!"] \\
W \arrow[dr, mapsto] & & \mathcal{D} \\
 & \mathrm{iso}(\mathcal{D}) &
\end{tikzcd}$$

construction of $\mathcal{C}[W^{-1}]$: ob = ob($\mathcal{C}$)

map: $x \to y$ in $\mathcal{C}[W^{-1}]$ are equivalence classes of chains

$$x \xleftarrow[u_1 \in W]{} \cdot \xrightarrow{a_1 \in \mathcal{C}} \cdot \xleftarrow[u_2 \in W]{} \cdot \xrightarrow{a_2 \in \mathcal{C}} \cdots \longleftarrow \longrightarrow y$$

relations: 1) $\cdots z_1 \xleftarrow{w} z_2 \xrightarrow{w} z_1 \cdots \simeq \cdots z_1 \cdots$

$w \in W$

$\cdots z_1 \xrightarrow{w} z_2 \xleftarrow{w} z_1 \cdots \simeq \cdots z_1 \cdots$

2) $\cdots z_1 = z_1 \cdots \simeq \cdots z_1 \cdots$

3) $\xrightarrow{a} \xrightarrow{b} \simeq \xrightarrow{ba}$

$\xleftarrow{u} \xleftarrow{v} \simeq \xleftarrow{uv}$

Remark:
$$\begin{tikzcd}
x \arrow[d, equal] & \cdot \arrow[l] \arrow[r] \arrow[d] & \cdot \arrow[d] & \cdot \arrow[l] \arrow[r] \arrow[d] & \cdot \arrow[d] & y \arrow[l] \arrow[d, equal] \\
x \arrow[d, equal] & \cdot \arrow[l] \arrow[r] \arrow[d] & \cdot \arrow[d] & \cdot \arrow[l] \arrow[r] \arrow[d] & \cdot \arrow[d] & y \arrow[l] \arrow[d, equal] \\
x \arrow[d, equal] & \cdot \arrow[l] \arrow[r] \arrow[d] & \cdot \arrow[d] & \cdot \arrow[l] \arrow[r] \arrow[d] & \cdot \arrow[d] & y \arrow[l] \arrow[d, equal] \\
x & \cdot \arrow[l] \arrow[r] & \cdot & \cdot \arrow[l] \arrow[r] & \cdot & y \arrow[l]
\end{tikzcd}$$

$\vdots$

The explicit construction of $\mathcal{C}[W^{-1}]$ factors through a simplicial category $L(\mathcal{C}, W)$ = Dwyer-Kan localization

OTHER CONSTRUCTIONS:

1) If W = homotopy equivalences

Example: Top and $C(\mathbb{Z})$

$$\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(x,y) = \mathrm{Hom}(x,y)/\text{homotopy}$$

Ex: $C(\mathbb{Z})[\text{h.equiv}^{-1}] = K(\mathbb{Z})$

2) Calculus of fractions

reduce chains $\leftarrow \rightarrow \leftarrow \rightarrow \cdots$

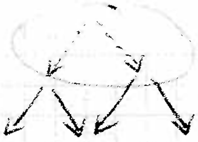
to length two chains!

$\xleftarrow{u} \xrightarrow{a} = a u^{-1}$ (right fraction)

or $\xrightarrow{a} \xleftarrow{u} = u^{-1} a$ (left fraction)

"axiom" replace $\rightarrow \leftarrow$ by $\leftarrow \rightarrow$

$\rightsquigarrow$ can reduce any chain to $\leftarrow \rightarrow$

Find 

category; nerve of this is D-K local

$$\Rightarrow \mathrm{Hom}_{\mathcal{C}[W^{-1}]}(x,y) = \{x \leftarrow z \rightarrow y\}/\sim \qquad \sim_{\text{gen}} \quad \begin{tikzcd}
 & z \arrow[dl] \arrow[d] \arrow[dr] & \\
x & z' \arrow[l] \arrow[r] & y
\end{tikzcd}$$

$\sim$ is generated by
$$\begin{tikzcd} & z \arrow[dl] \arrow[d] \arrow[dr] & \\ x & z' \arrow[l] \arrow[r] & y \end{tikzcd}$$

Rem: The category of fractions consisting of ob $x \xleftarrow{} z \rightarrow$
mor diagrams above, taking nerve gives ?
Dwyer-Kan localization

Remark: $\mathcal{C} \xrightarrow{\ell} \mathcal{C}[W^{-1}]$

$W \subset \overline{W} = \ell^{-1}(\mathrm{iso})$ saturation

Lemma: $\overline{W}$ has the 3 for 2 property.

Thus, in the above,
$$\begin{tikzcd} & z \arrow[dl] \arrow[d, "w"] \arrow[dr] & \\ x & z' \arrow[l] \arrow[r] & y \end{tikzcd}$$
$w \in \overline{W}$.

3) model structure = enhancement of the data $(\mathcal{C}, W)$
every chain is equivalent to
$$x \leftarrow Qx \longrightarrow Ry \leftarrow y$$
cofibrant replacement
fibrant replacement

Example of 2)

$$\begin{tikzcd} \mathrm{Pr}(\mathcal{C}) \arrow[r, shift left] & \mathrm{Sh}^{\tau}(\mathcal{C}) \arrow[l, shift left] \arrow[d, phantom, "\parallel"] \\ & \mathrm{Pr}(\mathcal{C})[W_{\tau}^{-1}] \end{tikzcd}$$
, W_τ = bicovering maps
($\to$ SGA)

a map is in W_τ if it
becomes invertible under
sheafification functor

Ref: Gabriel-Zisman

Ex: $(\mathrm{Ch}(\mathcal{R}), q\text{-iso.})$ does not have a calculus of fractions!

2 options:
- it has a model structure (next week)
- hom.eq. $\subset$ q. iso

$(K(\mathcal{R}), q\text{-iso.})$ has a calculus of fractions

$K(\mathcal{R})[W^{-1}] = D(\mathcal{R})$ derived category

Ex: atlas $[\text{refinements}^{-1}]$ = Man

② Factorization systems

Ex: Set

- $$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, two heads, "\text{surj.}"'] & & F \\ & \operatorname{im} f \arrow[ur, hook, "\text{inj}"'] & \end{tikzcd}$$ is unique fact. sys b/c im f det. up to iso.

$(\mathrm{Surj}, \mathrm{Inj}) \subset \mathrm{Set}$

- $(\mathrm{Inj}, \mathrm{Surj}) \subset \mathrm{Set}$ is a fact. system

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, tail, "\text{inj}"'] & & F \\ & E \amalg F \setminus f(E) \arrow[ur, "\text{surj}"'] & \end{tikzcd}$$

<u>not</u> unique!

<u>Def</u>: If $a, b \in \mathcal{C}$ are maps,

$a \pitchfork b$, "a is orthogonal to b", if

$$\forall \begin{tikzcd} \cdot \arrow[r] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowleft"] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}, \quad \exists\, u \text{ st } \begin{tikzcd} \cdot \arrow[r] \arrow[d, "a"'] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r] \arrow[ur, dashed, "u"] & \cdot \end{tikzcd} \text{ commutes}$$

u is called a "<u>filler</u>"

7.10.201

$A \subset C$
$B \subset C$ classes of maps

$A \pitchfork B$ if $\forall a \in A,\ b \in B : a \pitchfork b$

$A^{\pitchfork} = \{ b \in \underset{\text{maps}}{\mathcal{C}} \mid \forall a \in A,\ a \pitchfork b \}$ right orthogonal to A

${}^{\pitchfork}A = \{ \qquad b \pitchfork a \}$ left —"—

A and B are said to be <u>strongly orthogonal</u> if $A^{\pitchfork} = B$, $A = {}^{\pitchfork}B$
$(\Rightarrow A \pitchfork B)$

<u>Example</u>: $\mathcal{C} = Set$ $\quad I = \{ (\colon) \longrightarrow (\cdot) \}$

$I^{\pitchfork} = Inj.$

$$\begin{tikzcd} (\colon) \arrow[r] \arrow[d] & E \arrow[d, "f"] \\ (\cdot) \arrow[r] \arrow[ur, dashed] & F \end{tikzcd}$$

fibers of f are empty or $\{*\}$

${}^{\pitchfork}(I^{\pitchfork}) = {}^{\pitchfork}Inj = Surj.$

<u>Prop</u>: If I is a set of maps in $\mathcal{C}$, then
$A = {}^{\pitchfork}(I^{\pitchfork})$, $B = I^{\pitchfork}$ are strongly orthogonal.

<u>Def'n</u>: A (<u>functorial</u>) <u>factorization system</u> in $\mathcal{C}$ is a lifting system (A, B) s.t. every map $x \xrightarrow{f} y$ in $\mathcal{C}$ factors as (<u>functorially</u>)

$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr, "A \ni \alpha"'] & & y \\ & z \arrow[ur, "\beta \in B"'] & \end{tikzcd}$$

<u>Def'n</u> A <u>lifting system</u> in $\mathcal{C}$ is a pair (A, B) of strongly orthogonal classes of maps.

Example CommRings

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\text{localization}"'] & & B \\ & A[S^{-1}] \arrow[ur, "\text{conservative}"'] & \end{tikzcd}$$

alternatively: étale henselian

where $S = f^{-1}(B^*)$

and $g : C \to B$ is **conservative** if
$a \in C^* \iff f(a) \in B^*$

Example $\mathcal{C} = Top$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\text{cofibration}"'] & & Y \\ & Cyl(f) \arrow[ur, "\sim"', "\text{weak equiv.} + \text{Serre fibration}"] & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\text{weak eq.} \sim"', "+ \text{cofibration}"'] & & Y \\ & P(f) \arrow[ur, "\text{Serre fibration}"'] & \end{tikzcd}$$

$Cyl(f) = X \times I \sqcup_{X \times \{1\}} Y$ cylinder object

$P(f) = Y^I \times_Y X$ path object

Both are obtained by construction in prop. above:

(Cof, trivial Serre fibrations) (trivial cofibrations, Serre fib.)

$J = I \cup J$ I

where

$I = \{ D^n \longrightarrow D^n \times I, n \}$

$J = \{ S^{n-1} \longrightarrow D^n, n \}$

Serre fibration:

$$\begin{tikzcd} D^n \arrow[r] \arrow[d, "C_n"'] & X \arrow[d, "f"] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

is f st. it has right lifting property for all elements of I

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Small object argument:

= construction of a fact. syst. from a lifting system.

Prop: If (A,B) is a lifting system generated by a set I, $(A,B) = ({}^{\pitchfork}(I^{\pitchfork}), I^{\pitchfork})$ and satisfies a smallness condition and if $\mathcal{C}$ has all colimits (e.g. $\mathcal{C}$ loc pres) then (A,B) can be enhanced into a functorial fact. sys.

Idea of pf:

$$\begin{tikzcd}
\bigsqcup_{I_f} a \arrow[r] \arrow[d] & x \arrow[d, "f"] \\
\bigsqcup_{I_f} b \arrow[r] \arrow[ur, dashed] & y
\end{tikzcd}$$

Start with $x \xrightarrow{f} y$.
Want to factorize.

~~I_x = set of maps $a \longrightarrow x$, where a = domain of some~~

$I_f = \left\{ \begin{tikzcd} a \arrow[r] \arrow[d, "\in I"'] & x \arrow[d, "f"] \\ b \arrow[r] & y \end{tikzcd} \right\}$ is set b/c I is set ~~map in I~~.

$$\begin{tikzcd}
\bigsqcup_{I_f} a \arrow[r] \arrow[d] & x \arrow[d, "\alpha_1"] \arrow[ddr, bend left, "f"] & \\
\bigsqcup_{I_f} b \arrow[r] \arrow[drr, bend right] & X_1 \arrow[dr, "\beta_1"] \arrow[ul, phantom, "\lrcorner", very near start] & \\
 & & y
\end{tikzcd}$$

$\alpha_1 \in A$

β_1 may not be in B.

Iterate this construction for β_1 instead of f

Smallness condition ensures that this terminates.

Ref. for details: Joyal-Tierney "Simplicial homotopy theory" Appendix.

Lemma: If (A,B) is a lifting system, A is stable by pushout and B is stable by pullback, i.e.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "A \ni a"'] & \cdot \arrow[d, "a'"] \\ \cdot \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \cdot \end{tikzcd}$$

$a \in A \Rightarrow a' \in A$

Ex: $J = \{ S^{n-1} \longrightarrow D^n \}$

$f: X \longrightarrow *$

$$J_f = \left\{ \begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & X \\ D^n & \end{tikzcd} \right\}$$

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod D^n \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & X' \end{tikzcd}$$

X_1 is gluing of D^n along all S^{n-1} in X.

Convention
$S^{-1} = \emptyset$

$g: \emptyset \longrightarrow X$

$$J_g = \left\{ \begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & \emptyset \arrow[d] \\ D^n \arrow[r] & X \end{tikzcd} \right\}$$

$$= \left\{ \begin{tikzcd} \emptyset = S^{-1} \arrow[r] \arrow[d] & \emptyset \arrow[d] \\ \{*\} = D^0 \arrow[r] & X \end{tikzcd} \right\} = \{\text{pts of } X\}$$

$$\begin{tikzcd} \coprod S^{-1} \arrow[r] \arrow[d] & \emptyset \arrow[d] \arrow[ddr, bend left] & \\ \coprod D^0 \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] \arrow[drr, bend right] & X_{dis} \arrow[dr, "g_1"] & \\ & & X \end{tikzcd}$$

Iterate: $g_1: X_{dis} \longrightarrow X$

$$J_{g_1} = \left\{ \begin{tikzcd} S^0 \arrow[r] \arrow[d] & X_{dis} \arrow[d] \\ D^1 \arrow[r] & X \end{tikzcd} \right\} \cup \left\{ \begin{tikzcd} S^{-1} \arrow[r] \arrow[d] & X_{dis} \arrow[d] \\ D^0 \arrow[r] & X \end{tikzcd} \right\}$$

$\leadsto$ iteration builds a CW-cplx approximation

in the end get relative CW cplx

7.10.2013

③ Model categories

Let $\mathcal{M}$ be a category, $W \subset \mathcal{M}$

A model structure on $(\mathcal{M}, W)$ is the data of 2 classes of maps.

$\mathcal{C}$ cofibrations
F fibrations

(W is called weak equivalences)

s.t. - $\mathcal{M}$ is bicomplete
- W has the 3 for 2 property (always true if $\overline{W} = W$)
- $(C \cap W, F)$
  $(C, F \cap W)$ are factorization systems.

Rem: Weaker than "usual" defn ...

Localization: $\mathcal{M}[W^{-1}] = Ho(\mathcal{M})$ ... see literature.

Remark:

$$\begin{tikzcd}(M, W) \arrow[r] & \text{Segal space } X_\bullet = X^{(M,W)} \arrow[d] \\ & \text{Homotopy cat. } h_1(X_\bullet)\end{tikzcd}$$

$$Hom_{Ho(M)}(x, y) = Hom_{M}(Qx, Ry) / \text{homotopy}$$

Actually, Dwyer-Kan localization is "the real deal" in localization, we loose info.

BUT: model cat. are too weak, that's why you go to Segal spaces instead of simplicial categories.

21.10.2013

$\boxed{1}$

Homotopy (Co-)limits

- Simon Häberli

Ref: 1) D. Dugger, A primer on homotopy colimits

2) N. Gambino, Weighted limits in simplicial theo

1. Motivation/Computation

Recall: Let M be a cat., I small cat.

Then
$$\begin{tikzcd} \mathcal{F} \in M^I \arrow[r, bend right, "\operatorname{colim}"'] & M \arrow[l, "\text{const}"'] \end{tikzcd}$$
colim := left adjoint to const.

Equivalently,
$$\operatorname{colim}(\mathcal{F}) = \operatorname{coeq}\Big(\underbrace{\coprod_{\underset{\in I}{i \to j}} \mathcal{F}(i)}_{=:(A)} \rightrightarrows \underbrace{\coprod_{\in I} \mathcal{F}(i)}_{=:(B)} \Big)$$
$$= \underset{\underset{\underset{\tilde{I}}{\parallel}}{0 \bullet \rightrightarrows \bullet 1}}{\operatorname{colim}} (\tilde{\mathcal{F}})$$

$$\tilde{\mathcal{F}} : \tilde{I} \longrightarrow M$$
$$\begin{cases} 0 \longmapsto (A) \\ 1 \longmapsto (B) \end{cases}$$

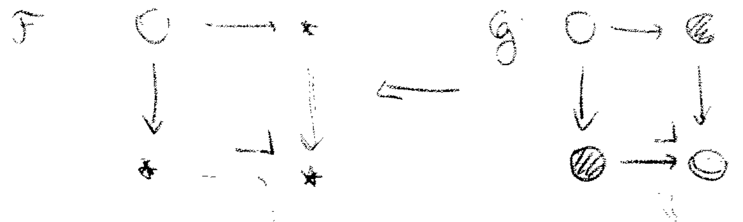
Problem: $\mathcal{F}, \mathcal{G} \in M^I$, $\mathcal{F} \xrightarrow{\sim} \mathcal{G}$ natural weak equiv.

$\rightsquigarrow \operatorname{colim}(\mathcal{F}) \longrightarrow \operatorname{colim}(\mathcal{G})$

↑ need not be a weak equivalence

Example $\mathcal{F} : I \longrightarrow Top$

$$I = \begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \\ 2 & \end{tikzcd}$$



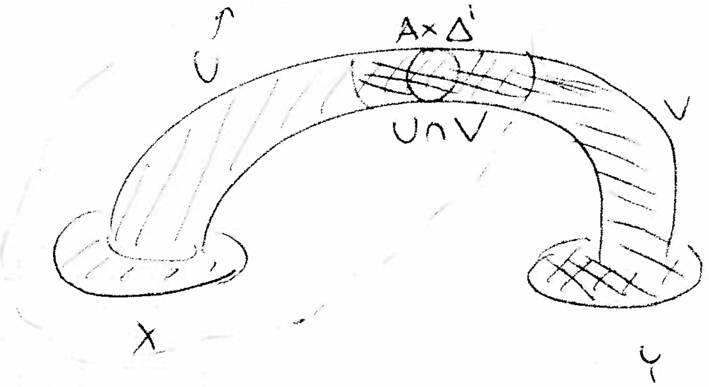
Example: $\mathcal{F}$:

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

$$\operatorname{hocolim} \mathcal{F} = X \amalg A \times \Delta^1 \amalg Y /\sim$$

can replace by
$\{0,1\} \xrightarrow{cof} Z$
$Z \xrightarrow{fib} *$
i.e. interval in sense that Paul defined last week.

$\forall a:$ $(a,0) \sim f(a)$
$(a,1) \sim g(a)$



Def'n: (1) I small category $\mathcal{F}: I \longrightarrow Top$

The nerve of $\mathcal{F}$ / simplicial replacement $srep(\mathcal{F})$ is

$$N\mathcal{F}: \quad \coprod_{i_0 \in I} \mathcal{F}(i_0) \leftleftarrows \coprod_{i_0 \leftarrow i_1} \mathcal{F}(i_1) \ \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\ \coprod_{i_0 \leftarrow i_1 \leftarrow i_2} \mathcal{F}(i_2) \ \cdots$$

(2) $\operatorname{hocolim}_I \mathcal{F} := |N\mathcal{F}|$

Recall: simpl. space $X: \Delta^{op} \longrightarrow Top$

$$|X| := \coprod_{n \geq 0} X_n \times \Delta^n /\sim$$

$$(x_n, t_n) \sim (x_k, t_k) \iff f: [k] \to [n]$$

s.t.

$t_k \xrightarrow{\Delta(f)} t_n$

$x_n \xrightarrow{x(f)} x_k$

$$|X| = coeq\left(\coprod_{[n] \to [k]} X_k \times \Delta^n \overset{g_1}{\underset{g_2}{\rightrightarrows}} \coprod_{[n]} X_n \times \Delta^n \right)$$

where $g_1: X_k \times \Delta^n \xrightarrow[id \times \Delta([n] \to [k])]{} X_k \times \Delta^k$, $g_2: X_k \times \Delta^n \xrightarrow[X([n] \to [k]) \times id]{} X_n \times \Delta^n$

Remark:

$$\underset{I}{\operatorname{hocolim}}\, \mathcal{F}$$

$$\|$$

$$\operatorname{coeq}\left(\coprod_{[n]\to[k]} (N\mathcal{F})_k \times \Delta^n \rightrightarrows \coprod_n (N\mathcal{F})_n \times \Delta^n \right)$$

$$\downarrow$$

$$\operatorname{coeq}\left(\coprod_{[n]\to[k]} (N\mathcal{F})_k \times * \rightrightarrows \coprod_n (N\mathcal{F})_n \times * \right)$$

$$\|$$

$$\underset{I}{\operatorname{colim}}\, N\mathcal{F}$$

$$\| \text{ Thm (MacLane)}$$

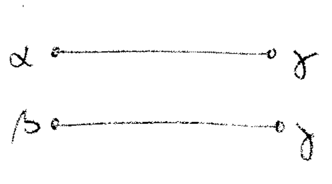
$$\underset{I}{\operatorname{colim}}\, \mathcal{F}$$

Example 1:

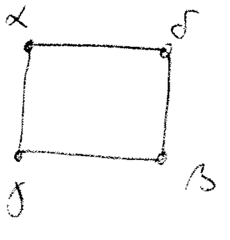
$$I: \begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \\ 2 & \end{tikzcd} \qquad \mathcal{F}: \begin{tikzcd} \{\alpha,\beta\} \arrow[r] \arrow[d] & \{\delta\} \\ \{\gamma\} & \end{tikzcd}$$

$$N\mathcal{F}: \quad \{\alpha,\beta,\gamma,\delta\} \leftleftarrows \{\alpha,\beta\} \sqcup \{\alpha,\beta\}$$

$|N\mathcal{F}|$:



$\sim$



$=$

$$\Rightarrow \underset{I}{\operatorname{hocolim}}(\mathcal{F}) = ○$$

$$\underset{I}{\operatorname{colim}}(\mathcal{F}) = *$$

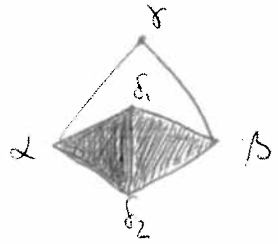
Example 2: I:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \\
2 \arrow[d] & \\
3 &
\end{tikzcd}$$

F:

$$\begin{tikzcd}
\{\alpha,\beta\} \arrow[r] \arrow[d] & \{\gamma\} \\
\{\delta_1\} \arrow[d] & \\
\{\delta_2\} &
\end{tikzcd}$$

$|NF| =$



Example I small category

$$NI: \quad \coprod_{i_0 \in I} * \leftleftarrows \coprod_{i_0 \leftarrow i_1} * \Lleftarrow \coprod_{i_0 \leftarrow i_1 \leftarrow i_2} * \quad \cdots$$

$|NI|$ ... classifying space of I

Example G groupoid $\quad G_0 \leftleftarrows G_1$

Note: $\operatorname{coeq}(G_0 \leftleftarrows G_1)$ = isom classes of G

$$NG: \quad \coprod_{g_0 \in G_0} * \leftleftarrows \coprod_{\substack{g_0 \leftarrow g_1 \\ \uparrow \\ G}} * \Lleftarrow \coprod_{g_0 \leftarrow g_1 \leftarrow g_2} *$$

$\exists$ equiv. $\quad G \longrightarrow \pi_1(|NG|)$

objects: pts $\in |NG|$

mor: homotopy classes of paths

21.10.2013

$\boxed{3}$

2. Hocolim as a derived functor

(M, W, C, F)

(M', W', C', F') two model categories

$$\begin{tikzcd} M \arrow[r, "\mathcal{F}"] \arrow[d, "i"'] & M' \arrow[d, "i'"] \\ Ho(M) \arrow[r, dashed, "\exists ? G"] & Ho(M') \end{tikzcd}$$

Remark: $\exists\, G$ s.t. $G \circ i = i' \circ \mathcal{F}$

$\iff \mathcal{F}(W) \subseteq W'$

Definition: Any G s.t. $\exists$ [illegible] $i' \circ \mathcal{F} \rightleftarrows G \circ i$ which is initial among such . is called <u>right derived functor</u> of $\mathcal{F}$ $(= R\mathcal{F})$
(left) $(= L\mathcal{F})$

Def^n: $W_c = \{$w. eq. btw cofibrant obj$\}$

$W_F = \{$ " fibrant " $\}$

Thm: • $\mathcal{F}(W_c) \subseteq W' \Rightarrow \underset{G \circ i}{\overset{\shortparallel}{L\mathcal{F}}}$ exact and $L\mathcal{F}(X) = \mathcal{F}(Q(X))$

• $\mathcal{F}(W_F) \subseteq W' \Rightarrow R\mathcal{F}$ exact and $R\mathcal{F}(X) := \mathcal{F}(R(X))$

Example: $M = M' = Ch(R)$, $E \in Ch(R)$

Fact: $M \xrightarrow{E \otimes_R -} M$

$E \otimes_R (W) \not\subseteq W$

Still: $E \otimes_R (W_c) \subseteq W$ for the proj. model structure

Prop:

$\rightsquigarrow$ get derived tensor product.

Def'n: $M \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} M'$ $\mathcal{F} \dashv \mathcal{G}$ ($\mathcal{F}$ left adj. of $\mathcal{G}$)

$\mathcal{F}$ is left Quillen, if moreover it "preserves" $C \cap W$ and C

right Quillen —"— F' and $F' \cap W'$

Fact: $\mathcal{F} \dashv \mathcal{G}$ then $\mathcal{F}$ left Quillen $\Leftrightarrow$ $\mathcal{G}$ right Quillen

⇝ "$\mathcal{F}$ and $\mathcal{G}$ are a Quillen adjunction"

Thm: If $\mathcal{F} \dashv \mathcal{G}$ is a Quillen adjunction,

$\exists\ L\mathcal{F}, R\mathcal{G}$ and $L\mathcal{F} \dashv R\mathcal{G}$

$$M^I \underset{\text{const}}{\overset{\text{colim}}{\rightleftarrows}} M$$

2 model structures on M^I:

	W	C	F
inj.	W_I	term-w. cofibr.	right lifting prop
proj.	W_I	left lifting prop	term-w. fibr.

termwise w. equiv.

Rem: Don't always exist!

If cofibrantly gen, proj. model structure exists.

Prop: (1) colim $\longrightarrow\!\!\!\!\dashv$ const is a Quillen adjunction for the projective model structure.

(2) $\overset{\text{const} \dashv \lim}{\lim \vdash\!\!\longrightarrow \text{const}}$ ——"——

—"— inj. ——"——

Def'n hocolim = L colim
holim = R lim

Remark: This coincides with what we defined ad hoc before!

Claudio - dgAlg

28.10

1

only for char=0

$$\begin{tikzcd}
\underline{Alg.Geom.} & \boxed{Alg_k} \arrow[r, bend left, "2"] \arrow[rrr, bend left=40, "2"] & \underline{H/DAG2} \arrow[l, bend left, "\pi_0(-)"] & \boxed{SimpAlg_k} \arrow[r, bend left, "D.-K."] & \boxed{dgAlg_k} \arrow[l, bend left] \arrow[lll, bend left=40, "H_0(-)"]
\end{tikzcd}$$

affine schemes (Zar. top) der (-) .

schemes (Zariski open immersions) der (-) .

algebraic spaces (étale maps) der (-) .

stack (smooth maps) der (-) .

Use homological degrees ..! $\cdots \to A_1 \to A_0 \to 0$

Extension by module

Let A be an (dg-) algebra, M an (A,A)-mod
An extension of A by M is

$$0 \longrightarrow M \longrightarrow M \oplus A \xrightarrow{alg.hom.} A \longrightarrow 0$$

(e.g. square-zero extension)

- A square-zero ext. is an extension of A by M s.t. $M^2 = 0$ in $M \oplus A$
- A split extension is

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & M \oplus A \arrow[r] & A \arrow[r] \arrow[l, bend right, "alg.hom"'] & 0
\end{tikzcd}$$

- A split square-zero extension $\rightsquigarrow$ $M|_{Hom} = 0$

$$\begin{tikzcd}
0 \arrow[r] & M[i] \arrow[r] & M[i] \oplus A \arrow[r] & A \arrow[r] \arrow[l, bend left, "d"] & 0
\end{tikzcd}$$

$d \leftrightarrow$ graded derivation

A derivation $d : A \longrightarrow M$

$$d(ab) = d(a)b \pm a d(b)$$

$$\begin{tikzcd}
\iff d : A \arrow[r] \arrow[dr, "id"'] & A \oplus M_{sq\,0\,ext.} \arrow[d] \\
& A
\end{tikzcd}$$

$$\begin{tikzcd} A \oplus_d M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_h"] & A \arrow[d, "i"] \\ A \arrow[r, "d"'] & M[1] \oplus A \end{tikzcd}$$

universal!

$\rightsquigarrow$ $A \oplus_d M$ is a square-zero extension

Postnikov tower

Def'n $A \in dg_{\geq 0}$

$$\cdots \to A_i \xrightarrow{d} A_{i-1} \to \cdots$$

A Postnikov tower is a sequence of dgas s.t.

$$A \to \cdots \to T_n(A) \to T_{n-1}(A) \to \cdots \to T_0(A)$$

$$\begin{tikzcd} & \vdots \arrow[d] \\ \vdots & T_n(A) \arrow[dd] \\ A \arrow[ur] \arrow[dr] & \\ \vdots & T_{n-1}(A) \arrow[d] \\ & \vdots \end{tikzcd}$$

s.t. 1) $H_i(T_j(A)) \cong \begin{cases} 0 & i > j \\ H_i(A) & i \leq j \end{cases}$

2) The map $A \to T_j(A)$ induces an isomorphism between homologies for $i \leq j$ $H_j(A) = H_i(T_j(A))$

Is unique up to homotopy.

Construction: naive PT

$N(A)_i$

$$\begin{tikzcd} \underline{i=0}: & \cdots \arrow[r] & 0 \arrow[r] & A_0/dA_1 \arrow[r] & 0 & & & \\ \underline{i=1}: & \cdots \arrow[r] & 0 \arrow[r] & A_1/dA_2 \arrow[r] \arrow[u] & A_0 \arrow[r] & 0 & & \\ \vdots & & & & & & & \\ \underline{i=n}: & \cdots \arrow[r] & 0 \arrow[r] & A_n/dA_{n+1} \arrow[r] \arrow[uu] & A_{n-1} \arrow[r] & \cdots \arrow[r] & A_0 \arrow[r] & 0 \end{tikzcd}$$

$\nwarrow$ sq-zero ext!

28.10.20
$\boxed{2}$

Construction "non-naive PT" (better for computation).

$$\begin{tikzcd}
i=0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & d(A_1) \arrow[r] & A_0 \arrow[r] & 0 \\
i=1 \arrow[r] & 0 \arrow[r] & d(A_2) \arrow[r] & A_1 \arrow[r] \arrow[u] & A_0 \arrow[r] & 0 \\
 & & & \vdots \arrow[u] & &
\end{tikzcd}$$

Fact: • $A \in dg_{\geq 0}Alg \Rightarrow H_n(A)$ is an $H_0(A)$-module

• For a $dg_{\geq 0}alg$ A, $n \geq 1$
$H_n(A)$ is a $T_{n-1}(A)$-module,
where the action is given by the projection $T_{n-1}(A) \to T_0(A)$

Note that we can consider

$$0 \to H_n(A)[n+1] \to H_n(A)[n+1] \oplus T_{n-1}(A) \to T_{n-1}(A) \to 0$$

Lemma: Let A be a $dg_{\geq 0}Alg$, $(T_n(A))_{n\geq 0}$ its PT. Then there exists an extension of $T_{n-1}(A)$ by $H_n(A)[n+1]$ and a splitting d:

$$\begin{tikzcd}
0 \arrow[r] & H_n(A)[n+1] \arrow[r] & H_n(A)[n+1] \oplus T_{n-1}(A) \arrow[r] & T_{n-1}(A) \arrow[r] \arrow[l, bend right, "d"'] & 0
\end{tikzcd}$$

s.t.

$$\begin{tikzcd}
T_n(A) \arrow[r, "p_n"] \arrow[d, "p_n"'] \arrow[dr, phantom, "\lrcorner_h" very near start] & T_{n-1}(A) \arrow[d] \\
T_{n-1}(A) \arrow[r, "d"] & H_n(A)[n+1] \oplus T_{n-1}(A)
\end{tikzcd}$$

So $T_n(A)$ is a square-zero extension (up to quasi-iso), i.e.

$T_n(A)$ is quasi-iso to a square-zero extension
$$T_{n-1}(A) \oplus_d H_n(A)[n] \to T_{n-1}(A)$$

Take T_n:

$$\begin{tikzcd}
n=0 & T_0(A) & \overset{\text{choose...}}{=} & H_0(A) \\
& T_1(A) \arrow[u, "p_1"] & \simeq & H_1(A)[1] \underset{d_1}{\oplus} H_0(A) \arrow[d] \\
& T_2(A) \arrow[u, "p_2"] & \simeq & H_2(A)[2] \underset{d_2}{\oplus} \left(\qquad\qquad \right)
\end{tikzcd}$$

So PT is a sequence of square-zero extensions

Towers of sq-zero ext. are exactly nilpotent extensions!

$$\begin{tikzcd}
Alg \arrow[r, bend right, "i"'] & dg_{\geq}Alg \arrow[l, bend right, "H_0"']
\end{tikzcd}$$

So what we get more in dgAlg world is just "nilpotents". So underlying top space Spec A of a dgAlg will be just Spec $H_0(A)$, the extra information in the derived world lies in the structure sheaf!

Model structure on A-Mod...

$\rightsquigarrow \quad B \overset{\mathbb{L}}{\underset{A}{\otimes}} - : Ho(A\text{-Mod}) \longrightarrow Ho(B\text{-Mod})$

coproduct $\coprod_{E} : A\text{-Mod} \longrightarrow B\text{-Mod}$

$\rightsquigarrow \quad \coprod_{\epsilon}^{(\mathbb{L})} : Ho(A\text{-Mod}) \longrightarrow Ho(B\text{-Mod})$ don't need to derive!

Def'n:
- A-Mod is flat if $- \overset{\mathbb{L}}{\otimes} M$ preserves homotopy ~~[illegible]~~ pull-backs
- A-Mod is projective if it is a retract of $\coprod_{\epsilon} A$ in Ho(A-Mod)

$f: A \to B \rightsquigarrow f^*: A\text{-Mod} \longrightarrow B\text{-Mod}$, $M \longmapsto B \otimes_A M$, $f_*: B\text{-Mod} \to A\text{-Mo}$

- f is flat if $\mathbb{L}f^*$ commutes with homotopy pullback
- f is Zariski open immersion if f is flat and f_* is fully faithful and finitely presented.

Def'n An A-module M is strong if

$$H_\bullet(A) \otimes_{H_0(A)} H_0(M) \longrightarrow H_\bullet(M)$$

is an isomorphism (i.e. generated by elts in deg 0)
(of graded modules)

Lemma:

a) An A-Mod M is projective ⟺ (1) $H_0(M)$ is a proj. $H_0(A)$-module
&
(2) M is strong.

b) — " — flat ⟺ (1) — " — flat
(2) — " —

Def'n
- $f : A \to B$ is strong if

$$H_\bullet(A) \otimes_{H_0(A)} H_0(B) \longrightarrow H_\bullet(B) \quad \text{is an iso.}$$

i.e. B is strong as an A-module.

- f is strongly flat if f is strong and $\operatorname{Spec} H_0(B) \longrightarrow \operatorname{Spec} H_0(A)$ is flat

- f is strongly Zariski open immersion if f is strong and $\operatorname{Spec} H_0(B) \longrightarrow \operatorname{Spec} H_0(A)$ is Z. open imm.

Thm: f is flat ⟺ strongly flat
f is Zariski open imm ⟺ strongly Zariski open immersion

4.11.2013
1

Cotangent complexes – Georg

General Context / Framework:

$\mathcal{C}$ model category

$\mathcal{C}_{ab}$ abelian objects in $\mathcal{C}$

$Ab : \mathcal{C} \longrightarrow \mathcal{C}_{ab}$ abelianization functor

Homology of X in $\mathcal{C}$:= $(\text{Derived } Ab)(X)$

Example 1: $\mathcal{C} = sSet$
$\mathcal{C}_{ab} = sAb$

$$Ab : sSet \longrightarrow sAb$$
$$(X : \Delta \to Set) \longmapsto \mathbb{Z} \circ X,$$
$$\mathbb{Z} : S \longmapsto \mathbb{Z}[S]$$

$$\pi_n\big((LAb)(X)\big) = H_n(|X|; \mathbb{Z})$$

Example 2: B fixed ring, $B \to A$ fixed ring hom.

$\mathcal{C} = Alg_B/A$ - Obj = $$\left(\begin{tikzcd} X \arrow[r] & A \\ B \arrow[u] \arrow[ur] & \end{tikzcd}\right)$$

$\mathcal{C}_{ab} \cong A\text{-Mod}$

$$Ab : Alg_B/A \longrightarrow (Alg_B/A)_{ab} \cong A\text{-Mod}$$
$$X \longmapsto \Omega_{X/B} \otimes A$$
$$A \xrightarrow{id} A \longmapsto \Omega_{A/B}$$

cotangent cplx of $f : B \longrightarrow A$:= $(LAb)(A)$

$$LAb(X \to A) = \mathbb{L}_{X/B} \otimes A$$

C.C = "Left derived vs of Kähler differentials"

= Kähler differentials to A, seen as derived objects

Need to do:

1) Define a model structure on $dgAlg_{\cdot\geq 0} = \mathcal{C}$

2) Study cofibrant replacements

3) Kähler differentials / derivations for $\mathcal{C}$

4) Examples

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1) $dgAlg_{\cdot \geq 0}$ (over fixed field of char 0) $(\cdots A^{-4} \xrightarrow{d} A^{-3} \longrightarrow A^{-2} \longrightarrow A^{-1} \longrightarrow A^{0} \to 0)$

$\|$

obj: $A = (\cdots \to A_4 \xrightarrow{d} A_3 \xrightarrow{d} A_2 \xrightarrow{d} A_1 \xrightarrow{d} A_0 \to 0)$

+ multipl. $A_i \times A_j \longrightarrow A_{i+j}$

$$\left.\begin{array}{l} ab = (-1)^{ij} ba \\ d(ab) = d(a) b + (-1)^i a\, d(b) \end{array}\right\} \forall a \in A_i,\ b \in A_j$$

Remarks: (i) $|A| := \bigoplus_{i \geq 0} A_i$

(ii) $dgAlg_{\cdot \geq 0} = dgalg^{\cdot \leq 0}$

$$\begin{tikzcd} Ch_{\cdot \geq 0} \arrow[r, bend left, "S(.)"] & dgAlg_{\cdot \geq 0} \arrow[l, bend left, "FF"] \end{tikzcd}$$

to transfer model structure

V chain cplx of v-sp , $S(V) = \bigoplus_{n \geq 0} \left(V^{\otimes n} / S_n \right)$

differential by Leibniz rule

Ex: • x deg 0
$S(k\langle x \rangle) = k[x]$

• x deg 1
$$S(k\langle x\rangle) = \left(\begin{array}{ccc} 0 & & 1 \\ k & \xleftarrow{0} & k \\ 1 & & x \end{array} \right)$$

$\rightsquigarrow \exists$ MS on $dgAlg_{\cdot \geq 0}$

$\boxed{4.11.2013}$ $\boxed{2}$

MS on $dgAlg_{\geq 0}$:

W : q-iso

F : deg-wise surj. in deg ≥ 1

C : ?

Def'n: $A \in dgAlg_{\geq 0}$

- is free if $\cong S(V)$ for some chain cplx V
- is semi-free if $|A| \cong k[\overset{}{x_i}, i \in I]$ as graded rings
  $\uparrow$ homog.

Proposition: P semi-free $\Rightarrow$ cofibrant. (i.e. $(1 \to P) \in C$)

Def'n: If A is a dgAlg, if $P \to A$ is a quasi-iso from P semi-free, then this is called a semifree resolution (aka a cofibrant resolution by prop)

Prop: $\forall A$ $\exists$ semifree resolution

Example (Koszul resolution)

B comm. ring, $I \subset B$ ideal generated by $f_1, \dots, f_k$
Want: resolution of $A = B/I$ as $dgalg_B$

Let $E = B^{\oplus k} = Be_1 \oplus \cdots \oplus Be_k$ and

$$s : E \longrightarrow B, \quad e_i \longmapsto f_i$$

$$\Lambda^j_B E \xrightarrow{d_j} \Lambda^{j-1}_B E$$

$$d_j(e_{i_1} \wedge \dots \wedge e_{i_j}) = \sum_{k=0}^{j} (-1)^{j+1} e_{i_1} \wedge \dots \wedge \hat{e}_{i_k} \wedge \dots \wedge e_{i_j}$$

$$\rightsquigarrow \text{cplx } (\dagger)\ \Lambda^\bullet_B E(s) = \left(\Lambda^k E \to \cdots \to E \to \Lambda^0 E = B \right) \longrightarrow A = B/I$$

$$(\ddagger)\quad |\Lambda^\bullet E| = S(E[1]) = B[\underset{\text{deg } 1}{e_1, \dots, e_k}], \quad d(e_i) = f_i$$

Proposition: If $f_1, \dots, f_k$ is a regular sequence

$$\Rightarrow \begin{cases} H_0(\Lambda^\bullet_B(E)(s)) = B/\mathrm{im}(d_1) = B/I = A \\ H_i(\qquad) = 0 & i \geq 1 \end{cases}$$

$\Rightarrow$ (†) is a semifree resolution by (‡) (not free)

Derivations (relative to fixed base ring B)

Let A be a B-algebra, M an A-module.

Proposition (A) There is a 1-1 correspondence between

1) derivations $d: A \to M$, $d(ab) = da \cdot b + a\,db$
2) split square-zero extensions

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & A' \arrow[r, "p"] & A \arrow[l, bend left, "s"] \arrow[r] & 0 \end{tikzcd}$$

$(s(a) = (a, da))$

- s, p are B-alg homs
- exact as B-modules.

3) abelian objects in Alg_B/A + split map

(X is abelian if $Mor(\cdot, X)$ is naturally an ab gp)

Kähler differentials

Def/Prop (B) 1) Let $F = \bigoplus_{x \in A} A\delta_x$ (δ_x formal symbol)

equiv. are

I ~~ideal~~ s/module generated by
- $\delta(xy) - x\delta(y) - \delta(x)y$
- $\delta(x+y) - \delta x - \delta y$ $\quad \forall x, y \in A$
- $\delta b \quad \forall b \in B$

$\Omega_{A/B} := F/I$

2) $Der_B(A, \cdot): \begin{cases} A\text{-Mod} \longrightarrow A\text{-mod} \\ M \longmapsto Der_B(A, M) \end{cases}$

$Der_B(A, \cdot)$ is (co)represented by $\Omega_{A/B}$.

$$Der_B(A, M) = Hom_{A\text{-mod}}(\Omega_{A/B}, M)$$

3)

$$\begin{tikzcd} \mathrm{Alg}_B/A \arrow[r, bend left, "Ab"] & (\mathrm{Alg}_B/A)_{ab} \arrow[l, bend left, "FF"] \cong A\text{-Mod} \\ & (A\oplus M \to A) \arrow[r, phantom, "\longleftarrow\!\shortmid"] & M \end{tikzcd}$$

∃ left adj $Ab \dashv FF$ given by

$$Ab(X \to A) = \Omega_{X/B} \otimes A$$

(4) $\Omega_{A/B} := I/I^2$, $I = \ker(A \otimes A \longrightarrow A)$)

For dgAlgs, Prop (A) still holds

For Prop (B), in 1) add x homogeneous

The derivation on F is defined as follows.

$$\begin{tikzcd} A_n \arrow[r, "\delta"] \arrow[d, "d"'] \arrow[dr, phantom, "\circlearrowleft"] & F_n \arrow[d, "d"] \\ A_{n-1} \arrow[r, "\delta"'] & F_{n-1} \end{tikzcd}$$

- $d(\delta x) := \delta(dx)$
- extend by Leibniz.

Check: d preserves I ✓

2) $\mathrm{Der}^*_B(A, M) = \underset{\text{internal Hom}}{\mathrm{Hom}^*}(\quad)$ (= degree n derivations)

3) same

Example: A semi-free/B , $|A| = B[x_i, i \in I]$

Calculation ⇝ $\Omega_{A/B} = \bigoplus_{i \in I} A\,\delta x_i$ + differential as before

$V = \mathbf{B}\langle x_i, i \in I\rangle$

$S^*_B(V) = |A|$ $\qquad \Omega_{A/B} = |A| \otimes V$ + diff

Def'n (cotangent complex)

Let $B \to A$ be dg Algs

$$\mathbb{L}_{A/B} = (LAb)(A)$$

$= \Omega_{P/B} \otimes_P A$, where $P \to A$ is a B-semifree resolution

Example: ① $A = k[x_1, \ldots, x_n]$, $B = k$ $\quad \leftrightarrow \quad \mathbb{A}^n \longrightarrow pt$

$$\Omega_{k[x_1,\ldots,x_n]/k} = \mathbb{L}_{A/B} = A^{\oplus n}$$

② $A = k$, $B = k[x_1, \ldots, x_n]$ $\quad \leftrightarrow \quad pt \hookrightarrow \mathbb{A}^n$

Ω_k

③ $B = k[x_1, \ldots, x_n]$, $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_k)$, $(f_1, \ldots, f_k)$ regular sequence

$$\mathbb{L}_{A/B} = \Big(B[e_1, \ldots, e_k] \otimes \langle \delta e_1, \ldots, \delta e_k\rangle\Big) \otimes_{B[e_1,\ldots,e_k]} A \cong \underbrace{A\delta e_1 \oplus \cdots \oplus A\delta e_k}$$

Koszul r: $P = S^*(\underset{\deg 1}{Be_1} \oplus \cdots \oplus \underset{\deg 1}{Be_k})$

concentrated in degree -1 $= \mathbb{L}_{A/B}$

Let $x : A \to k$ be a point

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-2} \arrow[r] & A^{-1} \arrow[r] & A^0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

$$\Rightarrow \Omega_{A/B} \otimes k = k^{\oplus k}[1]$$

In ③,

$$\Rightarrow \quad H^i(\mathbb{L}_{A/B}) = \begin{cases} A^{\oplus k} & i = -1 \\ 0 & i \neq 0 \end{cases}$$

$\mathbb{L}_{A/k}$ - $P = k[x_1, \dots, x_n, e_1, \dots, e_k]$

$$\mathbb{L}_{A/k} = \Omega_{P/k} \underset{P}{\otimes} A = \left(\underset{-1}{A\langle \delta e_1, \dots, \delta e_k \rangle} \xrightarrow{d} \underset{0}{A\langle \delta x_1, \dots, \delta x_n \rangle} \right),$$

$$d(\delta e_i) = \delta(de_i) = \delta(f_i)$$
$$= \sum_j \frac{\partial f_i}{\partial x_j} dx_j$$

$$\mathbb{L}_{A/k} \underset{x}{\otimes} k = \left(\underset{-1}{k\langle \delta e_1, \dots, \delta e_k \rangle} \xrightarrow{d} \underset{0}{k^{\oplus n}} \right)$$

$$\mathbb{L}_{A/k,x}^{\vee} = \left(\underset{0}{\left\langle \left(\frac{\partial}{\partial x_1}\right)_x, \dots, \left(\frac{\partial}{\partial x_n}\right)_x \right\rangle} \xrightarrow{\text{Jac} F} \underset{1}{\left\langle \frac{\partial}{\partial e_1}, \dots, \frac{\partial}{\partial e_n} \right\rangle} \right)$$

$F = (f_1, \dots, f_k)$

$H^0(\mathbb{L}_{A/k,x}^{\vee})$ = tangent space at x

$H^{-1}(\qquad)$ = "excess dimension"

④ $B = k[x,y]$ $f_1, f_2 \in B$ $A_i = B/(f_i)$ $\rightsquigarrow X_i = \operatorname{Spec} A_i$

$$X_1 \cap X_2 = \operatorname{Spec}(A_1 \underset{B}{\otimes} A_2)$$

$$X_1 \underset{\text{der}}{\cap} X_2 = \operatorname{Spec}(A_1 \overset{\mathbb{L}}{\underset{B}{\otimes}} A_2)$$ derived intersection.

Resolve A_2 by $B \xrightarrow{\cdot f_2} B$

$\Rightarrow$ $A_1 \overset{\mathbb{L}}{\underset{B}{\otimes}} A_2 := A_1 \xrightarrow{\cdot f_2} A_1$ $H^0_a(\quad) = A_1 \underset{B}{\otimes} A_2$

$\underline{\underline{f_1 = f_2}}$ $A_1 \xrightarrow{0} A_1$ $H^{-1}(\quad) = \operatorname{Tor}_B(A_1, A_2)$

cotangent cplx: $P = B[e, z]$ $\leftarrow$ deg -1 $de = f_1$, $dz = 0$

$\varphi(e) = 0$ $\varphi(z) = 1$

$$\begin{tikzcd} P_2 \arrow[r] & P_1 \arrow[r, "d"] \arrow[d, "\varphi"] & P_0 \arrow[d, "\varphi"] \\ & {} \arrow[r, "0"] & A \end{tikzcd}$$

$\rightsquigarrow$ semi free resol.

$$\longrightarrow \quad \mathbb{L}_{A/k,x} = \left(\Omega_{P/k} \otimes A\right) \otimes_{x} k$$

$$= \underset{-1}{\langle \delta z \rangle} \oplus \left(\underset{-1}{\langle \delta y \rangle} \xrightarrow{df_1} \underset{0}{\langle \delta x_1, \delta x_2 \rangle} \right)$$

excess dimension

if $df_1(x) \neq 0 \Rightarrow H^0(-) =$ 1-dim'l

$H^{-1}(-) =$ 1-dim'l

$$A = (A_1 \xrightarrow{0} A_1)$$

$A \xrightarrow{j} H_0(A) \qquad \mathrm{Spec}(H_0(A)) = X_1 \cap X_2$

$$j : X_1 \cap X_2 \hookrightarrow X_1 \underset{der}{\cap} X_2$$

$$j^* \mathbb{L}_{A/k,x} \longrightarrow \mathbb{L}_{H_0(A)/k,x}$$

Def: $f : A \to B$ is

formally étale $\Leftrightarrow \mathbb{L}_A \overset{\mathbb{L}}{\underset{A}{\otimes}} B \simeq \mathbb{L}_B$

" smooth $\Leftrightarrow L_{B/A}$ proj. + $\mathbb{L}_A \overset{\mathbb{L}}{\underset{A}{\otimes}} B \xrightarrow{} \mathbb{L}_B$ has a retraction s

Def'n $(P) =$ formally (P) + finitely presented $\qquad (P) = \begin{cases} \text{étale} \\ \text{smooth} \end{cases}$

Thm: f is smooth $\Leftrightarrow$ f is strong

+ $H_0(A) \longrightarrow H_0(B)$ smooth

Example: B smooth$/k \Rightarrow \ell \otimes_{k} H_0(B) \xrightarrow{\sim} H_\bullet(B)$

$\Rightarrow H_i(B) = 0 \quad \forall i \neq 0$

$\Rightarrow$ [illegible]$^{B} \cong H_0(B) \qquad \Rightarrow B$ is smooth affine scheme$/k$

DERIVED STACKS – Giorgia

1. der Stacks
2. cotangent cplx
3. Examples

$$\begin{tikzcd}
k\text{-alg} \arrow[r, "\text{schemes}"] \arrow[dr, "\text{1-stacks}"] \arrow[ddr, "(\infty\text{-})\text{stacks}"'] & \text{Set} \arrow[d] \\
& \text{Gpds} \arrow[d] \\
& \text{sSets}
\end{tikzcd}$$

- Grothendieck topology

$$F : k\text{-alg} \longrightarrow \text{Gpd}$$

"2-limit" taken in Gpd

1-stack: $\text{st. } F(X) \xrightarrow{\sim} \lim\left(\mathcal{F}(\mathcal{U}) \rightrightarrows \mathcal{F}(\mathcal{U} \times_X \mathcal{U}) \overset{\rightarrow}{\rightrightarrows} \mathcal{F}(\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U}) \right)$

$\{\mathcal{U} \to X\}$ covering

$$F : k\text{-alg} \longrightarrow \text{sSet}$$

$(\infty\text{-})$ stack: $\text{st. } F(X) \xrightarrow{\sim} \text{holim}\left(\mathcal{F}(\mathcal{U}) \rightrightarrows \mathcal{F}(\mathcal{U} \times_X \mathcal{U}) \overset{\rightarrow}{\rightrightarrows} \cdots \right)$

- <u>derived $(\infty\text{-})$ stacks</u>:

$$\begin{tikzcd}
k\text{-alg} \arrow[r, "\text{schemes}"] \arrow[d] & \text{Set} \arrow[d] \\
\text{dg-alg}_k \arrow[r, "\text{der stacks}"'] & \text{sSet}
\end{tikzcd}$$

<u>idea</u>: $F : \underbrace{\text{dg-alg}_k}_{} \longrightarrow \text{sSet}$

$A^\bullet, B^\bullet$ can be quasi-isom. for model structure on dg-alg

$F : \text{dg-alg}_k \longrightarrow \text{sSet}$ simplicial presheaf

Def'n: ① **Prestacks** are a subcategory of $\text{Ho}(\text{sPr}(\text{dg-alg}))$ corresponding to $F : \text{dg-alg}_k \longrightarrow \text{sSet}$ s.t.

$$A^{\cdot} \simeq B^{\cdot} \Rightarrow F(A) \simeq F(B)$$

i.e. we consider $\text{sPr}(\text{dg-alg}_k)$ and localize at $W = \{h_u : h_A \xrightarrow{\sim} h_B\}$, where $u : A \xrightarrow{\sim} B$ q-i.

$$\Rightarrow \quad F : \text{Ho}(\text{dg-alg}_k) \longrightarrow \text{sSet}$$

$\uparrow$

Need Grothendieck top. here.

Instead, we will define a "model topology" on dg-algs:

Def'n A **model topology** τ on a model category M is the following datum:

- $\forall X \in M$, $\text{Cov}_\tau(X)$ subset of objects in $\text{Ho}(M)/_X$ s.t.

(1) if $y \xrightarrow{\sim} X$ in $\text{Ho}(M)$, the 1-elt set $\{y \to X\} \in \text{Cov}_\tau(X$

(2) if $\{U_i \longrightarrow X\}_{i \in I} \in \text{Cov}_\tau(X)$ and $\{V_{ij} \to U_i\}_{j \in J_i} \in \text{Cov}_\tau(U_i)$

$$\to \{V_{ij} \longrightarrow X\}_{\substack{i \in I, \\ j \in J_i}} \in \text{Cov}_\tau(X)$$

Assume that $M = \text{dg-alg}_k$ has a model topology.

Def'n A **derived stack** is $F : \text{dg-alg} \longrightarrow \text{sSet}$ s.t.

(1) $\forall A^{\cdot} \xrightarrow{\sim} B^{\cdot} \Rightarrow F(A) \simeq F(B)$

(2) $\forall$ finite collections $\{A_i\}_{i \in I}$,

$$F(\sqcup \text{Spec} A_i) \xrightarrow[\text{in Ho(sSet)}]{\sim} \prod F(\text{Spec} A_i)$$

(3) $\{U \to X\}$ covering, then $F(X) \xrightarrow{\sim} \underset{n \in \Delta}{\text{holim}} \{F(U_n)\}$

$\mathcal{M}$ = dg-alge

Recall: $A^\cdot \in \mathcal{M}$. $M \in A$-mod is called strong if

$$H^\cdot(A) \otimes_{H^0(A)} H^0(M) \longrightarrow H^\cdot(M)$$ is an isomorphism

Def'n: M is projective (1) / flat (2) / perfect (3) is $\forall$ acyclic $C^\cdot$ we have that

(1) $\mathrm{Hom}_A(M, C^\cdot)$ is acyclic

(2) $M \otimes_A C^\cdot$ is acyclic

(3) $\mathrm{Hom}(M, -)$ commutes w/ direct limits.

Proposition: $M \in A$-Mod

(1) M is projective iff M is strong and $H^0(M)$ is a proj. $H^0(A)$-mo

(2) " flat " flat

(3) " perfect " $H^0(M)$ proj. + finitely gen $H^0(A)$-mod

~~y~~/ **Proposition:** $f: A \to B$, A, B dg-alg.

f strongly flat / smooth / étale if B is strong and $\mathrm{Spec}\, H^0(B) \longrightarrow \mathrm{Spec}\, H^0(A)$ is flat / smooth / étale.

Prop $f: A \to B$ is flat / smooth / étale $\Leftrightarrow$ strongly flat / smooth / strong.

Prop: étale coverings define a model ~~category~~ topology on dgal

$\rightsquigarrow$ $\boxed{\text{derived stack.}}$

WANT: "geometric stacks"

Will define "n-geometric derived stacks" $\rightsquigarrow$ "n-geom. stacks"

Idea: 0-geom:

$$\begin{tikzcd} S \times_X U \arrow[r] \arrow[d] & U = \mathrm{Spec}\, A \arrow[d] \\ S \arrow[r] & X \end{tikzcd}$$

Def'n (1) A <u>(-1)-geometric</u> <u>(derived) stack</u> is a representable (derived) stack.

(2) $f : F \longrightarrow G$ is <u>(-1)-representable</u> if $\forall X = \operatorname{Spec} A$,

$$\begin{tikzcd} F \times_G X \arrow[r] \arrow[d] & F \arrow[d] \\ X \arrow[r] & G \end{tikzcd}$$

$F \times_G X$ is (-1)-representable.

(3) $P = \{\text{smooth morphisms}\}$
$f : F \longrightarrow G$ is <u>(-1)-P</u> if it is (-1)-representable and $\forall X = \operatorname{Spec} A$, $\begin{matrix} F \times_G X \\ \downarrow \\ X \end{matrix}$ is in P.

If $n \geq 0$, an <u>n-atlas</u> for F is $\{U_i \longrightarrow F\}_{i \in I}$ s.t.

(a) U_i are representable
(b) $U_i \longrightarrow F$ are in $(n-1)$-P $\rightsquigarrow$
(c) $\coprod U_i \longrightarrow F$ epimorphism

$$\begin{tikzcd} U_i \times_F X \arrow[r] \arrow[d, "\tilde{f}"'] & U_i \\ \operatorname{Spec} A = X \arrow[r] & F \arrow[from=1-2, d] \end{tikzcd}$$

& $U_i \times_F X$ $(n-1)^P$-repr., where

$\tilde{f}$ s.t. $\exists\, \tilde{U} \longrightarrow U_i \times_F X$ and

$$\begin{tikzcd} \tilde{U} \arrow[r] \arrow[dr, "\text{in } P"'] & U_i \times_F X \arrow[d] \\ & X \end{tikzcd}$$

F is <u>n-geometric</u> if (a) $F \longrightarrow F \overset{h}{\times} F$ is $(n-1)$-representable
(b) F admits an n-atlas

<u>Example</u> X projective algebraic curve.

$\underline{\text{Vect}}_n(X) : \text{dg-alg} \longrightarrow \text{sSet}$
$A^\cdot$ [illegible] $\longmapsto \underline{\text{Vect}}_n(X)(A^\cdot) = \left\{ \begin{matrix} \text{vector bundles of} \\ \text{rk } n \text{ on } X \times \operatorname{Spec}(A^\cdot) \end{matrix} \right\}$

~~[illegible]~~

Category w weak equiv. Take nerve wrt weq. $\rightsquigarrow$ sSet.

would need to make sense of this —

$\underline{\text{Vect}}_n(X)$ factors through Ho(dg-alg)

Non-derived vs:

$$\mathrm{Vect}_n(X) : \mathrm{alg} \longrightarrow \mathrm{sSet}$$

$$A \longmapsto \{\text{v.b. on } X \times \mathrm{Spec}\, A\} \quad \text{seen as cst. sSet.}$$

$$\mathrm{alg} \xhookrightarrow{i} \mathrm{dgalg}$$

$$t_0(\underline{\mathrm{Vect}}_n(X)) : \mathrm{alg} \longrightarrow \mathrm{sSet}$$

$$t_0(\underline{\mathrm{Vect}}_n(X))(A) := \underline{\mathrm{Vect}}_n(X)(i(A))$$

<u>Proposition</u>: $t_0(\underline{\mathrm{Vect}}_n(X)) \cong \mathrm{Vect}_n(X)$

Cotangent complex for derived stacks:

Usually $A \xrightarrow{f} B$, $\mathrm{Spec}\, B \xrightarrow{f} \mathrm{Spec}\, A$

We define $\mathbb{L}_A$ by defining $f^*(\mathbb{L}_A)$ by saying that $f^*(\mathbb{L}_A)$ is the object representing the functor of derivations:

$$M \in B\text{-Mod}, \qquad \mathrm{Hom}_B(f^*(\mathbb{L}_A), M) = \mathrm{Der}_f(A, M)$$

$$\Vert$$

$$\left\{ \begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "f"'] & B \oplus M \arrow[d] \\ & B \end{tikzcd} \right\}$$

<u>For shacks</u>: $\mathrm{Spec}\, B \xrightarrow{f} X$

<u>Idea</u>: To define $\mathbb{L}_X$, we define $f^*(\mathbb{L}_X)$.

$$\left\{ \begin{tikzcd} \mathrm{Spec}(B \oplus M) \arrow[r, "g"] & \cancel{\mathrm{Spec}\, A}\ X \\ \mathrm{Spec}\, B \arrow[u] \arrow[ur, "f"'] & \end{tikzcd} \right\}$$

Define $f^*(\mathbb{L}_X)$ to be the object representing the following:

$$\text{\textcircled{$*$}} \begin{cases} B\text{-Mod} \longrightarrow \mathrm{sSet} \\ M \longmapsto \mathrm{hofiber}\left(X(B \oplus M) \longrightarrow X(B) \right) \end{cases}$$

$$X(B) \longleftarrow X(B \oplus M)$$

$$f \uparrow \qquad \uparrow \qquad \mathrm{Der}(B, M)$$

$\forall f: \operatorname{Spec} B \longrightarrow X$, $f^*(\mathbb{L}_X)$ is the object (if it exists) s.t.

$$\operatorname{Hom}(f^*(\mathbb{L}_X), M) \cong \operatorname{Der}_f(B,M) := X(B\oplus M) \times^h_{X(B)} *$$

<u>Defn</u> X has a <u>global cotangent complex</u> if (1) $\forall B, f: \operatorname{Spec} B \longrightarrow X$ the functor ⊛ is representable, i.e. $\exists f^*(\mathbb{L}_X)$

(2) $\forall u: A \longrightarrow B$,

$$\begin{tikzcd} Y = \operatorname{Spec} B \arrow[rr, "u"] \arrow[dr, "y"'] & & \operatorname{Spec} A = Z \arrow[dl, "x"] \\ & X & \end{tikzcd}$$

$$u^*: x^*(\mathbb{L}_X) \overset{\mathbb{L}}{\underset{A}{\otimes}} B \xrightarrow{\sim} y^*(\mathbb{L}_X)$$

How to construct the cotangent for a 0-geometric derived stack?

$$\begin{tikzcd} U \arrow[d, "\pi"] \\ X \end{tikzcd}$$

We can construct $\pi^*(\mathbb{L}_X)$ and this will be enough. The construction uses an exact triangle:

$$Z \xrightarrow{f} X$$

cf for schemes, $f^*(\mathbb{L}_X) \longrightarrow \mathbb{L}_Z \longrightarrow \mathbb{L}_{Z/X}$

$\rightsquigarrow$ <u>Define</u> $\pi^*(\mathbb{L}_X) :=$ cone of $(\mathbb{L}_U \longrightarrow \mathbb{L}_{U/X})$

<u>Now:</u>

$$\begin{tikzcd} U \times_X S \arrow[r, "\pi_S"] \arrow[d] & U \arrow[d, "\pi"] \\ S = \operatorname{Spec} A^\cdot \arrow[r, "f"'] & X \end{tikzcd}$$

$\rightsquigarrow \pi_S^* \pi^*(\mathbb{L}_X)$ on $U \times_X S$

iterate $\rightsquigarrow U \times_X U \times_X \cdots \times_X S$

and $f^*(\mathbb{L}_X) := \operatorname{holim}(\;\text{——}\;)$

1-stacks ⟶ cotangent cplx concentrated in deg $[0,1]$

0-geom. der stacks ⟶ " $[-\infty, 1]$

1-geom. der. stacks ⟶ " $[-\infty, 2]$

<u>Example</u>. $\underline{\mathrm{Vect}}_n(X)$, E a v.b. on X, $S = pt$.

$\mathrm{Vect}_n(X)$

$$\mathbb{T}_E(\underline{\mathrm{Vect}}_n(X)) = C^\bullet(X, \mathrm{End}\, E)[1]$$

$$\mathbb{T}_E(\mathrm{Vect}_n(X)) = \tau^{\leq 0}(C^\bullet(X, \mathrm{End}\, E)[1])$$

degree -1 $\quad$ 0

$H^0(X, \mathrm{End}\, E)$ $\quad$ $H^1(X, \mathrm{End}\, E)$